Thomas Jech

Set Theory

Second Edition

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Second Corrected Edition

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Perspectives in Mathematical Logic

This series was founded in 1969 by the Omega Group consisting of R. O. Gandy, H. Hermes, A. Levy, G. H. Müller, G. E. Sacks and D. S. Scott. Initially sponsored by a grant from the Stiftung Volkswagenwerk, the series appeared under the auspices of the Heidelberger Akademie der Wissenschaften. Since 1986, *Perspectives in Mathematical Logic* is published under the auspices of the Association for Symbolic Logic.

Mathematical Logic is a subject which is both rich and varied. Its origins lie in philosophy and the foundations of mathematics. But during the last half century it has formed deep links with algebra, geometry, analysis and other branches of mathematics. More recently it has become a central theme in theoretical computer science, and its influence in linguistics is growing fast.

The books in the series differ in level. Some are introductory texts suitable for final year undergraduate or first year graduate courses, while others are specialized monographs. Some are expositions of well-established material, some are at the frontiers of research. Each offers an illuminating perspective for its intended audience.

For Paula, Pavel, and Susanna

Aus dem Paradies, das Cantor uns geschaffen,
soll uns niemand vertreiben können.

David Hilbert

PREFACE

The main body of this book consists of 106 numbered theorems and a dozen of examples of models of set theory. A large number of additional results is given in the exercises, which are scattered throughout the text. Most exercises are provided with an outline of proof in square brackets [ ], and the more difficult ones are indicated by an asterisk.

I am greatly indebted to all those mathematicians, too numerous to mention by name, who in their letters, preprints, handwritten notes, lectures, seminars, and many conversations over the past decade shared with me their insight into this exciting subject.

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PART I

Sets

CHAPTER 1

AXIOMATIC SET THEORY

1. AXIOMS OF SET THEORY

Axioms of Zermelo–Fraenkel:

I. Axiom of Extensionality. *If X and Y have the same elements, then $X = Y$.*

II. Axiom of Pairing. *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

III. Axiom Schema of Separation. *If φ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : \varphi(u, p)\}$ that contains all those $u \in X$ that have the property φ.*

IV. Axiom of Union. *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

V. Axiom of Power Set. *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

VI. Axiom of Infinity. *There exists an infinite set.*

VII. Axiom Schema of Replacement. *If $\mathbf{F}$ is a function, then for any X there exists a set $Y = \mathbf{F}[X] = \{\mathbf{F}(x) : x \in X\}$.*

VIII. Axiom of Regularity. *Every nonempty set has an $\in$-minimal element.*

IX. Axiom of Choice. *Every family of nonempty sets has a choice function.*

The theory with axioms I–VIII is ZF, Zermelo–Fraenkel axiomatic set theory; ZFC denotes the theory ZF with the axiom of choice.

Why Axiomatic Set Theory?

Intuitively, a set is a collection of all elements that satisfy a certain given property. In other words, we might be tempted to postulate the following rule of formation for sets.

Axiom Schema of Comprehension (false). *If φ is a property, then there exists a set* $Y = \{x : \varphi(x)\}$.

This principle, however, is false:

Russell's Paradox. Consider the set S whose elements are all those (and only those) sets that are not members of themselves: $S = \{X : X \notin X\}$. Question: Does S belong to S? If S belongs to S, then S is not a member of itself, and so $S \notin S$. On the other hand, if $S \notin S$, then S belongs to S. In either case, we have a contradiction.

Thus we must conclude that

$$\{X : X \notin X\}$$

is not a set, and we must somewhat revise the intuitive notion of a set.

The safe way to eliminate paradoxes of this type is to abandon the schema of comprehension and keep its weak version, the *schema of separation*:

If φ is a property, then for any X there exists a set $Y = \{x \in X : \varphi(x)\}$.

Once we give up the full comprehension schema, Russell's paradox is no longer a threat; moreover, it provides useful information: The set of all sets does not exist. (Otherwise, apply the separation schema to the property $x \notin x$.)

In other words, it is the concept of the set of all sets that is paradoxical, not the idea of comprehension itself.

Replacing full comprehension by separation presents us with a new problem. The separation axioms are too weak to develop set theory with its usual operations and constructions. Notably, these axioms are not sufficient to prove that, e.g., the union $X \cup Y$ of two sets exists, or to define the notion of a real number.

Thus we have to add further construction principles that postulate the existence of sets obtained from other sets by means of certain operations.

The axioms of ZFC are generally accepted as a correct formalization of those principles that mathematicians apply when dealing with sets.

Language of Set Theory, Formulas

The axiom schema of separation as formulated above uses the vague notion of a *property*. To give the axioms a precise form, we develop axiomatic set theory in the framework of the first order predicate calculus. Apart from the equality predicate $=$, the language of set theory consists of the binary predicate $\in$, the *membership relation.*

The *formulas* of set theory are built up from the *atomic formulas*

$$x \in y, \qquad x = y$$

by means of connectives

$$\varphi \wedge \psi, \qquad \varphi \vee \psi, \qquad \neg\varphi, \qquad \varphi \rightarrow \psi, \qquad \varphi \leftrightarrow \psi$$

(conjunction, disjunction, negation, implication, equivalence), and *quantifiers*

$$\forall x\, \varphi, \qquad \exists x\, \varphi$$

In practice, we shall use in formulas other symbols, namely defined predicates, operations, and constants, and even use formulas informally; but it will be tacitly understood that each such formula can be written in a form that only involves $\in$ and $=$ as nonlogical symbols.

Concerning formulas with free variables, we adopt the notational convention that all free variables of a formula

$$\varphi(u_1, \ldots, u_n)$$

are among $u_1, \ldots, u_n$ (possibly some u_i are not free, or even do not occur, in φ). A formula without free variables is called a *sentence.*

Classes

Although we work in ZFC which, unlike alternative axiomatic set theories, has only one type of object, namely sets, we introduce the informal notion of a *class.* We do this for practical reasons: It is easier to manipulate classes than formulas.

If $\varphi(x, p_1, \ldots, p_n)$ is a formula, we call

$$\mathbf{C} = \{x : \varphi(x, p_1, \ldots, p_n)\}$$

a *class.* Members of the class **C** are all those sets x that satisfy $\varphi(x, p_1, \ldots, p_n)$:

$$x \in \mathbf{C} \qquad \text{iff} \qquad \varphi(x, p_1, \ldots, p_n)$$

We say that **C** is *definable from* $p_1, \ldots, p_n$; if φ has just one free variable, then the class **C** is *definable.*

We shall use boldface capital letters to denote classes. We shall, however, make departures from this general rule in cases when the standard notation is different (e.g., V, L, *Ord*, $\aleph$, etc.)

Two classes are considered equal if they have the same elements: If

$$\mathbf{C} = \{x : \varphi(x, p_1, \ldots, p_n)\}, \qquad \mathbf{D} = \{x : \psi(x, q_1, \ldots, q_m)\}$$

then $\mathbf{C} = \mathbf{D}$ iff for all x

$$\varphi(x, p_1, \ldots, p_n) \quad \leftrightarrow \quad \psi(x, q_1, \ldots, q_m)$$

The *universal class*, or *universe*, is the class of all sets:

$$V = \{x : x = x\}$$

We define *inclusion* of classes

$$\mathbf{C} \subseteq \mathbf{D} \quad \leftrightarrow \quad \forall x(x \in \mathbf{C} \rightarrow x \in \mathbf{D})$$

and the following operations on classes:

$$\mathbf{C} \cap \mathbf{D} = \{x : x \in \mathbf{C} \wedge x \in \mathbf{D}\}$$

$$\mathbf{C} \cup \mathbf{D} = \{x : x \in \mathbf{C} \vee x \in \mathbf{D}\}$$

$$\mathbf{C} - \mathbf{D} = \{x : x \in \mathbf{C} \wedge x \notin \mathbf{D}\}$$

$$\bigcup \mathbf{C} = \{x : x \in S \text{ for some } S \in \mathbf{C}\} = \bigcup \{S : S \in \mathbf{C}\}$$

Every set can be considered a class. If S is a set, consider the formula $x \in S$ and the class

$$\{x : x \in S\}$$

That the set S is uniquely determined by its elements follows from the axiom of extensionality.

A class that is not a set is a *proper class.*

Extensionality

If X and Y have the same elements, then $X = Y$:

$$\forall u(u \in X \leftrightarrow u \in Y) \rightarrow X = Y$$

The converse, namely, if $X = Y$, then $u \in X \leftrightarrow u \in Y$, is an axiom of predicate calculus. Thus we have

$$X = Y \quad \leftrightarrow \quad \forall u(u \in X \leftrightarrow u \in X)$$

The axiom expresses the basic idea of a set: A set is determined by its elements.

Pairing

For any a and b there exists a set $\{a, b\}$ that contains exactly a and b:

$$\forall a \; \forall b \; \exists c \; \forall x(x \in c \leftrightarrow x = a \vee x = b)$$

By Extensionality, the set c is unique, and we can define the *pair*

$$\{a, b\} = \text{the unique } c \text{ such that } \forall x(x \in c \leftrightarrow x = a \vee x = b)$$

The *singleton* $\{a\}$ is the set

$$\{a\} = \{a, a\}$$

Since $\{a, b\} = \{b, a\}$, we further define an *ordered pair*

$$(a, b)$$

so as to satisfy the following condition:

$$(1.1) \qquad (a, b) = (c, d) \quad \text{iff} \quad a = c \quad \text{and} \quad b = d$$

For the formal definition of an ordered pair, we take

$$(a, b) = \{\{a\}, \{a, b\}\}$$

Exercise 1.1. Verify that the definition of an ordered pair satisfies (1.1).

We further define ordered triples, quadruples, etc., as follows:

$$(a, b, c) = ((a, b), c)$$

$$(a, b, c, d) = ((a, b, c), d)$$

$$\vdots$$

$$(a_1, \ldots, a_{n+1}) = ((a_1, \ldots, a_n), a_{n+1})$$

It follows that two ordered n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are equal iff $a_1 = b_1, \ldots, a_n = b_n$.

Separation Schema

Let $\varphi(u, p)$ *be a formula. For any* X *and* p, *there exists a set* $Y = \{u \in X : \varphi(u, p)\}$:

$$(1.2) \qquad \forall X\, \forall p\, \exists Y\, \forall u(u \in Y \leftrightarrow u \in X \wedge \varphi(u, p))$$

For each formula $\varphi(u, p)$, the formula (1.2) is an axiom (of separation). The set Y in (1.2) is unique by Extensionality.

Note that a more general version of separation axioms can be proved using ordered n-tuples: Let $\psi(u, p_1, \ldots, p_n)$ be a formula. Then

$$(1.3) \qquad \forall X\, \forall p_1 \cdots \forall p_n\, \exists Y\, \forall u(u \in Y \leftrightarrow u \in X \wedge \psi(u, p_1, \ldots, p_n))$$

Simply let $\varphi(u, p)$ be the formula

$$\exists p_1 \cdots \exists p_n (p = (p_1, \ldots, p_n) \text{ and } \psi(u, p_1, \ldots, p_n))$$

and then, given X and $p_1, \ldots, p_n$, let

$$Y = \{u \in X : \varphi(u, (p_1, \ldots, p_n))\}$$

We can give the separation axioms the following form: Consider the class $\mathbf{C} = \{u : \varphi(u, p_1, \ldots, p_n)\}$; then by (1.3)

$$\forall X\, \exists Y(\mathbf{C} \cap X = Y)$$

Thus the intersection of a class $\mathbf{C}$ with any set is a set; or, we can say even more informally

a subclass of a set is a set

One consequence of the separation axioms is that the intersection and the difference of two sets is a set, and so we can define the operations

$$X \cap Y = \{u \in X : u \in Y\}, \qquad X - Y = \{u \in X : u \notin Y\}$$

Similarly, it follows that the empty class

$$\emptyset = \{u : u \neq u\}$$

is a set—the *empty set*; this, of course, only under the assumption that at least one set X exists (because $\emptyset \subseteq X$):

$$\exists X (X = X) \tag{1.4}$$

We have not included (1.4) among the axioms, but it follows from the axiom of infinity.

Two sets X, Y are called *disjoint* if $X \cap Y = \emptyset$.

If $\mathbf{C}$ is a nonempty class of sets, we let

$$\bigcap \mathbf{C} = \bigcap \{X : X \in \mathbf{C}\} = \{u : u \in X \text{ for every } X \in \mathbf{C}\}$$

Note that $\bigcap \mathbf{C}$ is a set (it is a subset of any $X \in \mathbf{C}$). Also, $X \cap Y = \bigcap \{X, Y\}$.

Another consequence of the separation axioms is that the universal class V is a proper class; otherwise,

$$S = \{x \in V : x \notin x\}$$

would be a set.

Union

For any X there exists a set $Y = \bigcup X$:

$$\forall X \, \exists Y \, \forall u (u \in Y \leftrightarrow \exists z (z \in X \wedge u \in z)) \tag{1.5}$$

Let us introduce the abbreviations

$$(\exists z \in X)\, \varphi \qquad \text{for} \qquad \exists z (z \in X \wedge \varphi)$$

and

$$(\forall z \in X)\, \varphi \qquad \text{for} \qquad \forall z (z \in X \rightarrow \varphi)$$

By (1.5), for every X there is a unique set

$$Y = \{u : (\exists z \in X)\,[u \in z]\} = \bigcup \{z : z \in X\} = \bigcup X$$

the *union* of X.

Now we can define

$$X \cup Y = \bigcup \{X, Y\}, \qquad X \cup Y \cup Z = (X \cup Y) \cup Z, \qquad \text{etc.}$$

and also

$$\{a, b, c\} = \{a, b\} \cup \{c\}$$

and in general

$$\{a_1, \ldots, a_n\} = \{a_1\} \cup \cdots \cup \{a_n\}$$

Power Set

For any X there exists a set $Y = P(X)$:

$$\forall X\ \exists Y\ \forall u(u \in Y \leftrightarrow u \subseteq X)$$

A set U is a *subset* of X, $U \subseteq X$, if

$$\forall z(z \in U \rightarrow z \in X)$$

If $U \subseteq X$ and $U \neq X$, then U is a *proper subset*, $U \subset X$.

The set of all subsets of X

$$P(X) = \{u : u \subseteq X\}$$

is called the *power set* of X.

Exercise 1.2. There is no set X such that $P(X) \subseteq X$.
[Use Russell's paradox.]

Using the power set axiom we can define other basic notions of set theory.

The *Cartesian product* of X and Y is the set of all pairs (x, y) such that $x \in X$ and $y \in Y$:

$$X \times Y = \{(x, y) : x \in X \wedge y \in Y\} \tag{1.6}$$

The notation $\{(x, y) : \cdots\}$ in (1.6) is justified because

$$\{(x, y) : \varphi(x, y)\} = \{u : \exists x\ \exists y(u = (x, y) \wedge \varphi(x, y))\}$$

The product $X \times Y$ is a set because

$$X \times Y \subseteq PP(X \cup Y)$$

Further, we define

$$X \times Y \times Z = (X \times Y) \times Z$$

and in general

$$X_1 \times \cdots \times X_{n+1} = (X_1 \times \cdots \times X_n) \times X_{n+1}$$

Thus

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : x_1 \in X_1 \wedge \cdots \wedge x_n \in X_n\}$$

We also let

$$X^n = X \times \cdots \times X \quad (n \text{ times})$$

An *n-ary relation* R is a set of n-tuples. R is a relation *over* X if $R \subseteq X^n$. It is customary to write $R(x_1, \ldots, x_n)$ instead of

$$(x_1, \ldots, x_n) \in R$$

and in case that R is binary, then we also use

$$x \; R \; y$$

for $(x, y) \in R$.

If R is a binary relation, then the *domain of* R is the set

$$\operatorname{dom}(R) = \{u : \exists v[(u, v) \in R]\}$$

and the *range* of R is the set

$$\operatorname{ran}(R) = \{v : \exists u[(u, v) \in R]\}$$

Note that $\operatorname{dom}(R)$ and $\operatorname{ran}(R)$ are sets because

$$\operatorname{dom}(R) \subseteq \bigcup \bigcup R, \qquad \operatorname{ran}(R) \subseteq \bigcup \bigcup R.$$

The *field* of a relation R is the set

$$\text{field } (R) = \operatorname{dom}(R) \cup \operatorname{ran}(R).$$

In general, we call a class $\mathbf{R}$ an n-ary relation if all its elements are n-tuples; in other words, if

$$\mathbf{R} \subseteq V^n = \text{the class of all } n\text{-tuples}$$

where $\mathbf{C}^n$ (and $\mathbf{C} \times \mathbf{D}$) is defined in the obvious way.

A binary relation f is a *function* if

$$(x, y) \in f \quad \text{and} \quad (x, z) \in f \qquad \text{implies} \qquad y = z$$

The unique y such that $(x, y) \in f$ is the *value* of f at x; we use the standard notation

$$y = f(x)$$

or its variations

$$f\colon x \mapsto y, \qquad y = f_x, \qquad \text{etc.}$$

for $(x, y) \in f$.

f is a function *on* X if $X = \operatorname{dom}(f)$. If $\operatorname{dom}(f) = X^n$, then f is an *n-ary function* on X.

f is a function *from* X *to* Y,

$$f\colon X \to Y$$

if $\operatorname{dom}(f) = X$ and $\operatorname{ran}(f) \subseteq Y$. The set of all functions from X to Y is denoted by ${}^XY$. Note that ${}^XY$ is a set:

$${}^XY \subseteq P(X \times Y)$$

If $Y = \mathrm{ran}(f)$, then f is a function *onto* Y. A function f is *one-to-one* if

$$f(x) = f(y) \qquad \text{implies} \qquad x = y$$

An *n-ary operation over* X is a function $f\colon X^n \to X$.

The *restriction* of a function f to a set X (usually a subset of $\mathrm{dom}(f)$) is the function

$$f \,|\, X = \{(x, y) \in f : x \in X\}$$

A function g is an *extension* of a function f if $g \supseteq f$, i.e., $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $g(x) = f(x)$ for all $x \in \mathrm{dom}(f)$.

We denote the *image of* X *by* f by

$$f[X] = \{y : (\exists x \in X)\,[y = f(x)]\}$$

and the *inverse image* by

$$f_{-1}(X) = \{x : f(x) \in X\}$$

If f is one-to-one, then f^{-1} denotes the *inverse* of f:

$$f^{-1}(x) = y \qquad \text{iff} \qquad x = f(y)$$

These definitions also apply to functions that are classes, i.e., a relation $\mathbf{F}$ such that

$$(x, y) \in \mathbf{F} \quad \text{and} \quad (x, z) \in \mathbf{F} \qquad \text{implies} \qquad y = z$$

For instance, $\mathbf{F}[\mathbf{C}]$ denotes the image of the class $\mathbf{C}$ under the function $\mathbf{F}$.

It should be noted that a function is often called a *mapping* or a *correspondence* (and similarly, a set is called a *family* or a *collection*).

An *equivalence relation* over a set X is a binary relation $\equiv$ which is *reflexive*, *symmetric*, and *transitive*:

$$x \equiv x \text{ for all } x \in X$$

$$x \equiv y \text{ implies } y \equiv x$$

$$x \equiv y \text{ and } y \equiv z \text{ implies } x \equiv z$$

A family of sets is *disjoint* if any two of its members are disjoint. A *partition* of a set X is a disjoint family P of nonempty sets such that

$$X = \bigcup \{Y : Y \in P\}$$

Let $\equiv$ be an equivalence relation over X. For every $x \in X$, let

$$[x] = \{y \in X : y \equiv x\}$$

(the *equivalence class* of x). The set

$$X/{\equiv} \;= \{[x] : x \in X\}$$

is a partition of X (the *quotient* of X by $\equiv$). Conversely, each partition P of X defines an equivalence relation over X:

$$x \equiv y \quad \leftrightarrow \quad (\exists Y \in P)\,[x \in Y \wedge y \in Y]$$

If an equivalence relation is a class, then its equivalence classes may be proper classes. In Section 9 we shall introduce a trick that enables us to handle equivalence classes as if they were sets.

Infinity

There exists an infinite set.

To give a precise formulation of the axiom of infinity, we have to define first the notion of finiteness. The most obvious definition of finiteness uses the notion of a natural number, which is as yet undefined. We shall define natural numbers (as finite ordinals) in Section 2 and give only a quick treatment of natural numbers and finiteness in the exercises below.

In principle, it is possible to give a definition of finiteness that does not mention numbers, but such definitions necessarily look artificial. We give the most successful version below.

We therefore formulate the axiom of infinity differently:

$$\exists S[\emptyset \in S \wedge (\forall x \in S)\,[x \cup \{x\} \in S]]$$

We call a set S with the above property *inductive*. Thus we have:

Axiom of Infinity. *There exists an inductive set.*

The idea is that an inductive set is infinite. We note that using the replacement schema, one can show that an inductive set exists if there exists an infinite set (see Section 2).

A set S is *T-finite* if every nonempty $X \subseteq P(S)$ has a $\subset$-maximal element, i.e., $u \in X$ such that there is no $v \in X$ with $u \subset v$.

Let

$$N = \bigcap \{X : X \text{ is inductive}\}$$

N is the smallest inductive set. Let us use the following notation:

$$0 = \emptyset, \qquad 1 = \{0\}, \qquad 2 = \{0, 1\}, \qquad \ldots$$

If $n \in N$, let $n + 1 = n \cup \{n\}$. Let us define $<$ (over N) by

$$n < m \quad \leftrightarrow \quad n \in m$$

A set T is *transitive* if $x \in T$ implies $x \subseteq T$.

Exercise 1.3. If X is inductive, then the set $\{x \in X : x \subseteq X\}$ is inductive. Hence N is transitive, and for each n,

$$n = \{m \in N : m < n\}$$

Exercise 1.4. If X is inductive, then the set $\{x \in X : x \text{ is transitive}\}$ is inductive. Hence every $n \in N$ is transitive.

Exercise 1.5. If X is inductive, then the set $\{x \in X : x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in N$.

Exercise 1.6. N is T-infinite; the set $N \subseteq P(N)$ has no $\subset$-maximal element.

Exercise 1.7. If X is inductive, then $\{x \in X : x \text{ is transitive and every nonempty } z \subseteq x \text{ has an } \in\text{-minimal element}\}$ is inductive (t is *$\in$-minimal* in z if there is no $s \in z$ such that $s \in t$).

Exercise 1.8. Every nonempty $X \subseteq N$ has an $\in$-minimal element.
[Pick $n \in X$ and look at $X \cap n$.]

Exercise 1.9. If X is inductive then so is $\{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$. Hence each $n \neq 0$ is $m + 1$ for some m.

Exercise 1.10. *Induction.* Let A be a subset of N such that $0 \in A$, and $n \in A \rightarrow n + 1 \in A$. Then $A = N$.

Exercise 1.11. Each $n \in N$ is T-finite.

A set X is said to *have n elements* ($n \in N$) if there is a one-to-one mapping of n onto X. A set is *finite* if it has n elements for some $n \in N$.

Exercise 1.12. Every finite set is T-finite.

Exercise 1.13. Every infinite set is T-infinite.
[If S is infinite, consider $X = \{u \subset S : u \text{ is finite}\}$.]

Replacement Schema

If **F** *is a function, then for every* X, $\mathbf{F}[X]$ *is a set.*

For each formula $\varphi(x, y, p)$, the formula (1.7) is an axiom (of replacement):

$$\begin{aligned} &\forall x\, \forall y\, \forall z[\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y = z] \\ &\rightarrow \quad \forall X\, \exists Y\, \forall y[y \in Y \leftrightarrow (\exists x \in X)\, \varphi(x, y, p)] \end{aligned} \tag{1.7}$$

As in case of separation axioms, we can prove the version of replacement axioms with more than one parameter: Replace p by $p_1, \ldots, p_n$.

If $\mathbf{F} = \{(x, y) : \varphi(x, y, p)\}$, then the premise of (1.7) says that **F** is a function, and we get the formulation above. We can also formulate the axioms in the following ways:

If **F** *is a function and* dom(**F**) *is a set, then* ran(**F**) *is a set.*
If **F** *is a function, then* $\forall X\ \exists f[\mathbf{F} \mid X = f]$.

Exercise 1.14. The separation axioms follow from the replacement schema.
[Given φ, let $\mathbf{F} = \{(x, x) : \varphi(x)\}$. Then $\{x \in X : \varphi(x)\} = \mathbf{F}[X]$, for every X.]

Exercise 1.15. Instead of union, power set, and replacement axioms consider the following weaker versions:

(1.8) $\forall X\ \exists Y[\bigcup X \subseteq Y]$ (i.e., $\forall X\ \exists Y(\forall x \in X)(\forall u \in x)\,[u \in Y]$)

(1.9) $\forall X\ \exists Y[P(X) \subseteq Y]$ $(\forall X\ \exists Y\ \forall u[u \subseteq X \to u \in Y])$

(1.10) If $\mathbf{F}$ is a function, then $\forall X\ \exists Y(\mathbf{F}[X] \subseteq Y)$

Then axioms IV, V, and VII can be proved from (1.8), (1.9), and (1.10), using the separation schema (1.3).

The remaining two axioms, choice and regularity, will be dealt with later.

2. ORDINAL NUMBERS

In this section we introduce ordinal numbers and prove the theorems on transfinite induction and transfinite recursion.

Linear and Partial Ordering

A binary relation $<$ over a set P is a *partial ordering of P* if:

(i) $p \nless p$ for any $p \in P$;
(ii) if $p < q$ and $q < r$, then $p < r$.

$(P, <)$—and, by abuse of notation, P—is called a *partially ordered set.* A partial ordering $<$ of P is a *linear ordering* if moreover

(iii) $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$.

If $<$ is a partial (linear) ordering, then the relation $\leq$ is also called a partial (linear) ordering (and $<$ is sometimes called a *strict* ordering).

If P is a partially ordered set and X is a nonempty subset of P, and $a \in P$, then:

a is a *maximal* element of X if $a \in X \wedge (\forall x \in X)\,[a \nless x]$;
a is a *minimal* element of X if $a \in X \wedge (\forall x \in X)\,[x \nless a]$;
a is the *greatest* element of X if $a \in X \wedge (\forall x \in X)\,[x \leq a]$;
a is the *least* element of X if $a \in X \wedge (\forall x \in X)\,[a \leq x]$;
a is an *upper bound* of X if $(\forall x \in X)\,[x \leq a]$;
a is a *lower bound* of X if $(\forall x \in X)\,[a \leq x]$;
a is the *supremum* of X if a is the least upper bound of X;
a is the *infimum* of X if a is the greatest lower bound of X.

The supremum (infimum) of X (if it exists) is denoted by $\sup X$ ($\inf X$). Note that if X is linearly ordered by $<$, then a maximal element of X is its greatest element (similarly for a minimal element).

If $(P, <)$ and $(Q, <)$ are partially ordered sets and $f\colon P \to Q$, then f is *order-preserving* if

$$x < y \quad \leftrightarrow \quad f(x) < f(y)$$

If P and Q are linearly ordered, then an order-preserving function is also said to be *increasing.*

A one-to-one order-preserving function of P onto Q is an *isomorphism* of P and Q; $(P, <)$ is then *isomorphic* to $(Q, <)$.

Exercise 2.1. The relation "$(P, <)$ is isomorphic to $(Q, <)$" is an equivalence relation (over the class of all partially ordered sets).

Well-Ordering

A linear ordering $<$ of a set P is a *well-ordering* if every nonempty subset of P has a least element.

The concept of well-ordering is of fundamental importance in view of the induction theorems. It is shown below that well-ordered sets can be compared by their lengths; ordinal numbers will be introduced as order-types of well-ordered sets.

Lemma 2.1. *If $(W, <)$ is a well-ordered set and $f\colon W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$.*

Proof. Assume that the set $X = \{x \in W : f(x) < x\}$ is nonempty and let z be the least element of X. If $w = f(z)$, then $f(w) < w$, a contradiction. ■

Corollary 1. *The only isomorphism of a well-ordered set onto itself is the identity.*

Proof. By Lemma 2.1, $f(x) \geq x$ for all x, and $f^{-1}(x) \geq x$ for all x. ■

Corollary 2. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique.* ■

If W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an *initial segment* of W (given by u).

Lemma 2.2. *No well-ordered set is isomorphic to an initial segment of itself.*

Proof. If $\operatorname{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to Lemma 2.1. ■

Theorem 1. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

(a) *W_1 is isomorphic to W_2 ;*
(b) *W_1 is isomorphic to an initial segment of W_2 ;*
(c) *W_2 is isomorphic to an initial segment of W_1.*

Proof. For $u \in W_i$ $(i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 : W_1(x) \text{ is isomorphic to } W_2(y)\}$$

Using Lemma 2.2, it is easy to see that f is a one-to-one function. If h is an isomorphism between $W_1(x)$ and $W_2(y)$ and $x' < x$, then $W_1(x')$ and $W_2(h(x'))$ are isomorphic. It follows that f is order-preserving.

If $\operatorname{dom}(f) = W_1$ and $\operatorname{ran}(f) = W_2$, then case (a) holds.

If $y_1 < y_2$ and $y_2 \in \operatorname{ran}(f)$, then $y_1 \in \operatorname{ran}(f)$. Thus if $\operatorname{ran}(f) \neq W_2$ and y_0 is the least element of $W_2 - \operatorname{ran}(f)$, we have $\operatorname{ran}(f) = W_2(y_0)$. Necessarily, $\operatorname{dom}(f) = W_1$, for otherwise we would have $(x_0, y_0) \in f$, where $x_0 =$ the least element of $W_1 - \operatorname{dom}(f)$. Thus case (b) holds.

Similarly, if $\operatorname{dom}(f) \neq W_1$, then case (c) holds.

In view of Lemma 2.2, the three cases are mutually exclusive. ■

If W_1 and W_2 are isomorphic, we say that they have the same *order-type*:

$$\text{o.t. } W_1 = \text{o.t. } W_2$$

Similarly, we define

$$\text{o.t. } W_1 < \text{o.t. } W_2$$

if W_1 is isomorphic to an initial segment of W_2.

Exercise 2.2. Assume that each well-ordered set W is assigned an *ordinal* $\alpha = \text{o.t. } W$. Show that the relation $\alpha < \beta$ is a linear ordering of the class of all ordinals and that every nonempty set of ordinals has a least element.

We shall now give a formal definition of ordinal numbers.

Ordinals

The idea is to define ordinal numbers so that

$$\alpha < \beta \quad \text{iff} \quad \alpha \in \beta, \qquad \text{and} \qquad \alpha = \{\beta : \beta < \alpha\}$$

A set T is *transitive* if

$$\forall x(x \in T \rightarrow x \subseteq T)$$

In other words, $\bigcup T \subseteq T$, or $T \subset P(T)$.

A set is an *ordinal number* (an *ordinal*) if it is transitive and well-ordered by $\in$.

We shall denote ordinals by lowercase Greek letters $\alpha, \beta, \gamma, \ldots$. The class of all ordinals is denoted by *Ord*.

We define

$$\alpha < \beta \qquad \text{iff} \qquad \alpha \in \beta$$

Lemma 2.3.

(a) *$0 = \emptyset$ is an ordinal.*
(b) *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*
(c) *If α, β are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$.*
(d) *If α, β are ordinals, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

Proof. (a), (b) by definition.

(c) If $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ. Thus $\alpha = \{\xi \in \beta : \xi < \gamma\} = \gamma$, and so $\alpha \in \beta$.

(d) Clearly, $\alpha \cap \beta$ is an ordinal, $\alpha \cap \beta = \gamma$. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$, and $\gamma \in \beta$, by (c). Then $\gamma \in \gamma$, which contradicts the definition of an ordinal (namely that $\in$ is a *strict* ordering of α). ■

Using Lemma 2.3, one gets the following facts about ordinal numbers (the proofs are routine):

(2.1) $\alpha < \beta$ is a linear ordering of the class *Ord*.
(2.2) For each α, $\alpha = \{\beta : \beta < \alpha\}$.
(2.3) If $\mathbf{C}$ is a nonempty class of ordinals, then $\bigcap \mathbf{C}$ is an ordinal, $\bigcap \mathbf{C} \in \mathbf{C}$ and $\bigcap \mathbf{C} = \inf \mathbf{C}$.
(2.4) If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$.
(2.5) For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$.

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$ (the *successor* of α). In view of (2.4), the class *Ord* is a proper class; otherwise, consider $\sup(Ord) + 1$.

We can now prove that the above definition of ordinals provides us with order-types of well-ordered sets.

Theorem 2. *Every well-ordered set is isomorphic to a unique ordinal number.*

Proof. The uniqueness follows from Lemma 2.2. Given a well-ordered set W, we find an isomorphic ordinal as follows: Let $\mathbf{F}(x) = \alpha$ if α is isomorphic to the initial segment of W given by x. If such an α exists, then it is unique. By the replacement axioms, $\mathbf{F}[W]$ is a set. For each $x \in W$, such an α exists (otherwise consider the least x for which such an α does not exist). If γ is the least $\gamma \notin \mathbf{F}[W]$, then $\mathbf{F}[W] = \gamma$ and we have an isomorphism of W onto γ. ■

If $\alpha = \beta + 1$, then α is a *successor ordinal*. If α is not a successor ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$; α is called a *limit ordinal*. We also consider 0 a limit ordinal and define $\sup \emptyset = 0$.

Exercise 2.3. α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$, for every β.

The existence of limit ordinals other than 0 follows from the axiom of infinity:

Exercise 2.4. If a set X is inductive, then $X \cap Ord$ is inductive. The set $N = \bigcap \{X : X \text{ is inductive}\}$ is the least limit ordinal $\neq 0$.

We denote the least nonzero limit ordinal ω (or N). The ordinals less than ω (elements of N) are called *finite ordinals*, or *natural numbers*. Specifically,

$$0 = \emptyset, \qquad 1 = 0 + 1, \qquad 2 = 1 + 1, \qquad \text{etc.}$$

We use n, m, l, k, j, i (usually) to denote natural numbers.

Exercise 2.5. (without the axiom of infinity). Let $\omega =$ least limit $\alpha \neq 0$ if it exists, $\omega = Ord$ otherwise. Call a set X finite if X is in a one-to-one correspondence with some $n \in \omega$. Show:

$$\exists(\text{inductive set}) \rightarrow \exists(\text{infinite set}) \rightarrow \omega \text{ is a set} \rightarrow \exists(\text{inductive set})$$

[For the second implication, apply replacement to the set {all finite subsets of X}.]

Induction Theorems

Theorem 3 (*Transfinite Induction*). *Let* **C** *be a class of ordinals and assume that:*

(i) $0 \in \mathbf{C}$;
(ii) *if* $\alpha \in \mathbf{C}$, *then* $\alpha + 1 \in \mathbf{C}$;
(iii) *if* α *is a nonzero limit ordinal and* $\beta \in \mathbf{C}$ *for all* $\beta < \alpha$, then $\alpha \in \mathbf{C}$.

Then **C** *is the class of all ordinals.*

Proof. Otherwise, let $\alpha = \inf(Ord\text{-}\mathbf{C})$ and apply (i), (ii), or (iii). ■

A function whose domain is the set N is called an (*infinite*) *sequence*. (A *sequence in* X is a function $f: N \rightarrow X$.) The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

or variants thereof. A *finite sequence* is a function s such that $\text{dom}(s) = \{i : i < n\}$ for some $n \in N$; then s is a *sequence of length* n.

A *transfinite sequence* is a function whose domain is an ordinal:

$$\langle a_\xi : \xi < \alpha \rangle$$

It is also called an α-*sequence*, or a *sequence of length* α. We also say that a sequence $\langle \alpha_\xi : \xi < \alpha \rangle$ is an *enumeration* of its range $\{\alpha_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cup (\alpha, x)$$

Sometimes we shall call a "sequence"

$$\langle a_\alpha : \alpha \in Ord \rangle$$

a function (a proper class) on Ord.

"Definition by transfinite induction" usually takes the following form: Given a function **G** (on the class of transfinite sequences), then for every θ there exists a unique θ-sequence

$$\langle a_\alpha : \alpha < \theta \rangle$$

such that

$$a_\alpha = \mathbf{G}(\langle a_\xi : \xi < \alpha \rangle)$$

for every $\alpha < \theta$.

We shall give a general version of this theorem, so that we can also construct sequences $\langle a_\alpha : \alpha \in Ord \rangle$.

Theorem 4 (*Transfinite Recursion; Definition by Induction*). *For every* **G** *there is* **F** *such that: If* **G** *is a function on V, then* **F** *is the unique function on Ord such that*

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F} \restriction \alpha)$$

for each α.

In other words, if we let $a_\alpha = \mathbf{F}(\alpha)$, then for each α,

$$a_\alpha = \mathbf{G}(\langle a_\xi : \xi < \alpha \rangle)$$

(Note that we tacitly use replacement: $\mathbf{F} \restriction \alpha$ is a set for each α.)

Proof. We let

$$\begin{aligned} \mathbf{F}(\alpha) = x \leftrightarrow \text{there is a sequence } &\langle a_\xi : \xi < \alpha \rangle \text{ such that:} \\ &\text{(a) } (\forall \xi < \alpha)[a_\xi = \mathbf{G}(\langle a_\eta : \eta < \xi \rangle)]; \\ &\text{(b) } x = \mathbf{G}(\langle a_\xi : \xi < \alpha \rangle). \end{aligned}$$

For every α, if there is an α-sequence that satisfies (a), then such a sequence is unique: If $\langle a_\xi : \xi < \alpha \rangle$ and $\langle b_\xi : \xi < \alpha \rangle$ are two α-sequences satisfying (a), one shows $a_\xi = b_\xi$ by induction on ξ. Thus $\mathbf{F}(\alpha)$ is determined uniquely by (b), and therefore **F** is a function. It follows, again by induction, that for each α there is an α-sequence that satisfies (a) (at limit steps, we use replacement to get the α-sequence as the union of all the ξ-sequences, $\xi < \alpha$). Thus **F** is defined for all $\alpha \in Ord$. It obviously satisfies

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F} \restriction \alpha)$$

If $\mathbf{F}'$ is any function on *Ord* that satisfies

$$\mathbf{F}'(\alpha) = \mathbf{G}(\mathbf{F}' \restriction \alpha)$$

then it follows by induction that $\mathbf{F}'(\alpha) = \mathbf{F}(\alpha)$ for all α. ■

Exercise 2.6. If W is a well-ordered set, then there exists no sequence $\langle a_n : n \in N \rangle$ in W such that $a_0 > a_1 > a_2 > \cdots$.

(Using the axiom of choice, this condition implies that W is well-ordered.)

Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a *nondecreasing* sequence of ordinals (i.e., $\xi < \eta$ implies $\gamma_\xi \leq \gamma_\eta$). We define the *limit* of the sequence by

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}$$

Exercise 2.7. There are arbitrarily large limit ordinals; i.e., $\forall \alpha \, \exists \beta > \alpha$ (β is a limit).
[Consider $\lim_{n \to \omega} \alpha_n$, where $\alpha_{n+1} = \alpha_n + 1$.]

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in Ord \rangle$ is *normal* if it is increasing and *continuous*, i.e., if for every limit α, $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$.

Exercise 2.8. Every normal sequence has arbitrarily large *fixed points*, i.e., α such that $\gamma_\alpha = \alpha$.
[Let $\alpha_{n+1} = \gamma_{\alpha_n}$, and $\alpha = \lim_{n \to \omega} \alpha_n$.]

Ordinal Sums and Products

We shall briefly touch the subject of addition and multiplication of order-types, and define ordinal sum and product, $\alpha + \beta$ and $\alpha \cdot \beta$.

Let us assume that we define in some way the *order-type* of linearly ordered sets. That is, two linearly ordered sets have the same order-type just in case they are isomorphic. The order-type of a well-ordered set W is the unique ordinal isomorphic to W.

If τ_1 and τ_2 are order-types, then $\tau_1 + \tau_2$ is defined as the order-type of a set $X \cup Y$ where X and Y are disjoint, o.t. $X = \tau_1$, o.t. $Y = \tau_2$, and $X \cup Y$ is ordered so that all $x \in X$ precede all $y \in Y$.

In case of ordinal numbers, the *ordinal sum*

$$\alpha + \beta$$

is the unique ordinal $\gamma \geq \alpha$ such that β is isomorphic to the set $\gamma - \alpha = \{\xi : \alpha \leq \xi < \gamma\}$.

Exercise 2.9. Prove that

$$\alpha + 0 = \alpha, \qquad \alpha + (\beta + 1) = (\alpha + \beta) + 1$$

$$\alpha + \beta = \lim_{\xi \to \beta} (\alpha + \xi), \qquad \text{if } \beta \text{ is a limit}$$

Thus $\alpha + \beta$ is a normal function of β.

The ordinal sum is associative, $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, but *not* commutative:

$$1 + \omega = \omega \neq \omega + 1$$

To define ordinal product, we consider *lexicographic ordering* of Cartesian products. If A and B are linearly ordered sets, we define the (vertical) lexicographic ordering of $A \times B$ as follows:

$$(2.6) \quad (a_1, b_1) < (a_2, b_2) \quad \leftrightarrow \quad \text{either } a_1 < a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 < b_2)$$

It is easy to see that the relation $<$ defined by (2.6) is a linear ordering of $A \times B$. See Fig. 2.1.

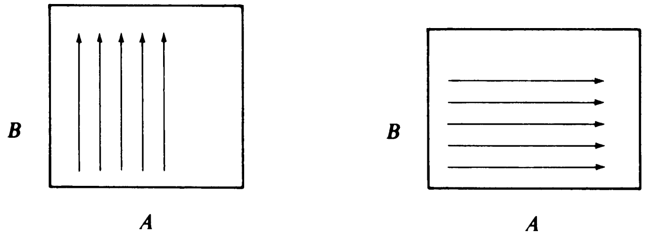

FIGURE 2.1 Vertical and horizontal lexicographic ordering.

Similarly, the *horizontal lexicographic ordering* is defined as follows:

$$(2.7) \quad (a_1, b_1) < (a_2, b_2) \quad \leftrightarrow \quad \text{either } b_1 < b_2 \text{ or } (b_1 = b_2 \text{ and } a_1 < a_2)$$

The *product* of order-types $\tau_1 \cdot \tau_2$ is defined as the order-type of the Cartesian product $A \times B$ of sets of order-types τ_1 and τ_2, respectively, under the horizontal lexicographic ordering.

If both A and B are well-ordered, it is easy to verify that $A \times B$ is well-ordered by either lexicographic ordering. Thus the *ordinal product*

$$\alpha \cdot \beta$$

is the unique ordinal isomorphic to $\alpha \times \beta$ under the horizontal lexicographic ordering.

Note that $\alpha \cdot 2 = \alpha + \alpha$.

Exercise 2.10. Prove that

$$\alpha \cdot 0 = 0, \qquad \alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$$
$$\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi, \qquad \text{if } \beta \text{ is a limit}$$

Thus for $\alpha \neq 0$, $\alpha \cdot \beta$ is a normal function of β.

Exercise 2.11. An ordinal α is a limit ordinal if and only if $\alpha = \omega \cdot \beta$ for some β.

The ordinal product is associative, but *not* commutative:

$$2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega$$

As for distributivity, we have

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

while, e.g.,

$$(\omega + 1) \cdot 2 = (\omega + 1) + (\omega + 1) = \omega \cdot 2 + 1$$

Exercise 2.12. Define by induction:

$$\alpha^0 = 1, \qquad \alpha^{\beta+1} = \alpha^\beta \cdot \alpha$$

$$\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi \qquad \text{if } \beta \text{ is a limit}$$

The function α^β is a normal function of β (for $\alpha > 1$).

Exercise 2.13. $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$, $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.

The Canonical Well-Ordering of $\alpha \times \alpha$

We define a well-ordering of the class *Ord* × *Ord* of ordinal pairs. Under this well-ordering, each $\alpha \times \alpha$ is an initial segment of Ord^2; the induced well-ordering of α^2 is called the *canonical well-ordering* of α^2. Moreover, the well-ordered class Ord^2 is isomorphic to the class *Ord*, and we have a one-to-one function Γ of Ord^2 onto *Ord*. As a bonus, for many α's the order-type of $\alpha \times \alpha$ is α.

If α, β are ordinals, let $\max(\alpha, \beta)$ be the greater ordinal of the two. We define:

$$(2.8) \quad (\alpha, \beta) < (\gamma, \delta) \quad \leftrightarrow \quad \begin{array}{l} \text{either } \max(\alpha, \beta) < \max(\gamma, \delta) \\ \text{or } (\max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma) \\ \text{or } (\max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta) \end{array}$$

The relation $<$ defined in (2.8) is obviously a linear ordering of the class *Ord* × *Ord*. Moreover, if $\mathbf{X} \subset Ord \times Ord$ is nonempty, then $\mathbf{X}$ has a least element (verify!). Also, for each α, $\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$. See Fig. 2.2. If we let

$$\Gamma(\alpha, \beta) = \text{the order type of the set } \{(\xi, \eta) : (\xi, \eta) < (\alpha, \beta)\}$$

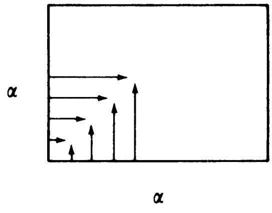

FIGURE 2.2 The canonical well-ordering of $\alpha \times \alpha$.

then Γ is a one-to-one mapping of Ord^2 onto Ord, and

$$(2.9) \qquad (\alpha, \beta) < (\gamma, \delta) \quad \leftrightarrow \quad \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta)$$

Note that $\Gamma[\omega \times \omega] = \omega$ and since $\gamma(\alpha) = \Gamma[\alpha \times \alpha]$ is an increasing function of α, we have $\gamma(\alpha) \geq \alpha$ for every α. However, $\gamma(\alpha)$ is easily shown to be normal, and so $\Gamma[\alpha \times \alpha] = \alpha$ for arbitrarily large α.

Exercise 2.14. Show that $\gamma(\alpha + 1) = \gamma(\alpha) + \alpha + \alpha + 1$, and hence $\gamma(\alpha) \leq \omega^\alpha$. Thus if $\alpha = \omega^\alpha$, we have $\Gamma[\alpha \times \alpha] = \alpha$.

Well-Founded Relations

Now we shall define an important generalization of well-ordered sets.

A binary relation E over a set P is *well-founded* if every nonempty $X \subseteq P$ has an *E-minimal* element, that is $a \in X$ such that there is no $x \in X$ with $x \; E \; a$.

Clearly, a well-ordering of P is a well-founded relation.

Exercise 2.15. If E is a well-founded relation over P, then there is no sequence $\langle a_n : n \in N \rangle$ in P such that

$$a_1 \; E \; a_0, \qquad a_2 \; E \; a_1, \qquad a_3 \; E \; a_2, \qquad \ldots$$

(Using the axiom of choice, this condition implies that E is well-founded.)

Given a well-founded relation E over a set P, we can define the *length* of E, and assign each $x \in P$ an ordinal number, the *rank* of x in E.

Theorem 5. *If E is a well-founded relation over P, then there exists a function f from P into the ordinals such that for any x, $y \in P$,*

$$x \; E \; y \quad \rightarrow \quad f(x) < f(y)$$

Moreover, there is a unique ordinal function ρ_E such that for each $x \in P$,

$$\begin{aligned} (2.10) \qquad \rho_E(x) &= \text{the least } \alpha \text{ such that } \alpha > \rho_E(y) \text{ for all } y \; E \; x \\ &= \sup\{\rho_E(y) + 1 : y \; E \; x\} \end{aligned}$$

The range of ρ_E is an initial segment of the ordinals (thus an ordinal). This ordinal is called the length of E:

$$(2.11) \qquad \text{length}(E) = \sup\{\rho_E(x) + 1 : x \in P\}$$

Proof. We shall define a function ρ_E satisfying (2.10) and then prove its uniqueness. By induction, let

$$P_0 = \emptyset, \qquad P_{\alpha+1} = \{x \in P : \forall y (y \; E \; x \rightarrow y \in P_\alpha)\}$$

$$P_\alpha = \bigcup_{\xi < \alpha} P_\xi \qquad \text{if } \alpha \text{ is a limit}$$

Let θ be the least ordinal such that $P_{\theta+1} = P_\theta$ (such θ exists by Replacement). First, it should be easy to see that $P_\alpha \subseteq P_{\alpha+1}$ for each α (by induction). Thus

$P_0 \subset P_1 \subset \cdots \subset P_\theta$. We claim that $P_\theta = P$: Otherwise, let a be an E-minimal element of $P - P_\theta$. It follows that each $x \mathrel{E} a$ is in P_θ, and so $a \in P_{\theta+1}$, a contradiction. Now we define

$$(2.12) \qquad \rho_E(x) = \text{the least } \alpha \text{ such that } x \in P_{\alpha+1}$$

It is obvious that if $x \mathrel{E} y$, then $\rho_E(x) < \rho_E(y)$; also, (2.10) is easily verified. The ordinal θ is the length of E.

The uniqueness of ρ_E is established as follows: Let ρ' be another function satisfying (2.10) and consider an E-minimal element of the set $\{x \in P : \rho_E(x) \neq \rho'(x)\}$. ■

3. CARDINAL NUMBERS

Two sets X, Y have the same *cardinality* (cardinal number, cardinal),

$$(3.1) \qquad |X| = |Y|$$

if there exists a one-to-one mapping of X onto Y.

The relation (3.1) is an equivalence relation over the universe. We assume that we can assign each set X its *cardinal number* $|X|$ so that two sets are assigned the same cardinal just in case they satisfy the condition of (3.1). The cardinal numbers can be defined either using the axiom of regularity (via equivalence classes of (3.1)), or using the axiom of choice. In this section we shall define cardinal numbers of well-orderable sets; and it follows from the axiom of choice that every set can be well-ordered.

We recall that a set X is *finite* if $|X| = |n|$ for some $n \in N$; then X is said to *have n elements*. Clearly, $|n| = |m|$ if and only if $n = m$, and so we define *finite cardinals* as natural numbers, i.e., $|n| = n$ for all $n \in N$.

Exercise 3.1.

(a) A subset of a finite set is finite.
(b) The union of a finite set of finite sets is finite.
(c) The power set of a finite set is finite.
(d) The image of a finite set (under a mapping) is finite.

The relation $|X| = |Y|$ for infinite sets is the most important notion of set theory. Most of set theory is practically the study of infinite cardinals. The following theorem tells us that the concept of a cardinal is not trivial.

Theorem 6 (Cantor). *For every set X,*

$$|P(X)| \neq |X|$$

Proof. Let f be a function from X into $P(X)$. The set

$$Y = \{x \in X : x \notin f(x)\}$$

is not in the range of f: If $z \in X$ were such that $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, a contradiction. Thus f is not a function of X onto $P(X)$. Hence $|P(X)| \neq |X|$. ■

The ordering of cardinal numbers is defined as follows:

$$|X| \leq |Y|$$

if there exists a one-to-one mapping of X into Y.

$$|X| < |Y| \quad \leftrightarrow \quad |X| \leq |Y| \wedge |X| \neq |Y|$$

The relation $\leq$ is clearly transitive. In view of the following theorem, $<$ is a partial ordering of cardinal numbers.

Theorem 7 (Cantor–Bernstein). *If* $|A| \leq |B|$ *and* $|B| \leq |A|$, *then* $|A| = |B|$.

Proof. If $f_1: A \to B$ and $f_2: B \to A$ are one-to-one, then if we let $B' = f_2[B]$ and $A_1 = f_2[f_1[A]]$, we have $A_1 \subseteq B' \subseteq A$ and $|A_1| = |A|$. Thus we may assume that $A_1 \subseteq B \subseteq A$ and that f is a one-to-one function of A onto A_1; we wish to show that $|A| = |B|$. See Fig. 3.1.

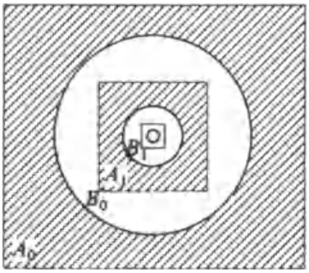

FIGURE 3.1

We define (by induction)

$$\begin{aligned} A_0 &= A, \qquad & A_{n+1} &= f[A_n] \\ B_0 &= B, \qquad & B_{n+1} &= f[B_n] \end{aligned} \qquad (n \in N)$$

Let g be the function on A defined as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n \\ x & \text{otherwise} \end{cases}$$

Then g is a one-to-one mapping of A onto B, as the reader will gladly verify. Thus $|A| = |B|$. ■

The arithmetic operations on cardinals are defined as follows:

$$(3.2)\quad \begin{aligned} \kappa+\lambda &= |A\cup B| &&\text{where } |A|=\kappa,\ |B|=\lambda, \\ & && \text{and } A, B \text{ are disjoint} \\ \kappa\cdot\lambda &= |A\times B| &&\text{where } |A|=\kappa,\ |B|=\lambda \\ \kappa^\lambda &= |{}^BA| &&\text{where } |A|=\kappa,\ |B|=\lambda \end{aligned}$$

Naturally, definition (3.2) is meaningful only if it is independent of the choice of A and B. Thus one has to check that, e.g., if $|A| = |A'|$ and $|B| = |B'|$, then $|A \times B| = |A' \times B'|$.

Lemma 3.1. *If* $|A| = \kappa$, *then* $|P(A)| = 2^\kappa$.

Proof. For every $X \subseteq A$, let χ_X be the function

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in A - X \end{cases}$$

The mapping $f\colon X \mapsto \chi_X$ is a one-to-one correspondence between $P(A)$ and ${}^A\{0, 1\}$. ■

Thus Cantor's theorem can be formulated as follows:

$$\kappa < 2^\kappa \qquad \text{for every cardinal } \kappa$$

(Clearly, $|A| \le |P(A)|$, for $x \mapsto \{x\}$ is one-to-one.)

A few simple facts about cardinal arithmetic:

(3.3) $+$ and $\cdot$ are associative, commutative and distributive.
(3.4) $(\kappa\cdot\lambda)^\mu = \kappa^\mu\cdot\lambda^\mu$.
(3.5) $\kappa^{\lambda+\mu} = \kappa^\lambda\cdot\kappa^\mu$.
(3.6) $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.
(3.7) If $\kappa \le \lambda$, then $\kappa^\mu \le \lambda^\mu$.
(3.8) If $0 < \lambda \le \mu$, then $\kappa^\lambda \le \kappa^\mu$.
(3.9) $\kappa^0 = 1$; $1^\kappa = 1$; $0^\kappa = 0$ if $\kappa > 0$.

To prove (3.3)–(3.9), one has only to find the appropriate one-to-one functions.

Alephs

An ordinal α is called a *cardinal number* (a cardinal) if $|\alpha| \ne |\beta|$ for all $\beta < \alpha$. We shall use $\kappa, \lambda, \mu, \ldots$ to denote cardinal numbers and *Card* the class of all cardinals.

If W is a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$. Thus we let

$$|W| = \text{the least ordinal such that } |W| = |\alpha|$$

Obviously, $|W|$ is a cardinal.

Every natural number is a cardinal (a *finite cardinal*); and if S is a finite set, then $|S| = n$ for some n.

The ordinal ω is the least infinite cardinal. Note that all infinite cardinals are limit ordinals.

Lemma 3.2.

(a) *For every α there is a cardinal greater than α.*
(b) *If X is a set of cardinals, then $\sup(X)$ is a cardinal.*

For every α, we let α^+ denote the least cardinal greater than α (the *cardinal successor* of $|\alpha|$).

Proof. (a) For any set X, let the *Hartogs number* of X be

$$h(X) = \text{the least } \alpha \text{ such that there is no one-to-one function of } \alpha \text{ into } X \tag{3.10}$$

There is only a set of possible well-orderings of subsets of X. Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

If α is an ordinal, then $|\alpha| < |h(\alpha)|$ by (3.10). That proves (a). (In fact, $h(\alpha) = \alpha^+$.)

(b) Let $\alpha = \sup X$. If f is one-to-one mapping of α onto some $\beta < \alpha$, let $\kappa \in X$ be such that $\beta < \kappa \leq \alpha$. Then $|\kappa| = |f[\kappa]| = |\text{order type } (f[\kappa])| \leq \beta$, a contradiction. Thus α is a cardinal. ■

Using Lemma 3.2, we define the normal sequence of all infinite cardinals (*alephs*). We usually use $\aleph_\alpha$ to denote the cardinal number, and ω_α to denote the order-type:

$$\aleph_0 = \omega_0 = \omega, \qquad \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+$$

$$\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \qquad \text{if } \alpha \text{ is a limit}$$

Sets whose cardinality is $\aleph_0$ are called *countable*; a set is *at most countable* if it is either finite or countable. Set of other cardinalities are *uncountable*.

A cardinal $\aleph_{\alpha+1}$ is a *successor cardinal*. A cardinal $\aleph_\alpha$ whose index is a limit ordinal is a *limit cardinal*.

Addition and multiplication of infinite cardinal numbers is a trivial matter, due to the following fact:

Theorem 8. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

Proof. Consider the canonical one-to-one mapping Γ of *Ord* × *Ord* onto *Ord* defined in Section 2. We shall show that $\Gamma[\omega_\alpha \times \omega_\alpha] = \omega_\alpha$. This is true for $\alpha = 0$. Thus let α be the least ordinal such that $\Gamma[\omega_\alpha \times \omega_\alpha] \supset \omega_\alpha$. Let $\beta, \gamma < \omega_\alpha$ be such that $\Gamma(\beta, \gamma) = \omega_\alpha$. Pick $\delta < \omega_\alpha$ such that $\delta > \beta$ and $\delta > \gamma$. Since $\delta \times \delta$ is an initial segment of *Ord* × *Ord* in the canonical well-ordering and contains

(β, γ), we have $\Gamma[\delta \times \delta] \supset \omega_\alpha$, and so $|\delta \times \delta| \geq \aleph_\alpha$. However, $|\delta \times \delta| = |\delta| \cdot |\delta|$, and by the induction hypothesis, $|\delta| \cdot |\delta| = |\delta| < \aleph_\alpha$. A contradiction. ■

As a corollary, we have

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta) \tag{3.11}$$

Exponentiation of cardinals will be dealt with in Section 6. Without the axiom of choice, one cannot prove that $2^{\aleph_\alpha}$ is an aleph (or that $P(\omega_\alpha)$ can be well-ordered), and there is very little one can prove about $2^{\aleph_\alpha}$ or $\aleph_\alpha^{\aleph_\beta}$.

Exercise 3.2. There is a canonical well-ordering of Ord^3 such that for each α, ω_α^3 is an initial segment and its order-type is ω_α. Hence $\aleph_\alpha^3 = \aleph_\alpha$.

Exercise 3.3. There is a well-ordering of the class of all finite sequences of ordinals such that for each α, the set of all finite sequences in ω_α is an initial segment and its order type is ω_α.

We say that a set B is a *projection* of a set A if there is a mapping of A onto B. Note that B is a projection of A if and only if there is a partition P of A such that $|P| = |B|$. If $|A| \geq |B| > 0$, then B is a projection of A. Conversely, using the axiom of choice, one shows that if B is a projection of A, then $|A| \geq |B|$. This, however, cannot be proved without the axiom of choice.

Exercise 3.4. If B is a projection of ω_α, then $|B| \leq \aleph_\alpha$.

Exercise 3.5. The set of all finite subsets of ω_α has cardinality $\aleph_\alpha$.
[The set is a projection of the set of finite sequences.]

Exercise 3.6. If B is a projection of A, then $|P(B)| \leq |P(A)|$.
[Consider $g(X) = f_{-1}[X]$, where f maps A onto B.]

Exercise 3.7. $\omega_{\alpha+1}$ is a projection of $P(\omega_\alpha)$.
[Use $|\omega_\alpha \times \omega_\alpha| = \omega_\alpha$ and project $P(\omega_\alpha \times \omega_\alpha)$: If $R \subseteq \omega_\alpha \times \omega_\alpha$ is a well-ordering, let $f(R)$ be its order-type.]

Exercise 3.8. $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$.
[Use Exercises 3.7 and 3.6.]

Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, β a limit ordinal, is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subseteq \alpha$ is *cofinal* in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the *cofinality* of α is

$$\text{cf}\,\alpha = \text{the least limit ordinal } \beta \text{ such that there is an increasing } \beta\text{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha$$

Obviously, $\text{cf}\,\alpha$ is a limit ordinal, and $\text{cf}\,\alpha \leq \alpha$. Examples: $\text{cf}(\omega + \omega) = \text{cf}\,\aleph_\omega = \omega$.

Lemma 3.3. $\operatorname{cf}(\operatorname{cf}\alpha) = \operatorname{cf}\alpha$.

Proof. If $\langle \alpha_\xi : \xi < \beta \rangle$ is cofinal in α and $\langle \xi(\iota) : \iota < \gamma \rangle$ is cofinal in β, then $\langle \alpha_{\xi(\iota)} : \iota < \gamma \rangle$ is cofinal in α. ■

Two useful facts about cofinality:

Lemma 3.4. *Let* $\alpha > 0$ *be a limit ordinal.*

(a) *If* $A \subseteq \alpha$ *and* $\sup A = \alpha$, *then* $\text{o.t.}(A) \geq \operatorname{cf}\alpha$.
(b) *If* $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_\xi \leq \cdots$, $\xi < \gamma$, *is a nondecreasing* γ*-sequence of ordinals in* α *and* $\lim_{\xi \to \gamma} \beta_\xi = \alpha$, *then* $\operatorname{cf}\gamma = \operatorname{cf}\alpha$.

Proof. (a) Let $\langle \alpha_\xi : \xi < \text{o.t. } A \rangle$ be the increasing enumeration of A.

(b) If $\gamma = \lim_{\nu \to \operatorname{cf}\gamma} \xi(\nu)$, then $\alpha = \lim_{\nu \to \operatorname{cf}\gamma} \beta_{\xi(\nu)}$, and the nondecreasing sequence $\langle \beta_{\xi(\nu)} : \nu < \operatorname{cf}\gamma \rangle$ has an increasing subsequence of length $\leq \operatorname{cf}\gamma$, with the same limit. Thus $\operatorname{cf}\alpha \leq \operatorname{cf}\gamma$.

To show that $\operatorname{cf}\gamma \leq \operatorname{cf}\alpha$, let $\alpha = \lim_{\nu \to \operatorname{cf}\alpha} \alpha_\nu$. For each $\nu < \operatorname{cf}\alpha$, let $\xi(\nu)$ be the least ξ greater than all $\xi(\iota)$, $\iota < \nu$, such that $\beta_\xi > \alpha_\nu$. Since $\lim_{\nu \to \operatorname{cf}\alpha} \beta_{\xi(\nu)} = \alpha$, it follows that $\lim_{\nu \to \operatorname{cf}\alpha} \xi(\nu) = \gamma$, and so $\operatorname{cf}\gamma \leq \operatorname{cf}\alpha$. ■

An infinite cardinal $\aleph_\alpha$ is *regular* if $\operatorname{cf}(\omega_\alpha) = \omega_\alpha$. It is *singular* if $\operatorname{cf}(\omega_\alpha) < \omega_\alpha$.

Lemma 3.5. $\operatorname{cf}(\alpha)$ *is always a regular cardinal.*

Proof. It is easy to see that if α is not a cardinal, then using a mapping of $|\alpha|$ onto α, one can construct a cofinal sequence in α of length $\leq |\alpha|$, and therefore $\operatorname{cf}\alpha < \alpha$.

Since $\operatorname{cf}(\operatorname{cf}\alpha) = \operatorname{cf}\alpha$, it follows that $\operatorname{cf}\alpha$ is a cardinal, and is regular. ■

There are arbitrarily large singular cardinals. For each α, $\aleph_{\alpha+\omega}$ is a singular cardinal of cofinality ω.

Using the axiom of choice, we shall show that every $\aleph_{\alpha+1}$ is regular. It is an open problem whether one can prove without the axiom of choice that there exists a regular uncountable $\aleph_\alpha$. (The informed guess is that one cannot.)

Exercise 3.9. If $\aleph_\alpha$ is an uncountable limit cardinal, then $\operatorname{cf}\omega_\alpha = \operatorname{cf}\alpha$; ω_α is the limit of a cofinal sequence $\langle \omega_\alpha : \xi < \operatorname{cf}\alpha \rangle$ of cardinals.

Lemma 3.6. *An infinite cardinal* κ *is singular if and only if there exists a cardinal* $\lambda < \kappa$ *and a family* $\{S_\xi : \xi < \lambda\}$ *of subsets of* κ *such that* $|S_\xi| < \kappa$ *for each* $\xi < \lambda$, *and* $\kappa = \bigcup_{\xi<\lambda} S_\xi$. *(And the least cardinal* λ *that satisfies the condition is in fact equal to* $\operatorname{cf}\kappa$.)

Proof. (a) If κ is singular, then there is an increasing sequence $\langle \alpha_\xi : \xi < \operatorname{cf}\kappa \rangle$ with $\lim_\xi \alpha_\xi = \kappa$. Let $\lambda = \operatorname{cf}\kappa$, and $S_\xi = \alpha_\xi$ for all $\xi < \lambda$.

(b) If the condition holds, let $\lambda < \kappa$ be the least cardinal for which there is a family $\{S_\xi : \xi < \lambda\}$ such that $\kappa = \bigcup_{\xi<\lambda} S_\xi$ and $|S_\xi| < \kappa$ for each $\xi < \lambda$. For every $\xi < \lambda$, let β_ξ be the order-type of $\bigcup_{\nu<\xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is nondecreasing, and by the minimality of λ, $\beta_\xi < \kappa$ for all $\xi < \lambda$. We shall show that $\lim_\xi \beta_\xi = \kappa$, thus proving that $\operatorname{cf} \kappa \leq \lambda$.

Let $\beta = \lim_{\xi \to \lambda} \beta_\xi$. There is a one-to-one mapping f of $\kappa = \bigcup_{\xi < \lambda} S_\xi$ into $\lambda \times \beta$: If $\alpha \in \kappa$, let $f(\alpha) = (\xi, \gamma)$, where ξ is the least ξ such that $\alpha \in S_\xi$ and γ is the order-type of $S_\xi \cap \alpha$. Since $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, it follows that $\beta = \kappa$. ■

As mentioned earlier, one cannot prove without the axiom of choice that ω_1 is not a countable union of countable sets.

Exercise 3.10. Show that ω_2 is not a countable union of countable sets.
[If $\omega_2 = \bigcup_{n<\omega} S_n$, then $\sup\{\text{o.t. } S_n : n < \omega\} = \omega_2$.]

The only cardinal inequality we have proved so far is the Cantor theorem $\kappa < 2^\kappa$. It follows that $\kappa < \lambda^\kappa$ for every $\lambda > 1$, and in particular $\kappa < \kappa^\kappa$ (for $\kappa \neq 1$). The following theorem gives a better inequality. This and other cardinal inequalities will also follow from König's theorem, to be proved in Section 6.

Theorem 9. *If κ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf} \kappa}$.*

Proof. Let F be a collection of κ functions from $\operatorname{cf} \kappa$ to κ: $F = \{f_\alpha : \alpha < \kappa\}$. It is enough to find f: $\operatorname{cf} \kappa \to \kappa$ that is different from all the f_α. Let $\kappa = \lim_{\xi \to \operatorname{cf} \kappa} \alpha_\xi$. For $\xi < \operatorname{cf} \kappa$, let

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \neq f_\alpha(\xi) \text{ for all } \alpha < \alpha_\xi$$

Such a γ exists since $|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}| \leq |\alpha_\xi| < \kappa$. Obviously, $f \neq f_\alpha$ for all $\alpha < \kappa$. ■

Consequently, $\kappa^\lambda > \kappa$ whenever $\lambda \geq \operatorname{cf} \kappa$.

An uncountable cardinal κ is *weakly inaccessible* if it is a limit cardinal and is regular. There will be more about inaccessible cardinals later, but let me mention at this point that existence of (weakly) inaccessible cardinals is not provable in ZFC.

To get an idea of the size of an inaccessible cardinal, note that if $\aleph_\alpha > \aleph_0$ is a limit and regular, then $\aleph_\alpha = \operatorname{cf} \aleph_\alpha = \operatorname{cf} \alpha \leq \alpha$, and so $\aleph_\alpha = \alpha$.

Since the sequence of alephs is a normal sequence, it has arbitrarily large fixed points; the problem is whether some of them are regular cardinals. For instance, the least fixed point $\aleph_\alpha = \alpha$ has cofinality ω:

$$\begin{aligned} \kappa &= \lim \langle \omega, \omega_\omega, \omega_{\omega_\omega}, \ldots \rangle \\ &= \lim_{n \to \omega} \kappa_n \qquad \text{where} \quad \kappa_0 = \omega, \qquad \kappa_{n+1} = \omega_{\kappa_n} \end{aligned}$$

A set S is *Dedekind-finite* (D-finite) if there is no one-to-one mapping of S onto a proper subset of S. Every finite set is D-finite. Using the axiom of choice, one proves that every infinite set is D-infinite, and so D-finiteness is the same as finiteness. Without the axiom of choice, however, one cannot prove that every D-finite set is finite.

The set N of all natural numbers is obviously D-infinite and hence every S such that $|S| \geq \aleph_0$, is D-infinite.

Exercise 3.11. S is D-infinite if and only if S has a countable subset.

[If S is D-infinite, let $f: S \to X \subset S$ be one-to-one. Let $x_0 \in S - X$ and $x_{n+1} = f(x_n)$. Then $S \supseteq \{x_n : n < \omega\}$.]

Exercise 3.12.

(a) If A and B are D-finite, then $A \cup B$ and $A \times B$ are D-finite.
(b) The set of all finite one-to-one sequences in a D-finite set is D-finite.
(c) The union of a disjoint D-finite family of D-finite sets is D-finite.

On the other hand, one cannot prove without the axiom of choice that a projection, power set, or the set of all finite subsets of a D-finite set is D-finite, or that the union of a D-finite family of D-finite sets is D-finite.

Exercise 3.13. If A is an infinite set, then $PP(A)$ is D-infinite.

[Consider the set $\{\{X \subset A : |X| = n\} : n < \omega\}$.]

4. REAL NUMBERS

Theorem 10 (Cantor). *The set of all real numbers is uncountable.*

Proof. Let us assume that the set $\boldsymbol{R}$ of all reals is countable, and let $c_0, c_1, \ldots, c_n, \ldots, n \in N$, be an enumeration of $\boldsymbol{R}$. We shall find a real number different from each c_n.

We let $a_0 = c_0$, $b_0 = c_k$ where k is the least k such that $a_0 < c_k$, $a_{n+1} = c_k$ where k is the least k such that $a_n < c_k < b_n$, $b_{n+1} = c_k$ where k is the least k such that $a_{n+1} < c_k < b_n$ (see Fig. 4.1). If we let $a = \sup\{a_n : n \in N\}$, then $a \neq c_k$ for all k. ■

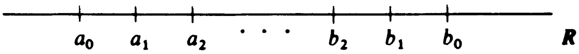

FIGURE 4.1

It is the *completeness* of the real line that accounts for uncountability of $\boldsymbol{R}$. The real numbers are usually defined as Dedekind cuts in the set of rational numbers. We will not bother with a formal definition of rational numbers, and note in passing only that the set $\boldsymbol{Q}$ of all rationals is countable (since $\aleph_0 \cdot \aleph_0 = \aleph_0$) and include another theorem of Cantor, which shows that the order-type of $(\boldsymbol{Q}, <)$ is the unique order-type of a countable dense unbounded linearly ordered set.

A linearly ordered set P is *dense* if for any $a < b$ there exists c such that $a < c < b$. P is *unbounded* if it has no least and no greatest element.

Theorem 11 (Cantor). *Any two countable linearly ordered sets that are dense and unbounded are isomorphic.*

Proof. Let $P_1 = \{a_n : n \in N\}$ and $P_2 = \{b_n : n \in N\}$ be dense unbounded linearly ordered sets. We construct an isomorphism $f: P_1 \to P_2$ in the following way (see Fig. 4.2): We first define $f(a_0)$, then $f^{-1}(b_0)$, then $f(a_1)$, then $f^{-1}(b_1)$, etc., so as to keep f order-preserving. For example, to define $f(a_n)$, if it is not yet defined, we let $f(a_n) = b_k$ where k is the least index such that f remains order-preserving (such a k exists because f has been defined for only finitely many $a \in P_1$, and because P_2 is dense and unbounded). ■

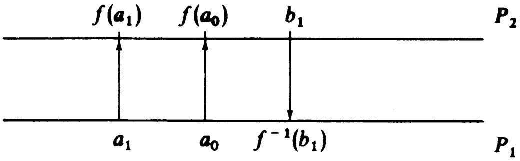

FIGURE 4.2

Exercise 4.1. Every countable linearly ordered set P can be embedded in the rationals; i.e., P is isomorphic to a set of rationals.

The real line is the unique dense, unbounded, linearly ordered set (up to isomorphism) that is complete and contains the rationals as a dense subset. A dense linearly ordered set is called *complete* if every nonempty bounded subset has a supremum. Let P be a dense linearly ordered set; a set $D \subseteq P$ is *dense in P* if for any $a < b$ there is $d \in D$ such that $a < d < b$.

Theorem 12 (*Completion of dense linear ordering*). *Let $(P, <)$ be a dense unbounded linearly ordered set. Then there is a complete unbounded linearly ordered set $(C, \prec)$ such that:*

(i) *$P \subseteq C$, and $<$ and $\prec$ agree on P;*
(ii) *P is dense in C.*

The set C is unique up to isomorphism; moreover, if C_1 and C_2 are two such sets, then there is an isomorphism h between C_1 and C_2 such that $h(p) = p$ for all $p \in P$.

Proof. A *Dedekind cut* in P is a pair (A, B) of disjoint nonempty subsets of P such that:

(a) $A \cup B = P$;
(b) $a < b$ for any $a \in A$ and $b \in B$;
(c) If $\inf B$ exists, then $\inf B \in B$ (note that $\sup A = \inf B$).

Let C be the set of all Dedekind cuts in P and let $(A_1, B_1) \prec (A_2, B_2)$ if $A_1 \subset A_2$ (and $B_1 \supset B_2$). The set C is complete: If $\{(A_i, B_i) : i \in I\}$ is a nonempty bounded subset of C, then $(\bigcup_{i \in I} A_i, \bigcap_{i \in I} B_i)$ is its supremum. For $p \in P$, let

$$A_p = \{x \in P : x < p\}, \qquad B_p = \{x \in P : x \geq p\}$$

Then $P' = \{(A_p, B_p) : p \in P\}$ is isomorphic to P and is dense in C.

To prove uniqueness of C, let C and C' be two complete dense unbounded linearly ordered sets, let P and P' be dense in C and C', respectively, and let f be an isomorphism of P onto P'. Then f can be extended (uniquely) to an isomorphism f^* of C and C': For $x \in C$, let $f^*(x) = \sup\{f(p) : p \in P \text{ and } p \leq x\}$. ■

The *real line* $\boldsymbol{R}$, or the *continuum*, is defined as the completion of the set $\boldsymbol{Q}$ of all rational numbers. It follows from the construction by Dedekind cuts that $|\boldsymbol{R}| \leq 2^{\aleph_0}$. To show that $|\boldsymbol{R}| = 2^{\aleph_0}$, it is enough to exhibit a set of reals of cardinality $2^{\aleph_0}$.

The Cantor Set

Let $\boldsymbol{C}$ be the set of all reals of the form $\sum_{n=1}^{\infty} a_n/3^n$, where each $a_n = 0$ or 2. $\boldsymbol{C}$ is obtained by removing from the closed interval $[0, 1]$, the open intervals $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{9}, \frac{2}{9})$, $(\frac{7}{9}, \frac{8}{9})$, etc. (the middle-third intervals). See Fig. 4.3. $\boldsymbol{C}$ is in a one-to-one correspondence with the set of all ω-sequences of 0's and 2's and so $|\boldsymbol{C}| = 2^{\aleph_0}$.

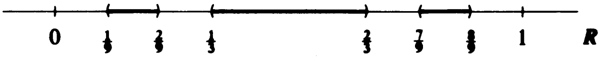

FIGURE 4.3 The Cantor set.

The Continuum Hypothesis

The cardinal $2^{\aleph_0}$ is also called the *continuum*. The *continuum hypothesis* (CH) is the conjecture that

$$2^{\aleph_0} = \aleph_1$$

Neither CH nor its negation is provable in ZFC.

Typical sets of cardinality $2^{\aleph_0}$ are the set of all sequences of natural numbers, the set of all sequences of real numbers, the set of all complex numbers. This is because $\aleph_0^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, $2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$.

Exercise 4.2. The set of all continuous functions f: $\boldsymbol{R} \to \boldsymbol{R}$ has cardinality $2^{\aleph_0}$ (while the set of all functions has cardinality $2^{2^{\aleph_0}}$).

[A continuous function on $\boldsymbol{R}$ is determined by its values at rational points.]

Exercise 4.3. There are at least $2^{\aleph_0}$ countable order-types of linearly ordered sets.

[For every sequence $a = \langle a_n : n \in \boldsymbol{N} \rangle$ of natural numbers consider the order type

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \cdots$$

where ξ is the order-type of integers. Show that if $a \neq b$, then $\tau_a \neq \tau_b$.]

A real number is called *algebraic* if it is a root of a polynomial whose coefficients are integers. Otherwise, it is *transcendental.*

Exercise 4.4. The set of all algebraic reals is countable.

Lemma 4.1. *If S is a countable set of reals, then* $|\boldsymbol{R} - S| = 2^{\aleph_0}$.

Proof. A trick: Use $\boldsymbol{R} \times \boldsymbol{R}$ rather than $\boldsymbol{R}$ (because $|\boldsymbol{R} \times \boldsymbol{R}| = 2^{\aleph_0}$). If $S \subseteq \boldsymbol{R} \times \boldsymbol{R}$ is countable, then for some $x \in \boldsymbol{R}$, $S \cap (\boldsymbol{R} \times \{x\}) = \emptyset$ and $|\boldsymbol{R} \times \{x\}| = 2^{\aleph_0}$. ■

Corollary.

(a) *There are* $2^{\aleph_0}$ *irrational numbers.*
(b) *There are* $2^{\aleph_0}$ *transcendental numbers* (*Cantor*).

Suslin's Problem

The real line is, up to isomorphism, the unique linearly ordered set that is dense, unbounded, complete and contains a countable dense subset.

Since $\boldsymbol{Q}$ is dense in $\boldsymbol{R}$, every nonempty open interval of $\boldsymbol{R}$ contains a rational number. Hence if S is a disjoint collection of open intervals, S is at most countable. (Let $\langle r_n : n \in \boldsymbol{N}\rangle$ be an enumeration of the rationals. To each $J \in S$ assign $r_n \in J$ with the least possible index n.)

Let P be a dense linearly ordered set. P is called *separable* if P has a countable dense subset. (Thus the real line is separable.) If every disjoint collection of open intervals in P is at most countable, then we say that P satisfies the *countable chain condition* (*c.c.c.*). Clearly, every separable linear ordering satisfies the c.c.c.

Suslin's Problem. Let P be a complete dense unbounded linearly ordered set that satisfies the countable chain condition. Is P isomorphic to the real line?

This question cannot be decided in ZFC; we shall return to the problem in Section 22.

The Topology of the Real Line

Open sets of reals are unions of open intervals; *closed sets* are complements of open sets. Since every nonempty interval has cardinality $2^{\aleph_0}$, it follows that every nonempty open set has cardinality $2^{\aleph_0}$. Every finite set is closed, and there are countable closed sets; it will follow from the Cantor–Bendixson theorem that every uncountable closed set has cardinality $2^{\aleph_0}$.

The open intervals with rational endpoints (the *rational intervals*) form a basis of the topology of the real line. Thus every open set G is the union of all rational intervals (q, r) such that $(q, r) \subseteq G$ (and every closed set is $F = \mathbf{R} - \bigcup \{(q, r) : F \cap (q, r) = \emptyset\}$). Hence the number of all open sets (= the number of all closed sets) is $2^{\aleph_0}$.

A set of reals is *compact* if it is closed and bounded. A function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ is *continuous* if and only if $f_{-1}[G]$ is open for every open set G.

Exercise 4.5.

(a) If $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq \cdots$, $n \in \boldsymbol{N}$ is a descending sequence of nonempty compact sets then $\bigcap_{n=0}^{\infty} C_n \neq \emptyset$.
(b) If $f\colon \boldsymbol{R} \to \boldsymbol{R}$ is continuous, and C is compact, then $f[C]$ is compact.

A *dense set of reals* means a set that is dense in $\boldsymbol{R}$.

Theorem 13 (*Baire Category Theorem*). *If $D_0, D_1, \ldots, D_n, \ldots, n \in \boldsymbol{N}$, are dense open sets of reals, then the intersection $D = \bigcap_{n=0}^{\infty} D_n$ is dense in $\boldsymbol{R}$.*

Proof. We show that D intersects every nonempty open interval I. First note that for each n, $D_0 \cap \cdots \cap D_n$ is dense and open. Thus let $\langle J_k : k \in \boldsymbol{N} \rangle$ be an enumeration of rational intervals. Let $I_0 = I$, and let, for each n, $I_{n+1} = J_k = (q_k, r_k)$, where k is the least k such that the closed interval $[q_k, r_k]$ is included in $I_n \cap D_n$. Then $a \in D \cap I$, where $a = \lim_{k\to\infty} q_k$. ■

Corollary. *The set $\boldsymbol{Q}$ of all rationals is not the intersection of a countable collection of open sets.*

Proof. If $\boldsymbol{Q} \subseteq \bigcap_{n=0}^{\infty} G_n$, where G_n are open sets, then each G_n is dense in $\boldsymbol{R}$. Thus each $G_n - \{q_n\}$, where q_n is the nth rational, is dense and open, and so $\bigcap_{n=0}^{\infty} G_n - \boldsymbol{Q}$ is nonempty. ■

Perfect Sets

Let A be a set of reals. A real number a is a *limit point* of A if every neighborhood of a contains some $x \in A$ other than a; $a \in A$ is an *isolated* point of A if there is a neighborhood U of a such that a is the only element of A in U.

A set of reals is *perfect* if it is nonempty, closed, and has no isolated points.

Exercise 4.6. The Cantor set is a perfect set.

Lemma 4.2. *Every perfect set has cardinality $2^{\aleph_0}$.*

Proof. Given a perfect set P, we want to find a one-to-one mapping F of $^{N}\{0, 1\}$ into P. The construction of F resembles the construction of the Cantor set. We construct, by induction on length, for every finite $\{0, 1\}$-sequence $s\colon n \to \{0, 1\}$, elements a_s of P such that:

(4.1) if $\operatorname{length}(s) = \operatorname{length}(t)$ and $s \neq t$, then $a_s \neq a_t$;
(4.2) if $f \in {}^{N}\{0, 1\}$, then $a_f = \lim_{s \subset f} a_s = \lim_{n\to\infty} a_{f|n}$ exists;
(4.3) if $f \neq g$, then $a_f \neq a_g$.

Since P is closed, each a_f is in P and $F\colon f \mapsto a_f$ is a one-to-one mapping of $^{N}\{0, 1\}$ into P.

Let $a_\emptyset \in P$ be arbitrary.

If $s = \langle s(0), \ldots, s(n-1)\rangle$, let $s0$ and $s1$ denote $s0 = \langle s(0), \ldots, s(n-1), 0\rangle$ and $s1 = \langle s(0), \ldots, s(n-1), 1\rangle$. To satisfy (4.2) and (4.3) it is enough to make sure that for each s, a_{s0} and a_{s1} are close to a_s. For instance, it suffices to let

$$a_{s0} = a_s \tag{4.4}$$

and

$$|a_{s1} - a_s| \le \varepsilon_n, \tag{4.5}$$

where $\varepsilon_0 = 1$ and

$$\varepsilon_n = \tfrac{1}{4}\min(|a_s - a_t| : s \ne t \in {}^n\{0, 1\})$$

We can find a_{s1} as follows:

$$a_{s1} = \begin{cases} \sup\{x \in P : a_s < x \le a_s + \varepsilon_n\} & \text{if it exists} \\ \inf\{x \in P : a_s - \varepsilon_n \le x < a_s\} & \text{otherwise} \end{cases} \tag{4.6}$$

Since a_s is a limit point of P and P is closed, there is some a_{s1} that satisfies (4.6). ■

We shall show that every closed set is either at most countable or else contains a perfect subset and thus has cardinality $2^{\aleph_0}$.

Lemma 4.3. *Let $F_0 \supset F_1 \supset \cdots \supset F_\alpha \supset \cdots$, $\alpha < \theta$, be a strictly descending sequence of closed sets. Then θ is countable.*

Proof. For each $\alpha < \theta$, there is a rational interval (q, r) that is disjoint from $F_{\alpha+1}$ but not from F_α. However, there are only countably many rational intervals. ■

Lemma 4.4. *Every set of reals has at most countably many isolated points.*

Proof. Let $A \subseteq \boldsymbol{R}$. Let J_n, $n \in N$, be an enumeration of rational intervals. For every isolated point a of A, let $k(a)$ be the least k such that a is the only element of A in J_k. Clearly $k(a) \ne k(b)$ if a, b are distinct isolated points of A. ■

Theorem 14 (Cantor–Bendixson). *If F is an uncountable closed set, then $F = P \cup S$, where P is perfect and S is countable.*

Corollary. *If F is a closed set, then either $|F| \le \aleph_0$ or $|F| = 2^{\aleph_0}$.*

Proof. For every $A \subseteq \boldsymbol{R}$, let

$$A' = \text{the set of all limit points of } A$$

(the *derivative* of A). It is easy to see that A' is closed, and if A is closed then $A' \subseteq A$. Thus we let

$$F_0 = F, \qquad F_{\alpha+1} = F'_\alpha$$

$$F_\alpha = \bigcap_{\gamma<\alpha} F_\gamma \qquad \text{if } \alpha > 0 \text{ is a limit}$$

Since $F_0 \supseteq F_1 \supseteq \cdots \supseteq F_\alpha \supseteq \cdots$, there exists an ordinal θ such that $F_\alpha = F_\theta$ for all $\alpha \geq \theta$. (In fact, the least θ with this property is countable, by Lemma 4.3.) We let $P = F_\theta$.

If $P \neq \emptyset$, then $P' = P$ and so it is perfect. Thus the proof is completed by showing that $F - P$ is at most countable.

We have $F - P = \bigcup_{\alpha<\theta} (F_\alpha - F'_\alpha)$; hence if $a \in F - P$, then there is a unique α such that a is an isolated point of F_α. As in Lemma 4.4, we let $k(a)$ denote the least k such that a is the only point of F_α in the interval J_k. Note that if $\alpha \leq \beta$ and $b \in F_\beta - F'_\beta$, then $b \notin J_{k(a)}$, and hence $k(b) \neq k(a)$. Thus the correspondence $a \mapsto k(a)$ is one-to-one, and it follows that $F - P$ is at most countable. ■

Exercise 4.7. If P is a perfect set and (a, b) is an open interval such that $P \cap (a, b) \neq \emptyset$, then $|P \cap (a, b)| = 2^{\aleph_0}$.

Exercise 4.8. If $P_2 \nsubseteq P_1$ are perfect sets, then $|P_2 - P_1| = 2^{\aleph_0}$.
[Use Exercise 4.7.]

If A is a set of reals, a real number a is called a *condensation point* of A if every neighborhood of a contains uncountably many elements of A. Let A^* denote the set of all condensation points of A.

Exercise 4.9. If P is perfect then $P^* = P$.
[Use Exercise 4.7.]

Exercise 4.10. If F is closed and $P \subseteq F$ is perfect, then $P \subseteq F^*$.
[$P = P^* \subseteq F^*$.]

Exercise 4.11. If F is an uncountable closed set and P is the perfect set constructed in Theorem 14, then $F^* \subseteq P$; thus $F^* = P$.
[Every $a \in F^*$ is a limit point of P since $|F - P| \leq \aleph_0$.]

Exercise 4.12. If F is an uncountable closed set, then $F = F^* \cup (F - F^*)$ is the unique partition of F into a perfect and an at most countable set.
[Use Exercise 4.8.]

The Baire Space

The *Baire space* is the space $\mathcal{N} = {}^N N$ of all infinite sequences of natural numbers, $\langle a_n : n \in N \rangle$, with the following topology: For every finite sequence $s = \langle a_k : k < n \rangle$, let

$$U_s = \{f \in \mathcal{N} : s \subset f\} = \{\langle c_k : k \in N \rangle : (\forall k < n)[c_k = a_k]\} \tag{4.7}$$

The sets (4.7) form a basis for the topology of $\mathcal{N}$. Note that each U_s is also closed.

A sequence $\langle a_n : n \in N \rangle$ is *eventually constant* if there is n_0 such that $a_n = a_{n_0}$ for all $n \geq n_0$. The set of all eventually constant sequences is countable and is *dense* in $\mathcal{N}$, i.e., every nonempty open set in $\mathcal{N}$ contains an eventually constant sequence.

If $\langle a_n : n \in N \rangle$ is a sequence of positive integers, let

$$b_0 = \frac{1}{a_0}, \qquad b_1 = \cfrac{1}{a_0 + \cfrac{1}{a_1}}, \qquad b_2 = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}},$$

etc. Notice that

$$b_1 < b_3 < b_5 < \cdots < b_4 < b_2 < b_0 \tag{4.8}$$

It can be shown that $\{b_n\}_{n=0}^{\infty}$ converges; let $\bar{a} = \lim_{n\to\infty} b_n$; $\bar{a}$ is the *continued fraction*

$$\bar{a} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}} \tag{4.9}$$

Obviously, $0 < \bar{a} < 1$. It can be verified that if a and a' are two different sequences, then $\bar{a} \neq \bar{a}'$.

Exercise 4.13. Every continued fraction is irrational.
[Assume that $\bar{a} = p/q$. Then

$$q = p \cdot a_0 + p \cdot \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

Remember the Euclidean algorithm.]

Exercise 4.14. If c is an irrational number, $0 < c < 1$, then there is a continued fraction such that $c = \bar{a}$. [Construct the sequence $\langle a_0, a_1, \ldots \rangle$, using (4.8).]

Thus if we assign to each $a \in \mathcal{N}$ the continued fraction $F(a) = \overline{a + 1}$, where $a + 1 = \langle a_n + 1 : n \in N \rangle$, we have a one-to-one correspondence F between the Baire space and the set of all irrational numbers in the unit interval $[0, 1]$.

Exercise 4.15. The function $F \colon \mathcal{N} \to [0, 1]$ is continuous, and its inverse is also continuous. Thus the Baire space is homeomorphic to the space of all irrational numbers in $[0, 1]$ (and hence to the set of all irrational numbers) with the topology of the real line.

Perfect Sets in the Baire Space

Let *Seq* denote the set of all finite sequences of natural numbers:

$$Seq = \bigcup_{n=0}^{\infty} {}^{n}N$$

A (*sequential*) *tree* is a set $T \subseteq Seq$ that satisfies

(4.10) if $t \in T$ and $s = t|n$ for some n, then $s \in T$.

If $T \subseteq Seq$ is a tree, let $[T]$ be the set of all *infinite paths* through T:

$$[T] = \{f \in \mathcal{N} : f|n \in T \text{ for all } n \in N\} \tag{4.11}$$

The set $[T]$ is a closed set in the Baire space: Let $f \in \mathcal{N}$ be such that $f \notin [T]$. Then there is $n \in N$ such that $f|n = s$ is not in T. In other words, the open set $U_s = \{g \in \mathcal{N} : g \supset s\}$, a neighborhood of f, is disjoint from $[T]$. Hence $[T]$ is closed.

Conversely, if F is a closed set in $\mathcal{N}$, then the set

$$T_F = \{s \in Seq : s \subset f \text{ for some } f \in F\} \tag{4.12}$$

is a tree, and it is easy to verify that $[T_F] = F$: If $f \in \mathcal{N}$ is such that $f|n \in T$ for all $n \in N$, then for each n there is some $g \in F$ such that $g|n = f|n$; and since F is closed, it follows that $f \in F$.

Exercise 4.16. The tree T_F in (4.12) has no maximal node, i.e., $s \in T$ such that there is no $t \in T$ with $s \subset t$. The map $F \mapsto T_F$ is a one-to-one correspondence between closed sets in $\mathcal{N}$ and sequential trees without maximal nodes.

If f is an isolated point of a closed set F in $\mathcal{N}$, then there is $n \in N$ such that there is no $g \in F$, $g \neq f$, such that $g|n = f|n$. Thus the following definition:

A nonempty sequential tree T is *perfect* if for every $t \in T$ there exist $s_1 \supset t$ and $s_2 \supset t$, both in T, that are *incompatible*, i.e., neither $s_1 \supseteq s_2$ nor $s_2 \supseteq s_1$.

Lemma 4.5. *A closed set $F \subseteq \mathcal{N}$ is perfect if and only if the tree T_F is a perfect tree.* ■

The Cantor–Bendixson analysis for closed sets in the Baire space is carried out as follows: For each tree $T \subseteq Seq$, we let

(4.13) $T' = \{t \in T : \text{there exist incompatible } s_1 \supset t \text{ and } s_2 \supset t \text{ in } T\}$

(Thus T is perfect if and only if $\emptyset \neq T = T'$.)

The set $[T] - [T']$ is at most countable: For each $f \in [T]$ such that $f \notin [T']$, let $s_f = f|n$ where n is the least number such that $f|n \notin T'$. If $f, g \in [T] - [T']$, then $s_f \neq s_g$, by (4.13). Hence the mapping $f \mapsto s_f$ is one-to-one and $[T] - [T']$ is at most countable.

Now we let

$$T_0 = T, \qquad T_{\alpha+1} = T_\alpha' \tag{4.14}$$
$$T_\alpha = \bigcap_{\beta<\alpha} T_\beta \qquad \text{if } \alpha > 0 \text{ is a limit}$$

Since $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_\alpha \supseteq \cdots$, and T_0 is at most countable, there is an ordinal $\theta < \omega_1$ such that $T_{\theta+1} = T_\theta$. If $T_\theta \neq \emptyset$, then it is perfect.

Now it is easy to see that $[\bigcap_{\beta<\alpha} T_\beta] = \bigcap_{\beta<\alpha} [T_\beta]$, and so

$$[T] - [T_\theta] = \bigcup_{\alpha<\theta} ([T_\alpha] - [T'_\alpha]) \tag{4.15}$$

hence (4.15) is at most countable. Thus if $[T]$ is an uncountable closed set in $\mathcal{N}$, the sets $[T_\theta]$ and $[T] - [T_\theta]$ constitute the decomposition of $[T]$ into a perfect and an at most countable set.

5. THE AXIOM OF CHOICE

Axiom of Choice (AC). *Every family of nonempty sets has a choice function.*

If S is a family of sets and $\emptyset \notin S$, then a *choice function* for S is a function f on S such that

$$f(X) \in X \tag{5.1}$$

for every $X \in S$.

The axiom of choice postulates that for every S such that $\emptyset \notin S$ there exists a function f on S that satisfies (5.1).

The axiom of choice differs from other axioms of ZF by stating existence of a set (i.e., a choice function) without defining it (unlike, for instance, the axiom of pairing or the axiom of power set). Thus it is often interesting to know whether a mathematical statement can be proved without using the axiom of choice. It turns out that the axiom of choice is independent of the other axioms of set theory and that many mathematical theorems are unprovable in ZF without AC.

In some trivial cases, the existence of a choice function can be proved outright in ZF:

(a) when every $X \in S$ is a singleton $X = \{x\}$;
(b) when S is finite: the existence of a choice function for S is proved by induction on the size of S;
(c) when every $X \in S$ is a finite set of real numbers: let $f(X) =$ the least element of X.

On the other hand, one cannot prove existence of a choice function (in ZF) just from the assumption that the sets in S are finite; even when every $X \in S$ has just two elements (e.g., sets of reals), we cannot necessarily prove that S has a choice function.

Using the axiom of choice, one proves that every set can be well-ordered, and therefore every infinite set has cardinality equal to some $\aleph_\alpha$. In particular, any two sets have comparable cardinals, and the ordering

$$|X| \le |Y|$$

is a well-ordering of the class of all cardinals.

Theorem 15 (*Zermelo's Well-Ordering Theorem*). *Every set can be well ordered.*

Proof. Let A be a set. To well-order A, it suffices to construct a transfinite one-to-one sequence $\langle a_\alpha : \alpha < \theta \rangle$ that enumerates A. That we can do by induction, using a choice function f for the family S of all nonempty subsets of A. We let

$$a_0 = f(A), \qquad a_\alpha = f(A - \{a_\xi : \xi < \alpha\})$$

if $A - \{a_\xi : \xi < \alpha\}$ is nonempty. Let θ be the least ordinal such that $A = \{a_\xi : \xi < \theta\}$. Clearly, $\langle a_\alpha : \alpha < \theta \rangle$ enumerates A. ■

In fact, Zermelo's theorem is equivalent to the axiom of choice: If every set can be well-ordered, then every family S of nonempty sets has a choice function. To see this, well-order $\bigcup S$ and let $f(X)$ be the least element of X for every $X \in S$.

Of particular importance is the fact that the set of all real numbers can be well-ordered. It follows that $2^{\aleph_0}$ is an aleph and so $2^{\aleph_0} \geq \aleph_1$.

The possibility of well-ordering the reals leads to some "unpleasant" constructions of sets of reals. Well known is Vitali's construction of a nonmeasurable set; another example is an uncountable set of reals without a perfect subset.

Exercise 5.1. There exists a set of reals of cardinality $2^{\aleph_0}$ without a perfect subset.

[Let $\langle P_\alpha : \alpha < 2^{\aleph_0} \rangle$ be an enumeration of all perfect sets of reals. Construct disjoint $A = \{a_\alpha : \alpha < 2^{\aleph_0}\}$ and $B = \{b_\alpha : \alpha < 2^{\aleph_0}\}$ as follows: Pick a_α such that $a_\alpha \notin \{a_\xi : \xi < \alpha\} \cup \{b_\xi : \xi < \alpha\}$, and b_α such that $b_\alpha \in P_\alpha - \{a_\xi : \xi \leq \alpha\}$. Then A is the set.]

If every set can be well-ordered, then every infinite set has a countable subset: Well-order the set and take the first ω elements. Thus every infinite set is Dedekind-infinite, and so finiteness and Dedekind finiteness coincide.

Dealing with cardinalities of sets is much easier when we have the axiom of choice. In the first place, any two sets have comparable cardinals. Another consequence is:

$$\text{(5.2)} \qquad \text{if } f[A] = B, \quad \text{then } |B| \leq |A|$$

To show (5.2), we have to find a one-to-one function from B to A. This is done by choosing one element from $f_{-1}(\{b\})$ for each $b \in B$.

Another thing one can show using the axiom of choice is:

(5.3) The union of a countable family of countable sets is countable.

(By the way, this often used fact cannot be proved in ZF alone.) To prove (5.3) let A_n be a countable set for each $n \in N$. For each n, let us *choose* an enumeration $\langle a_{nk} : k \in N \rangle$ of A_n. That gives us a projection of $N \times N$ onto $\bigcup_{n=0}^{\infty} A_n$:

$$(n, k) \mapsto a_{nk}$$

Thus $\bigcup_{n=0}^{\infty} A_n$ is countable.

In a similar fashion, one can prove a more general statement.

Lemma 5.1. $|\bigcup S| \le |S| \cdot \sup\{|X| : X \in S\}$.

Proof. Let $\kappa = |S|$ and $\lambda = \sup\{|X| : X \in S\}$. We have $S = \{X_\alpha : \alpha < \kappa\}$ and for each $\alpha < \kappa$, we choose an enumeration $X_\alpha = \{a_{\alpha\beta} : \beta < \lambda_\alpha\}$, where $\lambda_\alpha \le \lambda$. Again, we have a projection

$$(\alpha, \beta) \mapsto a_{\alpha\beta}$$

of $\kappa \times \lambda$ onto $\bigcup S$, and so $|\bigcup S| \le \kappa \cdot \lambda$. ■

In particular, the union of $\aleph_\alpha$ sets, each of cardinality $\aleph_\alpha$, has cardinality $\aleph_\alpha$.

Corollary. *Every $\aleph_{\alpha+1}$ is a regular cardinal.*

This is because otherwise $\omega_{\alpha+1}$ would be the union of at most $\aleph_\alpha$ sets of cardinality at most $\aleph_\alpha$. ■

Exercise 5.2. If X is an infinite set and S is the set of all finite subsets of X, then $|S| = |X|$.

[Use $|X| = \aleph_\alpha$.]

Axiom of Choice in Mathematics

In algebra and point set topology, one often uses the following version of the axiom of choice.

We recall that if $(P, <)$ is a partially ordered set, then $a \in P$ is called *maximal* in P if there is no $x \in P$ such that $a < x$. If X is a nonempty subset of P, then $c \in P$ is an *upper bound* of X if $x \le c$ for every $x \in X$.

We say that a nonempty $C \subseteq P$ is a *chain* in P if C is linearly ordered by $<$.

Theorem 16 (*Kuratowski-Zorn Lemma*). *If $(P, <)$ is a nonempty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

Proof. We construct (using a choice function for nonempty subsets of P), a chain in P that leads to a maximal element of P. We let, by induction,

$$a_\alpha = \text{an element of } P \text{ such that } a_\alpha > a_\xi \text{ for every } \xi < \alpha \text{ if there is one}$$

Clearly, if $\alpha > 0$ is a limit ordinal, then $C_\alpha = \{a_\xi : \xi < \alpha\}$ is a chain in P and a_α exists by the assumption. Eventually, there is θ such that there is no $a_{\theta+1} \in P$, $a_{\theta+1} > a_\theta$. Thus a_θ is a maximal element of P. ■

Like Zermelo's theorem, the Kuratowski-Zorn lemma is equivalent to the axiom of choice:

Exercise 5.3. Using Theorem 16, prove the axiom of choice.

[Let S be a family of nonempty sets. To find a choice function on S, let $P = \{f : f \text{ is a choice function on some } Z \subseteq S\}$, and apply the Kuratowski-Zorn lemma to the partially ordered set $(P, \subset)$.]

There are numerous examples of proofs using the Kuratowski–Zorn lemma. To mention only a few:

(a) Every vector space has a basis.
(b) Existence and uniqueness of algebraic closure of a field.
(c) Hahn–Banach extension theorem.
(d) Tychonoff product theorem for compact spaces.

Countable Choice and Dependent Choices

Many important consequences of the axiom of choice, particularly many concerning the real numbers, can be proved from a weaker version of the axiom of choice.

Countable Axiom of Choice. *Every* countable *family of nonempty sets has a choice function.*

For instance, the countable AC implies that the union of countably many countable sets is countable. In particular, the real line is not a countable union of countable sets. Similarly, it follows that $\aleph_1$ is a regular cardinal. On the other hand, the countable AC does not imply that the set of all reals can be well-ordered.

Exercise 5.4. The countable AC implies that every infinite set has a countable subset.

[If A is infinite, let $A_n = \{s : s$ is a one-to-one sequence in A of length $n\}$ for each n. Use a choice function for $S = \{A_n : n \in N\}$ to obtain a countable subset of A.]

In the next two exercises use the countable AC and note that the converse implication holds in ZF. Incidentally, neither of the two statements is provable in ZF alone. A counterexample is provided by an infinite, Dedekind-finite set of reals (in some model of ZF in which AC fails).

Exercise 5.5. If A is a set of reals and a is in the closure of A, then there is a sequence $\langle a_n : n \in N\rangle$ in A such that $a = \lim a_n$.

Hence a set $A \subseteq R$ is closed whenever an arbitrary convergent sequence $\langle a_n : n \in N\rangle$ of numbers in A converges to $a \in A$.

Exercise 5.6. In order that $f\colon R \to R$ be continuous at a, it suffices that $f(a) = \lim_n f(a_n)$ for every sequence that converges to a.

We briefly mention a related principle which follows from the axiom of choice.

Principle of Dependent Choices (DC). *If ρ is a binary relation over a nonempty set A, and if for every $a \in A$ there exists $b \in A$ such that $b\ \rho\ a$, then there is a sequence $a_0, a_1, \ldots, a_n, \ldots$ in A such that*

$$a_{n+1}\ \rho\ a_n \quad \text{for all} \quad n \in N \tag{5.4}$$

The principle of dependent choices is stronger than the countable axiom of choice.

***Exercise* 5.7.** Use DC to prove the countable AC.

[Given $S = \{A_n : n \in N\}$, consider the set A of all choice functions on some $S_n = \{A_i : i \le n\}$, with the binary relation $\supset$.]

The import of the principle of dependent choices lies in the following characterization of well-founded relations and well-orderings:

Lemma 5.2.

(a) *A linear ordering $<$ of a set P is a well-ordering of P if and only if there is no infinite descending sequence*

$$a_0 > a_1 > \cdots > a_n > \cdots$$

in A.

(b) *A relation E over P is well-founded if and only if there is no infinite sequence $\langle a_n : n \in N\rangle$ in P such that*

$$a_{n+1} \mathrel{E} a_n \qquad \text{for all} \quad n \in N \tag{5.5}$$

Proof. Note that (a) is a special case of (b) since a well-ordering is a well-founded linear ordering.

If $a_0, a_1, \ldots, a_n, \ldots$ is a sequence that satisfies (5.5), then the set $\{a_n : n \in N\}$ has no E-minimal element and hence E is not well-founded.

Conversely, if E is not well-founded, then there is a nonempty set $A \subseteq P$ with no E-minimal element. Using the principle of dependent choices we construct a sequence $a_0, a_1, \ldots, a_n, \ldots$ that satisfies (5.5). ■

6. CARDINAL ARITHMETIC

In the presence of the axiom of choice, every set can be well-ordered and so every infinite set has the cardinality of some $\aleph_\alpha$. Thus addition and multiplication of cardinal numbers is simple:

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta) \tag{6.1}$$

The exponentiation of cardinals is more interesting. This section is concerned with the operations $2^{\aleph_\alpha}$ and $\aleph_\alpha^{\aleph_\beta}$.

Lemma 6.1. *If $\alpha \le \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.*

Proof.

$$2^{\aleph_\beta} \le \aleph_\alpha^{\aleph_\beta} \le (2^{\aleph_\alpha})^{\aleph_\beta} \le 2^{\aleph_\alpha \cdot \aleph_\beta} = 2^{\aleph_\beta} \quad ■ \tag{6.2}$$

If κ is a cardinal less than $\aleph_\alpha$, then by (6.2)

$$\aleph_\alpha^\kappa \le \aleph_\alpha^{\aleph_\alpha} = 2^{\aleph_\alpha} = |P(\omega_\alpha)|$$

In fact, we can express $\aleph_\alpha^\kappa$ as the cardinality of a certain subfamily of $P(\omega_\alpha)$. If A is a set and $\kappa \leq |A|$, we define

$$[A]^\kappa = \text{the set of all } X \subseteq A \text{ such that } |X| = \kappa$$

Similarly,

$$[A]^{<\kappa} = \{X \subseteq A : |X| < \kappa\}$$

Lemma 6.2. *If* $\kappa \leq \aleph_\alpha$, *then* $\aleph_\alpha^\kappa = |[\omega_\alpha]^\kappa|$.

Proof. On the one hand, every $f\colon \kappa \to \omega_\alpha$ is a subset of $\kappa \times \omega_\alpha$, and $|f| = \kappa$. Thus

$$\aleph_\alpha^\kappa = |{}^\kappa\omega_\alpha| \leq |[\kappa \times \omega_\alpha]^\kappa| = |[\omega_\alpha]^\kappa|.$$

On the other hand, we construct a one-to-one function $F\colon [\omega_\alpha]^\kappa \to {}^\kappa\omega_\alpha$ to show that $|[\omega_\alpha]^\kappa| \leq \aleph_\alpha^\kappa$. If $X \subseteq \omega_\alpha$ and $|X| = \kappa$, let $F(X) = f$ be some function f on κ whose range is X. Clearly, F is one-to-one. ■

Infinite Sums and Products

Let $\{\kappa_i : i \in I\}$ be an indexed set of cardinal numbers. We define

$$\sum_{i \in I} \kappa_i = \left|\bigcup_{i \in I} X_i\right| \tag{6.3}$$

where $\{X_i : i \in I\}$ is a disjoint family of sets such that $|X_i| = \kappa_i$ for each $i \in I$.

Exercise 6.1. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are two disjoint families such that $|X_i| = |Y_i|$ for each $i \in I$, then $|\bigcup_{i \in I} X_i| = |\bigcup_{i \in I} Y_i|$.
[Use AC.]

Note that if κ and λ are cardinals and $\kappa_i = \kappa$ for each $i < \lambda$, then

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \kappa$$

In general, we have the following

Lemma 6.3. *If λ is an infinite cardinal and $\kappa_i > 0$ for each $i < \lambda$, then*

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i \tag{6.4}$$

Proof. Let $\kappa = \sup_{i<\lambda} \kappa_i$ and $\sigma = \sum_{i<\lambda} \kappa_i$. On the one hand, since $\kappa_i \leq \kappa$ for all i, we have $\sigma \leq \sum_{i<\lambda} \kappa = \lambda \cdot \kappa$. On the other hand, since $\kappa_i \geq 1$ for all i, we have $\lambda = \sum_{i<\lambda} 1 \leq \sigma$, and since $\sigma \geq \kappa_i$ for all i, we have $\sigma \geq \sup_{i<\lambda} \kappa_i = \kappa$. Therefore $\sigma \geq \lambda \cdot \kappa$. ■

In particular, if $\lambda \leq \sup_{i<\lambda} \kappa_i$, we have

$$\sum_{i<\lambda} \kappa_i = \sup_{i<\lambda} \kappa_i$$

Thus we can characterize singular cardinals as follows: An infinite cardinal κ is singular just in case

$$\kappa = \sum_{i<\lambda} \kappa_i$$

where $\lambda < \kappa$ and each $\kappa_i < \kappa$.

An infinite product of cardinals is defined using infinite cartesian products. If $\{X_i : i \in I\}$ is a family of sets, then the *Cartesian product* is defined as follows:

$$(6.5) \qquad \prod_{i \in I} X_i = \{f : f \text{ is a function on } I \text{ and } f(i) \in X_i \text{ for each } i \in I\}$$

In other words, elements of $\prod_{i \in I} X_i$ are functions

$$\langle x_i : i \in I \rangle$$

such that $x_i \in X_i$ for all $i \in I$. Note that if some X_i is empty, then the product is empty. If all the X_i are nonempty, then AC implies that the product is nonempty.

If $\{\kappa_i : i \in I\}$ is a family of cardinal numbers, we define

$$(6.6) \qquad \prod_{i \in I} \kappa_i = \left| \prod_{i \in I} X_i \right|$$

where $\{X_i : i \in I\}$ is a family of sets such that $|X_i| = \kappa_i$ for each $i \in I$. (We abuse the notation by using $\prod$ both for the Cartesian product and the product of cardinals.)

Exercise 6.2. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are such that $|X_i| = |Y_i|$ for each $i \in I$, then $|\prod_{i \in I} X_i| = |\prod_{i \in I} Y_i|$.

[Use AC.]

If $\kappa_i = \kappa$ for each $i \in I$, and $|I| = \lambda$, then $\prod_{i \in I} \kappa_i = \kappa^\lambda$. Also, infinite sums and products satisfy some of the rules satisfied by finite sums and products. For instance, $\prod_i \kappa_i^\lambda = (\prod_i \kappa_i)^\lambda$, or $\prod_i \kappa^{\lambda_i} = \kappa^{\sum_i \lambda_i}$. Or if I is a disjoint union $I = \bigcup_{j \in J} A_j$, then

$$(6.7) \qquad \prod_{i \in I} \kappa_i = \prod_{j \in J} \left(\prod_{i \in A_j} \kappa_i \right)$$

(the *associative law*). See Fig. 6.1.

If $\kappa_i \geq 2$ for each $i \in I$, then

$$(6.8) \qquad \sum_{i \in I} \kappa_i \leq \prod_{i \in I} \kappa_i$$

(The assumption $\kappa_i \geq 2$ is necessary: $1 + 1 > 1 \cdot 1$.) If I is finite, then (6.8) is certainly true; thus assume that I is infinite. Since $\prod_{i \in I} \kappa_i \geq \prod_{i \in I} 2 = 2^{|I|} > |I|$, it suffices to show that $\sum_i \kappa_i \leq |I| \cdot \prod_i \kappa_i$. If $\{X_i : i \in I\}$ is a disjoint

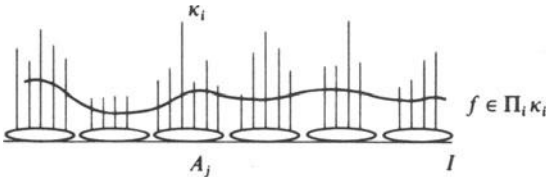

FIGURE 6.1

family, we assign to each $x \in \bigcup_i X_i$ a pair (i, f) such that $x \in X_i$, $f \in \prod_i X_i$ and $f(i) = x$. Thus we have (6.8).

The infinite product of cardinals can be evaluated using the following lemma:

Lemma 6.4. *If λ is an infinite cardinal and $\langle \kappa_i : i < \lambda \rangle$ is a nondecreasing sequence of nonzero cardinals, then*

$$\prod_{i<\lambda} \kappa_i = (\sup_i \kappa_i)^\lambda \tag{6.9}$$

Proof. Let $\kappa = \sup_i \kappa_i$. Since $\kappa_i \leq \kappa$ for each $i < \lambda$, we have

$$\prod_{i<\lambda} \kappa_i \leq \prod_{i<\lambda} \kappa = \kappa^\lambda$$

To prove that $\kappa^\lambda \leq \prod \kappa_i$, we consider a partition of λ into λ disjoint sets A_j, each of cardinality λ:

$$\lambda = \bigcup_{j<\lambda} A_j \tag{6.10}$$

(To get a partition (6.10), we can, e.g., use the canonical one-to-one mapping Γ: $\lambda \times \lambda \to \lambda$, and let $A_j = \Gamma[\lambda \times \{j\}]$.) Since a product of nonzero cardinals is greater than or equal to each factor, we have $\prod_{i \in A_j} \kappa_i \geq \sup_{i \in A_j} \kappa_i = \kappa$, for each $j < \lambda$. Thus, by (6.7),

$$\prod_{i<\lambda} \kappa_i = \prod_{j<\lambda} \left(\prod_{i \in A_j} \kappa_i \right) \geq \prod_{j<\lambda} \kappa = \kappa^\lambda. \quad \blacksquare$$

Exercise 6.3. $\prod_{0<n<\omega} n = 2^{\aleph_0}$.

Exercise 6.4. $\prod_{n<\omega} \aleph_n = \aleph_\omega^{\aleph_0}$.

Exercise 6.5. $\prod_{\alpha<\omega+\omega} \aleph_\alpha = \aleph_{\omega+\omega}^{\aleph_0}$.

The few strict inequalities in cardinal arithmetic proved in Section 3 can be obtained as special cases of the following general theorem.

Theorem 17 (König). *If $\kappa_i < \lambda_i$ for every $i \in I$, then*

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

Proof. We shall show that $\sum_i \kappa_i \not\geq \prod_i \lambda_i$. Let T_i, $i \in I$, be such that $|T_i| = \lambda_i$ for each $i \in I$. See Fig. 6.2. It suffices to show that if Z_i, $i \in I$, are subsets of $T = \prod_{i \in I} T_i$, and $|Z_i| \leq \kappa_i$ for each $i \in I$, then $\bigcup_{i \in I} Z_i \neq T$.

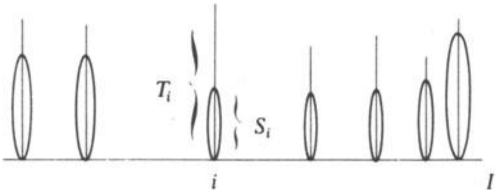

FIGURE 6.2

For every $i \in I$, let S_i be the projection of Z_i into the ith coordinate:

$$S_i = \{f(i) : f \in Z_i\}$$

Since $|Z_i| < |T_i|$, we have $S_i \subset T_i$. Now let $f \in T$ be a function such that $f(i) \notin S_i$ for every $i \in I$. Obviously, f does not belong to any Z_i, $i \in I$, and so $\bigcup_{i \in I} Z_i \neq T$. ■

Corollaries

1. $\kappa < 2^\kappa$ *for every* κ:

$$1 + 1 + \cdots \quad (\kappa \text{ times}) < 2 \cdot 2 \cdot \cdots \quad (\kappa \text{ times})$$

2. $\operatorname{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

It suffices to show that if $\kappa_i < 2^{\aleph_\alpha}$ for $i < \omega_\alpha$, then $\sum_{i < \omega_\alpha} \kappa_i < 2^{\aleph_\alpha}$. Let $\lambda_i = 2^{\aleph_\alpha}$.

$$\sum_{i < \omega_\alpha} \kappa_i < \prod_{i < \omega_\alpha} \lambda_i = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}$$

3. $\operatorname{cf}(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$.

We show that if $\kappa_i < \aleph_\alpha^{\aleph_\beta}$ for $i < \omega_\beta$, then $\sum_{i < \omega_\beta} \kappa_i < \aleph_\alpha^{\aleph_\beta}$. Let $\lambda_i = \aleph_\alpha^{\aleph_\beta}$.

$$\sum_{i < \omega_\beta} \kappa_i < \prod_{i < \omega_\alpha} \lambda_i = (\aleph_\alpha^{\aleph_\beta})^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}$$

4. $\kappa^{\operatorname{cf} \kappa} > \kappa$ *for every infinite cardinal* κ.

Let $\kappa_i < \kappa$, $i < \operatorname{cf} \kappa$, be such that $\kappa = \sum_{i < \operatorname{cf} \kappa} \kappa_i$. Then

$$\kappa = \sum_{i < \operatorname{cf} \kappa} \kappa_i < \prod_{i < \operatorname{cf} \kappa} \kappa = \kappa^{\operatorname{cf} \kappa}$$

The Continuum Function

By Cantor's theorem, $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$ for every α.

Generalized Continuum Hypothesis (GCH)

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \qquad \text{for all } \alpha$$

The generalized continuum hypothesis is independent of the axioms of ZFC. At this point, we shall give some properties of the function

$$\aleph_\alpha \mapsto 2^{\aleph_\alpha}$$

called the *continuum function.*

To start, it is obvious that

$$\text{(6.11)} \qquad \text{if} \quad \alpha \le \beta \quad \text{then} \quad 2^{\aleph_\alpha} \le 2^{\aleph_\beta}$$

By Cantor's theorem and by Corollary 2 of König's theorem, we have

$$\text{(6.12)} \qquad 2^{\aleph_\alpha} > \aleph_\alpha, \qquad \operatorname{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$$

It can be shown (Easton's theorem) that for *regular* cardinals, conditions (6.11) and (6.12) are the only restrictions on the continuum function provable in ZFC.

The case of singular cardinals is more interesting. For example, in the absence of large cardinals (namely if $0^\#$ does not exist), the continuum function is determined by its values at regular cardinals (Jensen). Some of the recent results are presented in Section 8.

In the following case, the value of 2^κ is determined by the values of the continuum function below κ.

Theorem 18 (Bukovský–Hechler). *Let κ be a singular cardinal such that the continuum function below κ is eventually constant, i.e., there exists $\gamma_0 < \kappa$ such that*

$$2^\gamma = 2^{\gamma_0} \qquad \text{for all } \gamma, \quad \gamma_0 \le \gamma < \kappa$$

Then $2^\kappa = 2^{\gamma_0}$.

The theorem is a consequence of the following lemma which gives us a formula for 2^κ at a singular cardinal κ. If κ is a limit cardinal, let us define

$$2^{<\kappa} = \sup\{2^\lambda : \lambda < \kappa\}$$

Note that if the GCH holds, then $2^{<\kappa} = \kappa$ for every limit cardinal κ.

Lemma 6.5. *If κ is a limit cardinal, then*

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa}$$

Proof. Let $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$, where $\kappa_i < \kappa$ for each i. We have

$$2^\kappa = 2^{\sum_i \kappa_i} = \prod_i 2^{\kappa_i} \le \prod_i 2^{<\kappa} = (2^{<\kappa})^{\operatorname{cf}\kappa} \le (2^\kappa)^{\operatorname{cf}\kappa} = 2^\kappa \quad \blacksquare$$

Proof of Theorem 18. If κ is a singular cardinal that satisfies the assumption of the theorem, then there is γ_0 such that $\operatorname{cf}\kappa \le \gamma_0 < \kappa$ and that $2^{<\kappa} = 2^{\gamma_0}$. Thus

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa} = (2^{\gamma_0})^{\operatorname{cf}\kappa} = 2^{\gamma_0} \quad \blacksquare$$

If κ is a limit cardinal and if the continuum function below κ is not eventually constant, then the cardinal $\lambda = 2^{<\kappa}$ is a limit of a nondecreasing sequence

$$\lambda = 2^{<\kappa} = \lim_{\alpha \to \kappa} 2^{|\alpha|}$$

of length κ. By Lemma 3.4b, we have

$$\operatorname{cf} \lambda = \operatorname{cf} \kappa$$

Using Lemma 6.5, we get

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf} \kappa} = \lambda^{\operatorname{cf} \lambda} \tag{6.13}$$

If κ is a regular cardinal, then $\kappa = \operatorname{cf} \kappa$; and since $2^\kappa = \kappa^\kappa$, we have

$$2^\kappa = \kappa^{\operatorname{cf} \kappa} \tag{6.14}$$

Thus (6.13) and (6.14) show that the continuum function can be defined in terms of the function

$$\gimel(\kappa) = \kappa^{\operatorname{cf} \kappa} \text{ (the \textit{gimel} function).}$$

(6.15) (a) If κ is a successor cardinal, then $2^\kappa = \gimel(\kappa)$.
(b) If κ is a limit cardinal and if the continuum function below κ is eventually constant, then $2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa)$.
(c) If κ is a limit cardinal and if the continuum function below κ is not eventually constant, then $2^\kappa = \gimel(2^{<\kappa})$.

Exercise 6.6. If β is such that $2^{\aleph_\alpha} = \aleph_{\alpha+\beta}$ for every α, then $\beta < \omega$.

[Let $\beta \geq \omega$. Let α be least such $\alpha + \beta > \beta$. We have $0 < \alpha \leq \beta$, and α is a limit. Let $\kappa = \aleph_{\alpha+\alpha}$; since $\operatorname{cf} \kappa = \operatorname{cf} \alpha \leq \alpha < \kappa$, κ is singular. For each $\xi < \alpha$, $\xi + \beta = \beta$, and so $2^{\aleph_{\alpha+\xi}} = \aleph_{\alpha+\xi+\beta} = \aleph_{\alpha+\beta}$. By Theorem 18, $2^\kappa = \aleph_{\alpha+\beta}$, a contradiction, since $\aleph_{\alpha+\beta} < \aleph_{\alpha+\alpha+\beta}$.]

Exponentiation of Cardinals

We are interested in the operation $\aleph_\alpha^{\aleph_\beta}$. If $\alpha \leq \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$ and so we shall concentrate on the case $\alpha > \beta$.

We start with the following observation: If κ is a regular cardinal and $\lambda < \kappa$, then every function f: $\lambda \to \kappa$ is bounded (i.e., $\sup f[\lambda] < \kappa$). Thus

$$^\lambda\kappa = \bigcup_{\alpha<\kappa} {}^\lambda\alpha, \tag{6.16}$$

and so

$$\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda \tag{6.17}$$

In particular, if κ is a successor cardinal, we obtain the *Hausdorff formula*

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1} \tag{6.18}$$

(Note that (6.18) holds for any α, β, also if $\beta > \alpha$.)

Exercise 6.7. $\prod_{\alpha<\omega_1+\omega} \aleph_\alpha = \aleph_{\omega_1+\omega}^{\aleph_1}$.

$$[\aleph_{\omega_1+\omega}^{\aleph_1} \leq \left(\prod_{n=0}^{\infty} \aleph_{\omega_1+n}\right)^{\aleph_1} = \prod_n \aleph_{\omega_1+n}^{\aleph_1} = \prod_n (\aleph_{\omega_1}^{\aleph_1} \cdot \aleph_{\omega_1+n}) = \aleph_{\omega_1}^{\aleph_1} \cdot \prod_n \aleph_{\omega_1+n}$$

$$= \prod_{\alpha<\omega_1+\omega} \aleph_\alpha.]$$

Exercise 6.8. If κ is a limit cardinal and $\lambda < \operatorname{cf} \kappa$, then $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda$.
[Note that (6.16) holds in this case.]

In general, we can compute κ^λ using the following lemma:

Lemma 6.6. *If κ is a limit cardinal, and $\lambda \geq \operatorname{cf} \kappa$, then*

$$\kappa^\lambda = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa}$$

Proof. Let $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$, where $\kappa_i < \kappa$ for each i. We have

$$\kappa^\lambda \leq \Big(\prod_{i<\operatorname{cf}\kappa} \kappa_i\Big)^\lambda = \prod_i \kappa_i^\lambda \leq \prod_i (\lim_{\alpha\to\kappa} \alpha^\lambda) = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa} \leq (\kappa^\lambda)^{\operatorname{cf}\kappa} = \kappa^\lambda \quad \blacksquare$$

Theorem 19 (*Inductive computation of $\aleph_\alpha^{\aleph_\beta}$*). *Let $\aleph_\beta$ be fixed. Then $\aleph_\alpha^{\aleph_\beta}$ can be computed as follows:*

(i) *If $\alpha \leq \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.*
(ii) *If there is $\gamma < \alpha$ such that $\aleph_\gamma^{\aleph_\beta} \geq \aleph_\alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\gamma^{\aleph_\beta}$.*
(iii) *If $\alpha > \beta$ and if $\aleph_\gamma^{\aleph_\beta} < \aleph_\alpha$ for all $\gamma < \alpha$, then:*
  (a) *if $\aleph_\alpha$ is regular or $\operatorname{cf} \aleph_\alpha > \aleph_\beta$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$;*
  (b) *if $\operatorname{cf} \aleph_\alpha \leq \aleph_\beta < \aleph_\alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$.*

Proof. (i) has been already proved.

(ii) $\aleph_\gamma^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta} \leq (\aleph_\gamma^{\aleph_\beta})^{\aleph_\beta} = \aleph_\gamma^{\aleph_\beta}$.

(iii) If $\aleph_\alpha$ is a successor cardinal, use the Hausdorff formula. If $\aleph_\alpha$ is a limit cardinal, then note that

$$\lim_{\gamma\to\alpha} \aleph_\gamma^{\aleph_\beta} = \aleph_\alpha \tag{6.19}$$

If $\operatorname{cf} \aleph_\alpha > \aleph_\beta$, then every $f\colon \omega_\beta \to \omega_\alpha$ is bounded, and (6.19) gives $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$. If $\operatorname{cf} \aleph_\alpha \leq \aleph_\beta < \aleph_\alpha$, then use Lemma 6.6 and (6.19) to conclude that $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$. $\blacksquare$

Corollary 1. *For all α, β, the value of $\aleph_\alpha^{\aleph_\beta}$ is always either:*

(a) $2^{\aleph_\beta}$ *or*
(b) $\aleph_\alpha$ *or*
(c) $\aleph_\gamma^{\operatorname{cf}\aleph_\gamma}$ *for some $\gamma \leq \alpha$, where $\aleph_\gamma$ is such that* $\operatorname{cf} \aleph_\gamma \leq \aleph_\beta < \aleph_\gamma$.

(If neither (a) nor (b) holds, let $\aleph_\gamma$ be the least κ such that $\kappa^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}$.)

Corollary 2. *If the* GCH *holds, then*

$$\aleph_\alpha^{\aleph_\beta} = \begin{cases} \aleph_\alpha & \text{if } \aleph_\beta < \operatorname{cf} \aleph_\alpha \\ \aleph_{\alpha+1} & \text{if } \operatorname{cf} \aleph_\alpha \leq \aleph_\beta \leq \aleph_\alpha \\ \aleph_{\beta+1} & \text{if } \aleph_\alpha \leq \aleph_\beta \end{cases} \tag{6.20}$$

(6.20) holds because if we assume the GCH, then

$$\aleph_\alpha^{\operatorname{cf}\aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

for all α.

In the course of investigation of the functions $2^{\aleph_\alpha}$ and $\aleph_\alpha^{\aleph_\beta}$ we have proved the following theorem.

Theorem 20 (Bukovský). *The continuum function can be defined from the gimel function. The exponentiation function* $\aleph_\alpha^{\aleph_\beta}$ *can be defined in terms of the gimel function and the cofinality function* cf κ. ■

This theorem shows the importance of the gimel function. We shall return to it below.

Exercise 6.9. $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

Exercise 6.10. If $\alpha < \omega_1$, then $\aleph_\alpha^{\aleph_1} = \aleph_\alpha^{\aleph_0} \cdot 2^{\aleph_1}$.

Exercise 6.11. If $\alpha < \omega_2$, then $\aleph_\alpha^{\aleph_2} = \aleph_\alpha^{\aleph_1} \cdot 2^{\aleph_2}$.

A cardinal κ is a *strong limit* cardinal if

$$2^\lambda < \kappa \qquad \text{for every} \quad \lambda < \kappa$$

Obviously, every strong limit cardinal is a limit cardinal. If the GCH holds, then every limit cardinal is a strong limit.

It should be easy to see that if κ is a strong limit, then

$$\lambda^\nu < \kappa \qquad \text{for all } \lambda, \nu < \kappa$$

An example of a strong limit cardinal is $\aleph_0$. Actually, the strong limit cardinals form a proper class: If α is an arbitrary cardinal, then the cardinal

$$\kappa = \sup\{\alpha, 2^\alpha, 2^{2^\alpha}, \ldots\}$$

(of cofinality ω) is a strong limit.

Another fact worth noticing is

(6.21) If κ is a strong limit, then $2^\kappa = \kappa^{\operatorname{cf}\kappa}$.

This need not be true if κ is a singular cardinal that is not a strong limit. If κ is a limit cardinal, let us define

$$\kappa^{<\kappa} = |[\kappa]^{<\kappa}| = \lim_{\alpha \to \kappa} \kappa^\alpha \tag{6.22}$$

Obviously,

$$\kappa^{<\kappa} \geq 2^{<\kappa} \geq \kappa$$

Exercise 6.12. If κ is regular and a limit, then $\kappa^{<\kappa} = 2^{<\kappa}$. If κ is regular and a strong limit then $\kappa^{<\kappa} = \kappa$.

Exercise 6.13. If κ is singular and is not a strong limit, then $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$.

Exercise 6.14. If κ is singular and a strong limit, then $2^{<\kappa} = \kappa$ and $\kappa^{<\kappa} = \kappa^{\operatorname{cf}\kappa}$.

For the sake of completeness, we define $\lambda^{<\kappa}$ for any cardinal λ and any infinite cardinal κ:

$$\lambda^{<\kappa} = \sup\{\lambda^{\mu} : \mu < \kappa\}$$

In particular, we have $\lambda^{<\kappa^+} = \lambda^{\kappa}$.

Properties of the Gimel Function

Theorem 20 shows the significance of the function $\gimel(\kappa) = \kappa^{\operatorname{cf}\kappa}$ in cardinal arithmetic. We shall discuss here some properties of the gimel function.

As shown earlier, we have

$$(6.23) \qquad \gimel(\kappa) > \kappa \quad \text{and} \quad \operatorname{cf}\gimel(\kappa) > \operatorname{cf}\kappa$$

for all κ. Also, if κ is regular then $\gimel(\kappa) = 2^{\kappa}$. If κ is singular, then in the following case, $\gimel(\kappa)$ is determined by the behavior of $\gimel$ below κ:

A singular cardinal κ is *bound* if

$$(6.24) \qquad \lambda^{\operatorname{cf}\kappa} \geq \kappa$$

for some $\lambda < \kappa$. Using the computation from Theorem 19, it follows that κ is bound just in case $\kappa \leq \gimel(\lambda)$ for some $\lambda < \kappa$ of cofinality $\leq \operatorname{cf}\kappa$ (see Exercise 6.15 below). A strong limit singular cardinal is not bound, but there may exist singular cardinals that are neither strong limits nor bound (e.g., $\aleph_\omega$ if $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_\omega$).

If κ is a bound singular cardinal and (6.24) holds for some $\lambda < \kappa$, then

$$(6.25) \qquad \kappa^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\kappa}$$

and the value of $\gimel(\kappa)$ is determined by exponentiation of smaller cardinals. ((6.25) is proved as follows: $\lambda^{\operatorname{cf}\kappa} \leq \kappa^{\operatorname{cf}\kappa} \leq (\lambda^{\operatorname{cf}\kappa})^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\kappa}$.)

Exercise 6.15. If κ is a bound singular cardinal, then $\gimel(\kappa)$ is either $2^{\operatorname{cf}\kappa}$, or $\gimel(\lambda)$ where λ is the least cardinal $\lambda \geq \operatorname{cf}\kappa$ that satisfies (6.24).

[Use Corollary 1 of Theorem 19.]

Exercise 6.16. If $2^{\aleph_0} > \aleph_\omega$, then $\aleph_\omega^{\aleph_0} = 2^{\aleph_0}$.

Exercise 6.17. If $2^{\aleph_1} = \aleph_2$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$, then $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_0}$.

Exercise 6.18. If $2^{\aleph_0} \geq \aleph_{\omega_1}$, then $\gimel(\aleph_\omega) = 2^{\aleph_0}$ and $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$.

If a singular cardinal κ is not a strong limit, then $\gimel(\kappa)$ is subject to the following condition: If $\kappa \leq \gimel(\lambda)$ for some $\lambda < \kappa$ such that $\operatorname{cf}\kappa \leq \operatorname{cf}\lambda$, then

$$(6.26) \qquad \gimel(\kappa) \leq \gimel(\lambda)$$

(This is clear since $\kappa^{\operatorname{cf}\kappa} \leq (\lambda^{\operatorname{cf}\lambda})^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\lambda}$.)

Apart from this condition (6.26) and a further condition on cofinality (Exercise 6.19), cardinal arithmetic alone does not seem to impose further restrictions on the behavior of $\gimel$ (or of the continuum function).

In Section 8 we present a result of Silver that gives additional substantial information on the continuum function of singular cardinals of uncountable cofinality.

And a recent result of Jensen shows that unless $0^\#$ exists (see Section 30 for a formulation of this large cardinal axiom), then whenever κ is a singular cardinal such that $2^{\mathrm{cf}\,\kappa} < \kappa$, we have $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$, and so all cardinal exponentiation is determined by the continuum function on regular cardinals and the cofinality function).

Exercise 6.19. If κ is singular and not bound, then $\mathrm{cf}(\gimel(\kappa)) \geq$ least α such that $\lambda^\alpha \geq \kappa$ for some $\lambda < \kappa$. In particular, if κ is a strong limit, then $\mathrm{cf}(\gimel(\kappa)) > \kappa$.

[Since κ is not bound, we have $\alpha > \mathrm{cf}\,\kappa$. Let $\mathrm{cf}\,\kappa \leq \gamma < \alpha$. Since $\lambda^\gamma < \kappa$ for all $\lambda < \kappa$, we have $\kappa^\gamma = \kappa^{\mathrm{cf}\,\kappa}$. However, $\mathrm{cf}(\kappa^\gamma) > \gamma$.]

Exercise 6.20. If $2^{\aleph_1} = \aleph_2$, then $\aleph_\omega^{\aleph_0} \neq \aleph_{\omega_1}$.

Inaccessible Cardinals

We recall that κ is weakly inaccessible if it is uncountable, regular, and a limit. We say that a cardinal κ is *inaccessible* (strongly) if $\kappa > \aleph_0$, κ is regular, and κ is a strong limit.

Every inaccessible cardinal is weakly inaccessible. If the GCH holds, then every weakly inaccessible κ is inaccessible.

The inaccessible cardinals owe their name to the fact that they cannot be obtained by the usual set theoretical operations (a precise statement will be proved in Chapter 2):

If κ is inaccessible and $|X| < \kappa$, then $|P(X)| < \kappa$. If $|S| < \kappa$ and if $|X| < \kappa$ for every $X \in S$, then $|\bigcup S| < \kappa$.

In fact, $\aleph_0$ has this property too. Thus we can say that in a sense an inaccessible cardinal is to smaller cardinals as $\aleph_0$ is to finite cardinals. This is one of the main themes of the theory of large cardinals in Part III.

7. FILTERS AND IDEALS. CLOSED UNBOUNDED SETS

Filters and ideals play an important role in abstract mathematical disciplines (algebra, logic, measure theory). In this section we introduce the notion of filter (and ideal) over a given set. The notion of ideal extrapolates the notion of small sets: Given an ideal I over S, a set $X \subset S$ is deemed to be small if it belongs to I.

A *filter* over a set S is a collection F of subsets of S such that:

(7.1) (i) $S \in F$;
(ii) if $X \in F$ and $Y \in F$, then $X \cap Y \in F$;
(iii) if $X, Y \subseteq S$, $X \in F$, and $X \subseteq Y$, then $Y \in F$.

A filter F is *proper* if $\emptyset \notin F$. We shall consider only proper filters; thus we always assume $S \neq \emptyset$ and

(iv) $\emptyset \notin F$.

An *ideal* over a set S is a collection I of subsets of S such that:

(7.2) (i) $\emptyset \in I$;
(ii) if $X \in I$ and $Y \in I$, then $X \cup Y \in I$;
(iii) if $X, Y \subseteq S$, $X \in I$, and $Y \subseteq X$, then $Y \in I$.

We shall consider only *proper* ideals:

(iv) $S \notin I$

If F is a filter over S, then the set

$$I = \{S - X : X \in F\} \tag{7.3}$$

is an ideal over S; and conversely, if I is an ideal, then

$$F = \{S - X : X \in I\} \tag{7.4}$$

is a filter. If (7.3), or (7.4), holds, we say that F and I are *dual.*

Examples

1. A *trivial* filter: $F = \{S\}$.

2. A *principal* filter. Let X_0 be a nonempty subset of S. The filter

$$F = \{X \subseteq S : X \supseteq X_0\} \tag{7.5}$$

is a *principal* filter.

The dual notions are a trivial ideal and a principal ideal.

3. The *Fréchet filter.* Let S be an infinite set, and let I be the ideal of all finite subsets of S. The dual filter

$$F = \{X \subseteq S : S - X \text{ is finite}\}$$

is called the Fréchet filter over S.

Note that the Fréchet filter is not principal.

Exercise 7.1. Every filter over a finite set is principal.

Exercise 7.2. If F is a filter and $X \in F$, then $P(X) \cap F$ is a filter over X.

Exercise 7.3. Let κ be an infinite cardinal, $|S| \geq \kappa$. The set $\{X \subset S : |X| < \kappa\}$ is a nonprincipal ideal over S.

Exercise 7.4. Let A be an infinite set and let $S = [A]^{<\omega}$ be the set of all finite subsets of A. For each $P \in S$, let $\hat{P} = \{Q \in S : P \subseteq Q\}$. Let F be the set of all $X \subseteq S$ such that $X \supseteq \hat{P}$ for some $P \in S$. Then F is a nonprincipal filter over S.

Let $\{A_i : i \in I\}$ be an indexed family of nonempty sets and let us consider the cartesian product

$$P = \prod_{i \in I} A_i$$

Let further F be a filter over the set I of indices. We define the following relation over P:

$$f =_F g \quad \text{iff} \quad \{i \in I : f(i) = g(i)\} \in F \tag{7.6}$$

(that is, if there is only a small set of i's such that $f(i) \neq g(i)$). It follows easily from the properties (7.1) of F that $=_F$ is an equivalence relation over P. The set of all $=_F$ equivalence classes in P is called the *reduced product* of A_i, $i \in I$, mod F

$$\prod_F A_i = P/=_F \tag{7.7}$$

In case that $A_i = A$ for each $i \in I$, the reduced product (7.7) is called the *reduced power* of A mod F.

A family G of sets has the *finite intersection property* if every finite $H = \{X_1, \ldots, X_n\} \subseteq G$ has a nonempty intersection:

$$X_1 \cap \cdots \cap X_n \neq \emptyset$$

Every filter has the finite intersection property.

Lemma 7.1.

(a) *If $\mathscr{F}$ is a nonempty family of filters over S, then $\bigcap \mathscr{F}$ is a filter over S.*
(b) *If $\mathscr{C}$ is a $\subset$-chain of filters over S, then $\bigcup \mathscr{C}$ is a filter over S.*
(c) *If $G \subset P(S)$ has the finite intersection property, there is a filter F over S such that $G \subseteq F$.*

Proof. (a), (b) left to the reader; verify (7.1).

(c) Let F be the set of all $X \subseteq S$ such that there is a finite $H = \{X_1, \ldots, X_n\} \subseteq G$ such that

$$X_1 \cap \cdots \cap X_n \subseteq X \tag{7.8}$$

It is easy to verify that F is a filter. ■

Since every filter $F \supseteq G$ must contain all finite intersections of sets in G, it follows that the filter F constructed in the proof of Lemma 7.1c is the least filter over S that extends G:

$$F = \bigcap \{D : D \text{ is a filter over } S \text{ and } G \subseteq D\}$$

The filter F is *generated* by G.

Exercise 7.5. The filter from Exercise 7.4 is generated by the sets $\{a\}^\wedge$, $a \in A$.

A filter U over a set S is an *ultrafilter* if

(7.9) for every $X \subseteq S$, either $X \in U$ or $S - X \in U$

(but not both: $X \cap (S - X) = \emptyset$, and $\emptyset \notin U$). The dual notion is a *prime ideal*: For every $X \subseteq S$, either $X \in I$ or $S - X \in I$. Note that $I = P(S) - U$.

Exercise 7.6. If U is an ultrafilter and $X \cup Y \in U$, then either $X \in U$ or $Y \in U$.

A filter F over S is *maximal* if there is no filter F' over S such that $F \subset F'$.

Lemma 7.2. *A filter F over S is an ultrafilter if and only if it is maximal.*

Proof. (a) An ultrafilter U is clearly a maximal filter: Assume that $U \subset F$ and $X \in F - U$. Then $S - X \in U$, and so both $S - X \in F$ and $X \in F$, a contradiction.

(b) Let F be a filter that is not an ultrafilter. We find $F' \supset F$: Let $Y \subseteq S$ be such that neither Y nor $S - Y$ is in F. Consider the family $G = F \cup \{Y\}$; we claim that G has the finite intersection property. If $X \in F$, then $X \cap Y \neq \emptyset$, for otherwise we would have $S - Y \supseteq X$ and $S - Y \in F$. Thus, if $X_1, \ldots, X_n \in F$, we have $X_1 \cap \cdots \cap X_n \in F$ and so

$$Y \cap X_1 \cap \cdots \cap X_n \neq \emptyset$$

Hence G has the finite intersection property, and by Lemma 7.1c there is a filter $F' \supseteq G$. Since $Y \in F' - F$, we have $F \subset F'$. ■

Theorem 21 (*Tarski's Ultrafilter Theorem*). *Every filter over S can be extended to an ultrafilter.*

Proof. Let F_0 be a filter over S. Let P be the set of all filters F over S such that $F \supseteq F_0$ and consider the partially ordered set $(P, \subset)$. If $\mathscr{C}$ is a chain in P, then by Lemma 7.1b, $\bigcup \mathscr{C}$ is a filter and hence an upper bound of $\mathscr{C}$ in P. By the Kuratowski–Zorn lemma there exists a maximal element U in P. This U is an ultrafilter by Lemma 7.2. ■

For every $a \in S$, the principal filter

$$\{X \subseteq S : a \in X\} \tag{7.10}$$

is an ultrafilter. If S is finite, then every ultrafilter over S is of the form (7.10).

If S is infinite, then there is a nonprincipal ultrafilter over S: If U extends the Fréchet filter, then U is nonprincipal.

A filter F over S is *countably complete* (*σ-complete*) if whenever $\{X_n : n \in N\}$ is a countable family of subsets of S and $X_n \in F$ for every n, then

$$\bigcap_{n=0}^{\infty} X_n \in F \tag{7.11}$$

A *countably complete* ideal (a *σ-ideal*) is such that if $X_n \in I$ for every n, then

$$\bigcup_{n=0}^{\infty} X_n \in I$$

More generally, if κ is a regular uncountable cardinal, and F a filter over S, then F is called *κ-complete* if F is closed under intersections of less than κ sets, i.e., if whenever $\{X_\alpha : \alpha < \gamma\}$ is a family of subsets of S, $\gamma < \kappa$, and $X_\alpha \in F$ for every $\alpha < \gamma$, then

$$\bigcap_{\alpha<\gamma} X_\alpha \in F \tag{7.12}$$

The dual notion is a *κ-complete ideal.*

A σ-complete filter is the same as $\aleph_1$-complete. Every filter is closed under finite intersections. Every principal filter is closed under arbitrary intersections.

***Exercise** 7.7.* If ω_α is singular, then there is no nonprincipal ω_α-complete ideal over ω_α.

[Let $X = \{a : \{a\} \in I\}$. X is the union of $< \aleph_\alpha$ sets of cardinality $< \aleph_\alpha$, and so $X \in I$.]

There is no nonprincipal σ-complete filter over a countable set S. If S is uncountable, then

$$\{X \subseteq S : |X| \leq \aleph_0\}$$

is a σ-ideal over S.

Similarly, if $\kappa > \omega$ is regular and $|S| \geq \kappa$, then

$$\{X \subseteq S : |X| < \kappa\}$$

is the least κ-complete ideal over S containing all singletons $\{a\}$.

The question whether a nonprincipal *ultrafilter* over a set can be σ-complete gives rise to deep investigations of the metamathematics of set theory. In particular, if such ultrafilters exist, then there exist many large cardinals (inaccessible, etc.). We shall return to this problem in Part III.

Closed Unbounded Sets

Let κ be a regular uncountable cardinal. We call a set $C \subseteq \kappa$ *closed unbounded* in κ if

(7.13) (i) for every sequence $\alpha_0 < \alpha_1 < \cdots < \alpha_\xi < \cdots (\xi < \gamma)$ of elements of C, of length $\gamma < \kappa$, we have $\lim_{\xi \to \gamma} \alpha_\xi \in C$ (closed);

(ii) for every $\alpha < \kappa$, there is $\beta > \alpha$ such that $\beta \in C$ (unbounded).

Lemma 7.3. *If C and D are closed unbounded, then $C \cap D$ is closed unbounded.*

Proof. It is immediate that $C \cap D$ is closed. To show that $C \cap D$ is unbounded, let $\alpha < \kappa$. Since C is unbounded, there exists $\alpha_1 > \alpha$, $\alpha_1 \in C$. Similarly, there is $\alpha_2 > \alpha_1$, $\alpha_2 \in D$. In this fashion, we construct an increasing sequence

$$\alpha < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots \tag{7.14}$$

such that $\alpha_1, \alpha_3, \alpha_5, \ldots \in C$, and $\alpha_2, \alpha_4, \alpha_6, \ldots \in D$. If we let β be the limit of the sequence (7.14), then $\beta < \kappa$, and $\beta \in C$ and $\beta \in D$. ■

The collection of all closed unbounded subsets of κ has the finite intersection property. The filter generated by the closed unbounded sets consists of all $X \subseteq \kappa$ that contain a closed unbounded subset. We call this filter the *closed unbounded filter* over κ.

The set of all limit ordinals $\alpha < \kappa$ is closed unbounded in κ. If A is a subset of κ, we say that a limit ordinal $\alpha < \kappa$ is a *limit point* of A if $0 < \alpha = \sup(A \cap \alpha)$. If A is an unbounded set of κ, then the set of all limit points of A is closed unbounded.

A function $f\colon \kappa \to \kappa$ is *normal* if it is increasing and continuous ($f(\alpha) = \lim_{\xi \to \alpha} f(\xi)$ for every nonzero limit $\alpha < \kappa$). The range of a normal function is a closed unbounded set. Conversely, if C is closed unbounded, there is a unique normal function that enumerates C.

Exercise 7.8. The set of all fixed points (i.e., $f(\alpha) = \alpha$) of a normal function is closed unbounded.

Exercise 7.9. If $f\colon \kappa \to \kappa$, then the set of all $\alpha < \kappa$ such that $f[\alpha] \subseteq \alpha$ is closed unbounded.

The closed unbounded filter over κ is κ-complete:

Lemma 7.4. *The intersection of less than κ closed unbounded subsets of κ is closed unbounded.*

Proof. We prove, by induction on $\gamma < \kappa$, that the intersection of a sequence $\{C_\alpha : \alpha < \gamma\}$ of closed unbounded subsets of κ is closed unbounded. The induction step works at successor ordinals because of Lemma 7.3. If γ is a limit ordinal, we assume that the assertion is true for every $\alpha < \gamma$; then we can replace each C_α by $\bigcap_{\xi \le \alpha} C_\xi$ and obtain a decreasing sequence with the same intersection. Thus assume that

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_\alpha \supseteq \cdots \qquad (\alpha < \gamma)$$

are closed unbounded, and let $C = \bigcap_{\alpha < \gamma} C_\alpha$.

It is easy to see that C is closed. To show that C is unbounded, let $\alpha < \kappa$. We construct a γ-sequence

$$\beta_0 < \beta_1 < \cdots < \beta_\xi < \cdots \qquad (\xi < \gamma) \tag{7.15}$$

as follows: We let $\beta_0 \in C_0$ be such $\beta_0 > \alpha$, and for each $\xi < \gamma$, let $\beta_\xi \in C_\xi$ be such that $\beta_\xi > \sup\{\beta_\iota : \iota < \xi\}$. Since κ is regular and $\gamma < \kappa$, such a sequence (7.15) exists and its limit β is less than κ. For each $\eta < \gamma$, β is the limit of a sequence $\langle \beta_\xi : \eta \le \xi < \gamma \rangle$ in C_η, and so $\beta \in C_\eta$. Hence $\beta \in C$. ■

Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence of subsets of κ. The *diagonal intersection* of X_α, $\alpha < \kappa$, is defined as follows:

$$\triangle \{X_\alpha : \alpha < \kappa\} = \{\xi < \kappa : \xi \in \bigcap_{\alpha < \xi} X_\alpha\} \tag{7.16}$$

Note that $\triangle X_\alpha = \triangle Y_\alpha$ where $Y_\alpha = \{\xi \in X_\alpha : \xi > \alpha\}$; see Fig. 7.1. Note also that $\triangle X_\alpha = \bigcap_\alpha (X_\alpha \cup \{\xi : \xi \leq \alpha\})$.

Lemma 7.5. *The diagonal intersection of a κ-sequence of closed unbounded sets is closed unbounded.*

Proof. Let $\langle C_\alpha : \alpha < \kappa \rangle$ be a sequence of closed unbounded sets. It is clear from the definition that if we replace each C_α by $\bigcap_{\xi \leq \alpha} C_\xi$, the diagonal intersection is the same. In view of Lemma 7.4 we may thus assume that

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_\alpha \supseteq \cdots \qquad (\alpha < \kappa)$$

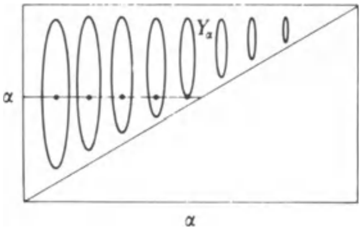

FIGURE 7.1

Let $C = \triangle\{C_\alpha : \alpha < \kappa\}$. To show that C is closed, let α be a limit point of C. We want to show that $\alpha \in C$, or that $\alpha \in C_\xi$ for all $\xi < \alpha$. If $\xi < \alpha$, let $X = \{\nu \in C : \xi < \nu < \alpha\}$. Every $\nu \in X$ is in C_ξ, by (7.16). Hence $X \subset C_\xi$ and $\alpha = \sup X \in C_\xi$. Therefore $\alpha \in C$ and C is closed.

To show that C is unbounded, let $\alpha < \kappa$. We construct a sequence $\langle \beta_n : n < \omega \rangle$ as follows: Let $\beta_0 > \alpha$ be such that $\beta_0 \in C_0$, and for each n, let $\beta_{n+1} > \beta_n$ be such that $\beta_{n+1} \in C_{\beta_n}$. Let us show that $\beta = \lim_n \beta_n$ is in C: If $\xi < \beta$, let us show that $\beta \in C_\xi$. Since $\xi < \beta$, there is an n such that $\xi < \beta_n$. Each β_k, $k > n$, belongs to C_{β_n} and so $\beta \in C_{\beta_n}$. Therefore $\beta \in C_\xi$. Thus $\beta \in C$, and C is unbounded. ■

Corollary. *The closed unbounded filter over κ is closed under diagonal intersections.*

We say that a set $S \subseteq \kappa$ is *stationary* in κ if $S \cap C \neq \emptyset$ for every closed unbounded subset C of κ. A set $T \subseteq \kappa$ that is not stationary will be called *thin*. Note that the collection of all thin sets is the ideal dual to the closed unbounded filter; we shall call this ideal the *thin ideal* over κ.

Example. The set $S = \{\alpha < \kappa : \text{cf } \alpha = \omega\}$ is stationary in κ. Similarly, if $\kappa > \omega_1$, the set $S = \{\alpha < \kappa : \text{cf } \alpha = \omega_1\}$ is stationary in κ.

Exercise 7.10. If S is stationary and C is closed unbounded, then $S \cap C$ is stationary.

An ordinal function f on a set S is called *regressive* if $f(\alpha) < \alpha$ for every nonzero $\alpha \in S$.

Theorem 22 (Fodor). *If f is a regressive function on a stationary set $S_0 \subseteq \kappa$, then there exists a stationary set $S \subseteq S_0$ and some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in S$.*

Proof. Let us assume that for each $\gamma < \kappa$, the set $\{\alpha \in S_0 : f(\alpha) = \gamma\}$ is thin, and choose a closed unbounded set C_γ such that $f(\alpha) \neq \gamma$ for each $\alpha \in S_0 \cap C_\gamma$. Let $C = \triangle\{C_\gamma : \gamma < \kappa\}$. The set $S_0 \cap C$ is stationary and if $\alpha \in S_0 \cap C$, we have $f(\alpha) \neq \gamma$ for every $\gamma < \alpha$; in other words, $f(\alpha) \geq \alpha$. This is a contradiction. ■

Exercise 7.11. If $T \subset \kappa$ is thin, then there exists a regressive function f on T such that $\{\alpha : f(\alpha) \leq \gamma\}$ is bounded, for every $\gamma < \kappa$.

[Let $C \cap T = \emptyset$, and let $f(\alpha) = \sup(C \cap \alpha)$.]

The closed unbounded filter over κ is not an ultrafilter. This is because there is a stationary subset of κ whose complement is stationary. If $\kappa > \omega_1$, this is clear: The sets $\{\alpha : \operatorname{cf} \alpha = \omega\}$ and $\{\alpha : \operatorname{cf} \alpha = \omega_1\}$ are disjoint. If $\kappa = \omega_1$, the decomposition of ω_1 into disjoint stationary sets uses the axiom of choice. It is an interesting problem (related to large cardinals) to show that the use of AC is necessary.

In fact, every stationary $S \subseteq \kappa$ can be decomposed into κ disjoint stationary sets. We shall prove this in Section 35. Here is a weaker statement:

Lemma 7.6. *Every regular uncountable κ is the union of κ disjoint stationary sets.*

Proof. Let $W = \{\alpha < \kappa : \operatorname{cf} \alpha = \omega\}$. For every $\alpha \in W$, we choose an increasing sequence $\{a_n^\alpha : n \in N\}$ such that $\lim_n a_n^\alpha = \alpha$. See Fig. 7.2. First we show that there is an n such that for all $\eta < \kappa$, the set

$$\{\alpha \in W : a_n^\alpha \geq \eta\} \tag{7.17}$$

is stationary. Otherwise there is η_n and a closed unbounded set C_n such that $a_n^\alpha < \eta_n$ for all $\alpha \in C_n \cap W$, for every n. If we let η be the supremum of the η_n and C the intersection of the C_n, we have $a_n^\alpha < \eta$ for all n, all $\alpha \in C \cap W$. This is a contradiction. Now let n be such that (7.17) is stationary for every $\eta < \kappa$. Let f be the following function on W:

$$f(\alpha) = a_n^\alpha$$

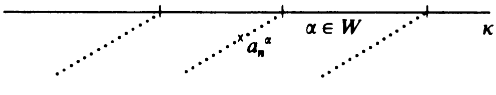

FIGURE 7.2

The function f is regressive; and so for every $\eta < \kappa$, we find by Fodor's theorem a stationary subset S_η of (7.17) and $\gamma_\eta \geq \eta$ such that $f(\alpha) = \gamma_\eta$ on S_η. If $\gamma_\eta \neq \gamma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$, and since κ is regular, we have $|\{S_\eta : \eta < \kappa\}| = |\{\gamma_\eta : \eta < \kappa\}| = \kappa$. ■

Let α be an ordinal. A set A is a *closed* set of ordinals of length α if it is the range of a normal sequence of length α (note that $\sup A$ is not in A if α is a limit ordinal).

Exercise 7.12. For every stationary $S \subseteq \omega_1$ and every $\alpha < \omega_1$ there is a closed set of ordinals A of length α such that $A \subset S$.

[By induction on α: $\forall\gamma\ \exists$closed $A \subset S$ of length α such that $\gamma < \min(A)$. The nontrivial step: If true for a limit α, find a closed $A \subset S$ of length α such that $\sup A \in S$. Let A_ξ, $\xi < \omega_1$, be closed subsets of S, of length α such that $\lambda_\xi = \sup \bigcup_{i<\xi} A_i < \min A_\xi$. There is ξ such that $\lambda_\xi \in S$. Let $\xi = \lim_n \xi_n$. Pick initial segments $B_{\xi_n} \subset A_{\xi_n}$ of length $\alpha_n + 1$ where $\lim_n \alpha_n = \alpha$. Let $A = \bigcup_{n=0}^{\infty} B_{\xi_n}$.]

Exercise 7.12 does not generalize to closed sets of uncountable length. It is not provable in ZFC that given $X \subseteq \omega_2$, either X or $\omega_2 - X$ contains a closed set of length ω_1. It is an open problem whether this statement is consistent.

Let F be a filter over a cardinal κ; F is *normal* if it is closed under diagonal intersections:

(7.18) $$\text{if}\quad X_\alpha \in F,\ \alpha < \kappa, \qquad \text{then}\quad \triangle\{X_\alpha : \alpha < \kappa\} \in F$$

An ideal I over κ is *normal* if the dual filter is normal.

Exercise 7.13. A κ-complete ideal I over κ is normal if and only if for every $S_0 \notin I$ and any regressive f on S_0 there is $S \subseteq S_0$, $S \notin I$, such that f is constant on S.

[One direction is like Fodor's theorem. For the other direction, let $X_\alpha \in F$ for each $\alpha < \kappa$. If $\triangle X_\alpha \notin F$, let $S_0 = \kappa - \triangle X_\alpha$ and let $f(\alpha) =$ some $\xi < \alpha$ such that $\alpha \notin X_\xi$. If $f(\alpha) = \gamma$ for all $\alpha \in S$, then $X_\gamma \cap S = \emptyset$, a contradiction.]

Exercise 7.14. There is no normal nonprincipal filter over ω.

[Use the regressive function $f(n+1) = n$.]

Exercise 7.15. If κ is singular, then there is no normal ideal over κ that contains all bounded subsets of κ.

The closed unbounded filter is κ-complete and normal, and contains all complements of bounded sets. It is the smallest such filter over κ:

Lemma 7.7. *If κ is regular and uncountable and if F is a normal filter over κ that contains all final segments $\{\alpha : \alpha_0 < \alpha < \kappa\}$, then F contains all closed unbounded sets.*

Proof. First we note that the set C_0 of all limit ordinals is in F: C_0 is the diagonal intersection of the sets

$$X_\alpha = \{\xi : \alpha + 1 < \xi < \kappa\}$$

Now let C be a closed unbounded set, and let $C = \{a_\alpha : \alpha < \kappa\}$ be its increasing enumeration. We let

$$X_\alpha = \{\xi : a_\alpha < \xi < \kappa\}$$

Then $C \supseteq C_0 \cap \triangle\{X_\alpha : \alpha < \kappa\}$. ■

Mahlo Cardinals

Let κ be an inaccessible cardinal. The set of all cardinals below κ is a closed unbounded subset of κ, and so is the set of its limit points, the set of all limit cardinals. In fact, the set of all strong limit cardinals below κ is closed unbounded (to find the next one, take the limit of λ, 2^λ, $2^{2^\lambda}, \ldots$).

If κ is the least inaccessible cardinal, then all strong limit cardinals below κ are singular, and so the set of all singular strong limit cardinals below κ is closed unbounded. If κ is the αth inaccessible, where $\alpha < \kappa$, then still the set of all regular cardinals below κ is thin.

An inaccessible cardinal κ is called a *Mahlo cardinal* if the set of all regular cardinals below κ is stationary.

(Then the set of all inaccessibles below κ is stationary, and κ is the κth inaccessible cardinal.)

Exercise 7.16. Let κ be the least inaccessible cardinal such that κ is the κth inaccessible cardinal. Then κ is not Mahlo.

[Use $f(\lambda) = \alpha$ where λ is the αth inaccessible.]

Similarly, we define a *weakly Mahlo* cardinal as a cardinal κ that is weakly inaccessible and the set of all regular cardinals below κ is stationary (then the set of all weakly inaccessibles is stationary in κ).

Exercise 7.17. If κ is a limit (weakly inaccessible, weakly Mahlo) cardinal and the set of all strong limit cardinals below κ is unbounded in κ, then κ is a strong limit (inaccessible, Mahlo).

8. SINGULAR CARDINALS

We have shown in Section 6 that if κ is singular, then the size of $P(\kappa)$ depends on the continuum function below κ (Theorem 18). Namely, if $2^{\aleph_\alpha}$ is eventually constant for $\aleph_\alpha < \kappa$, then $2^\kappa = 2^{<\kappa}$. This is just one of many examples when exponentiation involving singular cardinals depends substantially on the values of the continuum function at regular cardinals. In fact, unless strong axioms of infinity (so-called large cardinal axioms) are assumed, then all exponentiation can be computed in terms of $2^{\aleph_\alpha}$ for regular $\aleph_\alpha$.

Let us consider the following hypothesis.

The Singular Cardinals Hypothesis. *For every singular cardinal* κ, *if* $2^{\operatorname{cf}\kappa} < \kappa$, *then* $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.

Obviously, the singular cardinals hypothesis follows from the GCH. If $2^{\operatorname{cf}\kappa} \geq \kappa$ then $\kappa^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa}$. If $2^{\operatorname{cf}\kappa} < \kappa$, then κ^+ is the least possible value of $\kappa^{\operatorname{cf}\kappa}$. If we know the continuum function on regular cardinals and if we know the values of $\kappa^{\operatorname{cf}\kappa}$ on singular cardinals, then we can compute κ^λ for all κ and λ (Theorem 20). In case the singular cardinals hypothesis holds, cardinal arithmetic is particularly simple:

Lemma 8.1. *Assume the singular cardinals hypothesis.*

(a) *If* κ *is a singular cardinal, then*

(8.1)

$$2^\kappa = \begin{cases} 2^{<\kappa} & \textit{if the continuum function is eventually constant below } \kappa \\ (2^{<\kappa})^+ & \textit{otherwise} \end{cases}$$

(b) *If* κ, λ *are infinite cardinals, then*

$$\kappa^\lambda = \begin{cases} 2^\lambda & \textit{if } 2^\lambda \geq \kappa \\ \kappa & \textit{if } 2^\lambda < \kappa \textit{ and } \lambda < \operatorname{cf} \kappa \\ \kappa^+ & \textit{if } 2^\lambda < \kappa \textit{ and } \lambda \geq \operatorname{cf} \kappa \end{cases} \tag{8.2}$$

[Note that the size of κ^λ in (8.2) is least possible: It is necessary that $\kappa^\lambda \geq \kappa$ and $\kappa^\lambda \geq 2^\lambda$; moreover, since $\operatorname{cf}(\kappa^\lambda) > \lambda$, it is necessary that $\kappa^\lambda > \kappa$ whenever $\lambda \geq \operatorname{cf} \kappa$.]

Corollary. *If the singular cardinals hypothesis holds, then cardinal exponentiation is completely determined by the continuum function on regular cardinals* (*and the cofinality function*).

[Note that by (b), the singular cardinals hypothesis is equivalent to the assertion that $\kappa^\lambda \leq 2^\lambda \cdot \kappa^+$ for all infinite cardinals κ and λ.]

Proof. (a) If κ is a singular cardinal, then by Lemma 6.5, 2^κ is either λ or $\lambda^{\operatorname{cf} \lambda}$ where $\lambda = 2^{<\kappa}$. The latter occurs if 2^α is not eventually constant below κ. Then $\operatorname{cf} \lambda = \operatorname{cf} \kappa$, and since $2^{\operatorname{cf} \kappa} < 2^{<\kappa} = \lambda$, we have $\lambda^{\operatorname{cf} \lambda} = \lambda^+$ by the singular cardinals hypothesis.

(b) We prove (8.2) by induction on κ, for a fixed λ. Let $\kappa > 2^\lambda$. If κ is a successor cardinal, $\kappa = \nu^+$, then $\nu^\lambda \leq \kappa$ (by the induction hypothesis), and $\kappa^\lambda = (\nu^+)^\lambda = \nu^+ \cdot \nu^\lambda = \kappa$, by the Hausdorff formula.

If κ is a limit cardinal, then $\nu^\lambda < \kappa$ for all $\nu < \kappa$. By Theorem 19, $\kappa^\lambda = \kappa$ if $\lambda < \operatorname{cf} \kappa$, and $\kappa^\lambda = \kappa^{\operatorname{cf} \kappa}$ if $\lambda \geq \operatorname{cf} \kappa$. In the latter case, $2^{\operatorname{cf} \kappa} \leq 2^\lambda < \kappa$, and by the singular cardinals hypothesis, $\kappa^{\operatorname{cf} \kappa} = \kappa^+$. ■

It follows from a recent result of Jensen that the singular cardinals hypothesis holds unless a large cardinal axiom is true (namely, if the singular cardinals hypothesis fails, then $0^\#$ exists).

On the other hand, assuming a large cardinal axiom, one can show that the singular cardinals hypothesis is independent.

In this section we present a theorem of Silver and related results concerning the values 2^κ and $\kappa^{\operatorname{cf} \kappa}$ for singular cardinals of uncountable cofinality.

Theorem 23 (Silver).

(a) *Let κ be a singular cardinal of uncountable cofinality,* $\operatorname{cf}\kappa \geq \aleph_1$. *If the generalized continuum hypothesis holds below κ, then it holds at κ:*

$$\text{if} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1} \text{ for all } \aleph_\alpha < \kappa, \qquad \text{then} \quad 2^\kappa = \kappa^+$$

(b) *If the singular cardinals hypothesis holds for all singular cardinals of cofinality ω, then it holds for all singular cardinals.*

Theorem 24 (Galvin–Hajnal). *Let κ be a strong limit singular cardinal of uncountable cofinality such that $\kappa < \aleph_\kappa$. Then $2^\kappa < \aleph_\kappa$.*

[A better estimate: If $\kappa = \aleph_\eta$, then $2^{\aleph_\eta} < \aleph_\gamma$ where $\gamma = (2^{|\eta|})^+$.]

These two theorems are samples of a number of theorems that can be proved about singular cardinals of uncountable cofinality. We shall give a few more examples in the exercises. The assumption that $\operatorname{cf}\kappa$ is uncountable is necessary. It has been shown that the following is consistent (relative to a large cardinal assumption): $2^{\aleph_n} = \aleph_{n+1}$ for all n, and $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Both theorems can be proved in the same way; in fact, they are special cases of a more general theorem. We first consider a more general version of Theorem 23a:

Lemma 8.2. *Let κ be a singular cardinal, let* $\operatorname{cf}\kappa \geq \omega_1$. *If* $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa \rangle$ *is a normal sequence of cardinals such that* $\lim \kappa_\alpha = \kappa$, *and if the set*

$$\{\alpha < \operatorname{cf}\kappa : 2^{\kappa_\alpha} = \kappa_\alpha^+\}$$

is stationary in $\operatorname{cf}\kappa$, *then* $2^\kappa = \kappa^+$.

Part (b) of Theorem 23 follows from the next lemma, which is a generalization of Lemma 8.2:

Lemma 8.3. *Let κ be a singular cardinal, let* $\operatorname{cf}\kappa \geq \omega_1$, *and assume that* $\lambda^{\operatorname{cf}\kappa} < \kappa$ *for all* $\lambda < \kappa$. *If* $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa \rangle$ *is a normal sequence of cardinals such that* $\lim \kappa_\alpha = \kappa$, *and if the set*

$$\{\alpha < \operatorname{cf}\kappa : \kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+\}$$

is stationary, then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.

Proof of Theorem 23b. We prove by induction on the cofinality of κ that $2^{\operatorname{cf}\kappa} < \kappa$ implies $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. The assumption of the theorem is that this holds for each κ of cofinality ω. Thus let κ be of uncountable cofinality and let $2^{\operatorname{cf}\kappa} < \kappa$. Using the induction hypothesis and formulas (8.2), one easily verifies that $\lambda^{\operatorname{cf}\kappa} < \kappa$ for all $\lambda < \kappa$.

Let $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa \rangle$ be any normal sequence of cardinals such that $\lim \kappa_\alpha = \kappa$. The set

$$S = \{\alpha < \operatorname{cf}\kappa : \operatorname{cf}\kappa_\alpha = \omega \text{ and } 2^{\aleph_0} < \kappa_\alpha\}$$

is clearly stationary in cf κ, and for every $\alpha \in S$, $\kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+$ by the assumption. Hence $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. ■

We shall first prove Theorem 23 via Lemmas 8.2 and 8.3 and then generalize the proof to obtain Theorem 24.

To simplify the notation, we shall consider the special case when

$$\kappa = \aleph_{\omega_1}$$

The general case is proved in exactly the same way.

Let f and g be two functions on ω_1. We say that f and g are *almost disjoint* if there is $\alpha_0 < \omega_1$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \alpha_0$. A family $\mathscr{F}$ of functions on ω_1 is an *almost disjoint family* if any two distinct $f, g \in \mathscr{F}$ are almost disjoint.

Lemma 8.3 follows from

Lemma 8.4. *Assume that* $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ *for all* $\alpha < \omega_1$. *Let* $\mathscr{F}$ *be an almost disjoint family of functions*

$$\mathscr{F} \subset \prod_{\alpha < \omega_1} A_\alpha$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_{\alpha+1}\} \tag{8.3}$$

is stationary. Then $|\mathscr{F}| \leq \aleph_{\omega_1+1}$.

[In the general case, we consider almost disjoint functions on cf κ.]

Proof of Lemmas 8.2 and 8.3 from Lemma 8.4. We assume that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for a stationary set of α's; we want to show that $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$. For every $X \subseteq \aleph_{\omega_1}$, we let $f_X = \langle X_\alpha : \alpha < \omega_1 \rangle$ where $X_\alpha = X \cap \aleph_\alpha$, and let $\mathscr{F} = \{f_X : X \subseteq \aleph_{\omega_1}\}$. If $X \neq Y$, then f_X and f_Y are almost disjoint. Moreover,

$$\mathscr{F} \subseteq \prod_{\alpha < \omega_1} P(\omega_\alpha)$$

Since for a stationary set of α's, $|P(\omega_\alpha)| = \aleph_{\alpha+1}$, we have $|\mathscr{F}| \leq \aleph_{\omega_1+1}$, and so $|P(\aleph_{\omega_1})| = \aleph_{\omega_1+1}$.

To prove Lemma 8.3, we assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ and that $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha} = \aleph_{\alpha+1}$ for a stationary set of α's; we want to show that $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1+1}$. For every $h: \omega_1 \to \aleph_{\omega_1}$, we let $f_h = \langle h_\alpha : \alpha < \omega_1 \rangle$, where dom $h_\alpha = \omega_1$ and

$$h_\alpha(\xi) = \begin{cases} h(\xi) & \text{if } h(\xi) < \aleph_\alpha \\ 0 & \text{otherwise} \end{cases}$$

and let $\mathscr{F} = \{f_h : h \in {}^{\omega_1}\aleph_{\omega_1}\}$. If $h \neq g$, then f_h and f_g are almost disjoint. Moreover,

$$\mathscr{F} \subseteq \prod_{\alpha < \omega_1} {}^{\omega_1}\aleph_\alpha$$

Since for a stationary set of α's, $\aleph_\alpha^{\aleph_1} = \aleph_{\alpha+1}$ (namely for all α such that $\aleph_\alpha > 2^{\aleph_1}$ and $\aleph_\alpha^{\aleph_0} = \aleph_{\alpha+1}$), we have $|\mathscr{F}| \leq \aleph_{\omega_1+1}$, and so $|{}^{\omega_1}\aleph_{\omega_1}| = \aleph_{\omega_1+1}$.
[In the general case of Lemma 8.3 we have to show that

$$\{\alpha < \operatorname{cf} \kappa : \kappa_\alpha^{\operatorname{cf} \kappa} = \kappa_\alpha^+\}$$

is stationary. Note that the set

$$C = \{\alpha : \alpha \text{ is a limit ordinal and } (\forall \lambda < \kappa_\alpha) \quad \lambda^{\operatorname{cf} \kappa} < \kappa_\alpha\}$$

is closed unbounded in $\operatorname{cf} \kappa$; if $\alpha \in C$, then $\operatorname{cf} \kappa_\alpha < \operatorname{cf} \kappa$ and we have $\kappa_\alpha^{\operatorname{cf} \kappa} = \kappa_\alpha^{\operatorname{cf} \alpha}$.] ■

The first step in the proof of Lemma 8.4 is

Lemma 8.5. *Assume that* $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ *for all* $\alpha < \omega_1$. *Let* $\mathscr{F}$ *be an almost disjoint family of functions*

$$\mathscr{F} \subset \prod_{\alpha < \omega_1} A_\alpha$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_\alpha\} \tag{8.4}$$

is stationary. Then $|\mathscr{F}| \leq \aleph_{\omega_1}$.

(The assumption (8.3) is replaced by (8.4) and the bound for $|\mathscr{F}|$ is $\aleph_{\omega_1}$ rather than $\aleph_{\omega_1+1}$.)

Proof. We may as well assume that each A_α is a set of ordinals and that $A_\alpha \subseteq \omega_\alpha$ for all α in some stationary subset of $\aleph_1$. Let

$$S_0 = \{\alpha < \omega_1 : \alpha \text{ is a limit ordinal and } A_\alpha \subseteq \omega_\alpha\}$$

Thus if $f \in \mathscr{F}$, then $f(\alpha) < \omega_\alpha$ for all $\alpha \in S_0$. Given $f \in \mathscr{F}$, we can find for each $\alpha > 0$ in S_0 some $\beta < \alpha$ such that $f(\alpha) < \omega_\beta$; call this $\beta = g(\alpha)$. The function g is regressive on S, and by Fodor's theorem there is a stationary $S \subseteq S_0$ such that g is constant on S. In other words, the function f is bounded on S, by some $\omega_\gamma < \omega_{\omega_1}$.

We assign to each f a pair $(S, f \mid S)$ where $S \subseteq S_0$ is a stationary set and $f \mid S$ is a bounded function. For any S, if $f \mid S = g \mid S$, then $f = g$ since any two distinct functions in $\mathscr{F}$ are almost disjoint. Thus the correspondence

$$f \mapsto (S, f \mid S)$$

is one-to-one.

For a given S, the number of bounded functions on S is at most

$$\sum_{\gamma < \omega_1} \aleph_\gamma^{|S|} = \sup_\gamma \aleph_\gamma^{\aleph_1} = \aleph_{\omega_1}$$

Since $|P(\omega_1)| = 2^{\aleph_1} < \aleph_{\omega_1}$, the number of pairs $(S, f \mid S)$ is at most $\aleph_{\omega_1}$. Hence $|\mathscr{F}| \leq \aleph_{\omega_1}$. ■

Proof of Lemma 8.4. Let $\mathscr{F} \subset \prod_{\alpha<\omega_1} A_\alpha$ be an almost disjoint family of functions satisfying (8.3). We may assume that each A_α is a set of ordinals, and that the set

$$S_0 = \{\alpha < \omega_1 : A_\alpha \subseteq \omega_{\alpha+1}\}$$

is stationary.

First we shall prove that for every $f \in \mathscr{F}$, the set

$$\mathscr{F}_f = \{g \in F : \exists \quad \text{stationary } S \subseteq S_0 (\forall \alpha \in S)\, g(\alpha) \leq f(\alpha)\}$$

has cardinality $\leq \aleph_{\omega_1}$. See Fig. 8.1. If $S \subseteq S_0$ is stationary, let

$$\mathscr{F}_{f,S} = \{g \in \mathscr{F} : (\forall \alpha \in S)\, g(\alpha) \leq f(\alpha)\}$$

The family $\mathscr{F}_{f,S}$ is almost disjoint, and since $\mathscr{F}_{f,S} \subseteq \prod_{\alpha<\omega_1} B_\alpha$ where $B_\alpha = f(\alpha) + 1$ for all $\alpha \in S$, we use the fact that $|f(\alpha) + 1| \leq \aleph_\alpha$ and apply Lemma 8.5 to conclude that $|\mathscr{F}_{f,S}| \leq \aleph_{\omega_1}$. Then we have

$$\mathscr{F}_f = \bigcup \{\mathscr{F}_{f,S} : S \subseteq S_0 \text{ stationary}\}$$

and so $|\mathscr{F}_f| \leq \aleph_{\omega_1}$.

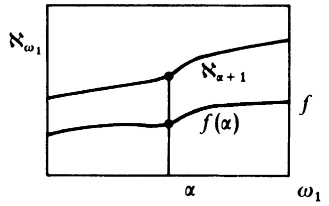

FIGURE 8.1

To complete the proof, we shall construct a sequence

$$(8.5) \qquad \langle f_\xi : \xi < \theta \rangle$$

of functions in $\mathscr{F}$, such that $\theta \leq \aleph_{\omega_1+1}$, and

$$(8.6) \qquad \mathscr{F} = \bigcup \{\mathscr{F}_{f_\xi} : \xi < \theta\}$$

We construct $\langle f_\xi : \xi < \theta \rangle$ by induction. Given f_ν, $\nu < \xi$, we look for $f_\xi \in \mathscr{F}$ such that $f_\xi \notin \mathscr{F}_{f_\nu}$, for all $\nu < \xi$. If there is no such f_ξ, then we let $\theta = \xi$ and have (8.6). If there is such f, then the set

$$\{\alpha \in S_0 : f_\xi(\alpha) \leq f_\nu(\alpha)\}$$

is thin, and so $f_\nu \in \mathscr{F}_{f_\xi}$, for each $\nu < \xi$.

Since $|\mathscr{F}_{f_\xi}| \leq \aleph_{\omega_1}$ and $\mathscr{F}_{f_\xi} \supseteq \{f_\nu : \nu < \xi\}$, it follows that f_ξ exists only if $\xi < \aleph_{\omega_1+1}$. That shows that the sequence (8.5) has length $\theta \leq \aleph_{\omega_1+1}$, and using (8.6), we get $|\mathscr{F}| \leq \aleph_{\omega_1+1}$. ■

This sequence of lemmas completes the proof of Theorem 23. Now we shall generalize Lemma 8.4 in order to derive Theorem 24. As before, we shall consider the special case

$$\kappa = \aleph_{\omega_1}$$

Lemma 8.6. *Assume that* $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ *for all* $\alpha < \omega_1$. *Let* $\mathscr{F}$ *be an almost disjoint family of functions*

$$\mathscr{F} \subset \prod_{\alpha<\omega_1} A_\alpha$$

such that $|A_\alpha| < \aleph_{\omega_1}$ *for all* $\alpha < \omega_1$. *Then* $|\mathscr{F}| < \aleph_\gamma$ *where* $\gamma = (2^{\aleph_1})^+$.

Theorem 24 (for $\kappa = \aleph_{\omega_1}$) follows from Lemma 8.6 in exactly the same way as Lemma 8.2 follows from Lemma 8.4 (see the proof). And the general case of Theorem 24 follows from the general version of Lemma 8.6 that is obtained by making only notational changes (namely, considering almost disjoint functions on cf κ).

To prove Lemma 8.6, we first introduce the following relation among functions $\varphi: \omega_1 \to \omega_1$

$$(8.7) \qquad \varphi < \psi \quad \text{iff} \quad \{\alpha < \omega_1 : \varphi(\alpha) \geq \psi(\alpha)\} \text{ is thin}$$

Since the closed unbounded filter is σ-complete, it follows that there is no infinite descending sequence

$$\varphi_0 > \varphi_1 > \varphi_2 > \cdots$$

Otherwise, the set $\{\alpha < \omega_1 : \varphi_n(\alpha) \leq \varphi_{n+1}(\alpha) \text{ for some } n\}$ is thin and so there is α such that

$$\varphi_0(\alpha) > \varphi_1(\alpha) > \varphi_2(\alpha) > \cdots$$

which is a contradiction.

Hence the relation $\varphi < \psi$ is well-founded and by Theorem 5, we can define the rank $\|\varphi\|$ of φ in this relation (called the *norm* of φ) such that

$$\|\varphi\| = \sup\{\|\psi\| + 1 : \psi < \varphi\}$$

(In fact, the relation $<$ induces a well-founded partial ordering of the reduced power ${}^{\omega_1}\omega_1$ modulo the closed unbounded filter.) Note that $\|\varphi\| = 0$ if and only if $\varphi(\alpha) = 0$ for a stationary set of α's (iff there is no ψ such that $\psi < \varphi$).

Lemma 8.6 follows from

Lemma 8.7. *Assume that* $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ *for all* $\alpha < \omega_1$. *Let* $\varphi: \omega_1 \to \omega_1$ *and let* $\mathscr{F}$ *be an almost disjoint family of functions*

such that

$$\mathscr{F} \subset \prod_{\alpha<\omega_1} A_\alpha$$

$$|A_\alpha| \leq \aleph_{\alpha+\varphi(\alpha)}$$

for every $\alpha < \omega_1$. *Then* $|\mathscr{F}| \leq \aleph_{\omega_1+\|\varphi\|}$.

To prove Lemma 8.6 from Lemma 8.7, we let φ be such that $|A_\alpha| \leq \aleph_{\alpha+\varphi(\alpha)}$. If θ is the length of the well-founded relation $\varphi < \psi$, then certainly $|\theta| \leq |{}^{\omega_1}\omega_1| = 2^{\aleph_1}$ and so $\theta < (2^{\aleph_1})^+$. Hence $\omega_1 + \|\varphi\| < (2^{\aleph_1})^+$ for every φ and Lemma 8.6 follows. Note also that Lemmas 8.4 and 8.5 are special cases of Lemma 8.7, namely when $\|\varphi\|$ is 1 or 0 respectively.

Proof of Lemma 8.7. By induction on $\|\varphi\|$. If $\|\varphi\| = 0$, then $\varphi(\alpha) = 0$ on a stationary set and the statement is precisely Lemma 8.5.

To handle the case $\|\varphi\| > 0$, we first generalize the definition of $\varphi < \psi$. Let $S \subseteq \omega_1$ be a stationary set. We define

$$(8.8) \qquad \varphi <_S \psi \quad \text{iff} \quad \{\alpha \in S : \varphi(\alpha) \geq \psi(\alpha)\} \text{ is thin}$$

The same argument as before shows that $\varphi <_S \psi$ is a well-founded relation and so we define the norm $\|\varphi\|_S$ accordingly. Note that if $S \subseteq T$, then $\|\varphi\|_T \leq \|\varphi\|_S$. In particular, $\|\varphi\| \leq \|\varphi\|_S$, for any stationary S. Moreover,

$$(8.9) \qquad \|\varphi\|_{S \cup T} = \min(\|\varphi\|_S, \|\varphi\|_T)$$

as can easily be verified.

For every $\varphi: \omega_1 \to \omega_1$, we let I_φ be the collection of all thin sets along with those stationary S such that $\|\varphi\| < \|\varphi\|_S$. If S is stationary and X is thin, then $\|\varphi\|_{S \cup X} = \|\varphi\|_S$. This and (8.9) imply that I_φ is a (proper) ideal over ω_1.

If $\|\varphi\|$ is a limit ordinal, then

$$S = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor}\} \in I_\varphi$$

because if $S \notin I_\varphi$, then $\|\varphi\| = \|\varphi\|_S = \|\psi\|_S + 1$, where $\psi(\alpha) = \varphi(\alpha) - 1$ for all $\alpha \in S$. Hence

$$\{\alpha < \omega_1 : \varphi(\alpha) \text{ is a limit}\} \notin I_\varphi$$

Similarly, if $\|\varphi\|$ is a successor ordinal, then

$$\{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor}\} \notin I_\varphi$$

Now we are ready to proceed with the induction.

(a) Let $\|\varphi\|$ be a limit ordinal, $\|\varphi\| > 0$. Let

$$S = \{\alpha < \omega_1 : \varphi(\alpha) > 0 \text{ and is a limit ordinal}\}$$

It follows that $S \notin I_\varphi$.

We may assume that $A_\alpha \subseteq \aleph_{\alpha+\varphi(\alpha)}$ for every α, and so we have $f(\alpha) < \aleph_{\alpha+\varphi(\alpha)}$ for every $f \in \mathscr{F}$. Given $f \in \mathscr{F}$, we can find for each $\alpha \in S$ some $\beta < \varphi(\alpha)$ such that $f(\alpha) < \omega_{\alpha+\beta}$; call this $\beta = \psi(\alpha)$. For $\alpha \notin S$, let $\psi(\alpha) = \varphi(\alpha)$. Since $S \notin I_\varphi$, we have $\|\psi\| \leq \|\psi\|_S < \|\varphi\|_S = \|\varphi\|$. We also have $f \in \mathscr{F}_\psi$, where

$$\mathscr{F}_\psi = \{f \in \mathscr{F} : f(\alpha) < \omega_{\alpha+\psi(\alpha)} \text{ for all } \alpha\}$$

and so

$$\mathscr{F} = \bigcup \{\mathscr{F}_\psi : \|\psi\| < \|\varphi\|\}$$

By the induction hypothesis, $|\mathscr{F}_\psi| \le \aleph_{\omega_1+\|\psi\|} < \aleph_{\omega_1+\|\varphi\|}$ for every ψ such that $\|\psi\| < \|\varphi\|$. Since the number of functions $\psi: \omega_1 \to \omega_1$ is $2^{\aleph_1}$, and $2^{\aleph_1} < \aleph_{\omega_1}$, we have $|\mathscr{F}| \le \aleph_{\omega_1+\|\varphi\|}$.

(b) Let $\|\varphi\|$ be a successor ordinal, $\|\varphi\| = \gamma + 1$. Let

$$S_0 = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor}\}$$

It follows that $S_0 \notin I_\varphi$.

Again, we may assume that $A_\alpha \subseteq \omega_{\alpha+\varphi(\alpha)}$ for each $\alpha < \omega_1$. The proof will follow closely the proof of Lemma 8.4.

First we prove that for every $f \in \mathscr{F}$, the set

$$\mathscr{F}_f = \{g \in \mathscr{F} : \exists S \subseteq S_0,\, S \notin I_\varphi,\, (\forall \alpha \in S)\, g(\alpha) \le f(\alpha)\}$$

has cardinality $\aleph_{\omega_1+\gamma}$. If $S \subseteq S_0$ and $S \notin I_\varphi$, let

$$\mathscr{F}_{f,S} = \{g \in \mathscr{F} : (\forall \alpha \in S)\, g(\alpha) \le f(\alpha)\}$$

Let $\psi: \omega_1 \to \omega_1$ be such that $\psi(\alpha) = \varphi(\alpha) - 1$ for $\alpha \in S$, and $\psi(\alpha) = \varphi(\alpha)$ otherwise. Since $S \notin I_\varphi$, we have $\|\psi\| \le \|\psi\|_S < \|\varphi\|_S = \|\varphi\| = \gamma + 1$ and so $\|\psi\| = \gamma$. Since $\mathscr{F}_{f,S} \subset \prod_{\alpha<\omega_1} B_\alpha$, where $|B_\alpha| \le \aleph_{\alpha+\psi(\alpha)}$ for all α, we use the induction hypothesis to conclude that $|\mathscr{F}_{f,S}| \le \aleph_{\omega_1+\gamma}$. Then it follows that $|\mathscr{F}_f| \le \aleph_{\omega_1+\gamma}$.

To complete the proof, we construct a sequence

$$\langle f_\xi : \xi < \theta \rangle \tag{8.10}$$

such that $\theta \le \aleph_{\omega_1+\gamma+1}$ and

$$\mathscr{F} = \bigcup \{\mathscr{F}_{f_\xi} : \xi < \theta\} \tag{8.11}$$

Given f_ν, $\nu < \xi$, we let $f_\xi \in \mathscr{F}$ (if it exists) be such that $f_\xi \notin \mathscr{F}_{f_\nu}$, for all $\nu < \xi$. Then the set

$$\{\alpha \in S_0 : f_\xi(\alpha) \le f_\nu(\alpha)\}$$

belongs to I_φ, and so $f_\nu \in \mathscr{F}_{f_\xi}$, for each $\nu < \xi$.

Since $|\mathscr{F}_{f_\xi}| \le \aleph_{\omega_1+\gamma}$ and $\mathscr{F}_{f_\xi} \supseteq \{f_\nu : \nu < \xi\}$, it follows that $\xi < \aleph_{\omega_1+\gamma+1}$ if f_ξ exists. Thus the sequence (8.10) has length $\theta \le \aleph_{\omega_1+\gamma+1}$. Then we have

$$\mathscr{F} = \bigcup \{\mathscr{F}_{f_\xi} : \xi < \theta\}$$

and so $|\mathscr{F}| \le \aleph_{\omega_1+\gamma+1}$. ■

In the following exercises, apply Lemma 8.7 (generalized, if necessary).

Exercise 8.1. If $\beta < \omega_1$ and if $2^{\aleph_\alpha} \le \aleph_{\alpha+\beta}$ for a stationary set of α's, then $2^{\aleph_{\omega_1}} \le \aleph_{\omega_1+\beta}$.

[By induction on β: If $\varphi(\alpha) \le \beta$ on a stationary set, then $\|\varphi\| \le \beta$.]

Exercise 8.2. If $\beta < \omega_1$, if $2^{\aleph_1} < \aleph_{\omega_1}$ and if $\aleph_\alpha^{\aleph_0} \le \aleph_{\alpha+\beta}$ for a stationary set of α's, then $\aleph_{\omega_1}^{\aleph_1} \le \aleph_{\omega_1+\beta}$.

Exercise 8.3. If $2^{\aleph_\alpha} \le \aleph_{\alpha+2}$ holds for all cardinals of cofinality ω, then the same holds for all singular cardinals.

Exercise 8.4. If $\aleph_1 \le \operatorname{cf} \aleph_\eta < \aleph_\eta$, if $\beta < \operatorname{cf} \aleph_\eta$, and if $2^{\aleph_\alpha} \le \aleph_{\alpha+\beta}$ for all $\alpha < \eta$, then $2^{\aleph_\eta} \le \aleph_{\eta+\beta}$.

Exercise 8.5. If $2^{\aleph_\alpha} \le \aleph_{\alpha+\alpha+1}$ for a stationary set of $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} \le \aleph_{\omega_1+\omega_1+1}$.
[If $\varphi(\alpha) = \alpha$ for all $\alpha < \omega_1$, then $\|\varphi\| = \omega_1$.]

Exercise 8.6. If $2^{\aleph_{\omega_1+\alpha}} < \aleph_{\omega_1+\alpha+\alpha}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1+\omega_1}} < \aleph_{\omega_1+\omega_1+\omega_1}$.
[Use the sets $A_\alpha = \omega_{\omega_1+\alpha}$.]

Exercise 8.7. If $2^{\aleph_1} < \aleph_{\omega_1}$ and if $\aleph_\alpha^{\aleph_0} \le \aleph_{\alpha+\alpha+1}$ for all $\alpha < \omega_1$, then $\aleph_{\omega_1}^{\aleph_1} \le \aleph_{\omega_1+\omega_1+1}$.

Exercise 8.8. If κ is a strong limit cardinal, $\kappa = \aleph_\eta$, and $\operatorname{cf} \kappa \ge \aleph_1$, then $2^\kappa < \aleph_\gamma$, where $\gamma = (|\eta|^{\operatorname{cf} \kappa})^+$.

Exercise 8.9. If $\aleph_1 \le \operatorname{cf} \kappa < \kappa$ and if $\lambda^{\operatorname{cf} \kappa} < \kappa$ for all $\lambda < \kappa$, then $\kappa^{\operatorname{cf} \kappa} < \aleph_\gamma$, where $\gamma = (|\eta|^{\operatorname{cf} \kappa})^+$.

It is an open problem to find a bound on 2^κ as in Theorem 24 if $\aleph_\kappa = \kappa$ (but see (35.14e)).

9. THE AXIOM OF REGULARITY

Axiom of regularity. Every nonempty set has an $\in$-minimal element.

$$\forall S[S \ne \emptyset \to (\exists x \in S)\, S \cap x = \emptyset]$$

In other words, the relation $\in$ over any family of sets is well-founded. As a consequence, there is no infinite sequence

$$x_0 \ni x_1 \ni x_2 \ni \cdots$$

(Consider the set $S = \{x_0, x_1, x_2, \ldots\}$ and apply the axiom.) In particular, there is no set x such that

$$x \in x$$

and there are no "cycles"

$$x_0 \in x_1 \in \cdots \in x_n \in x_0$$

Thus the axiom of regularity postulates that sets of certain type do not exist. This restriction on the universe of sets is not contradictory (i.e., the axiom is consistent with the other axioms) and is irrelevant for the development of ordinal and cardinal numbers, natural and real numbers, and in fact of all ordinary mathematics. However, it is extremely useful in the metamathematics of set theory, in construction of models. In particular, all sets can be assigned ranks and can be arranged in a cumulative hierarchy.

We recall that a set T is *transitive* if

$$x \in T \quad \text{implies} \quad x \subseteq T$$

Lemma 9.1. *For every set S there exists a transitive set $T \supseteq S$.*

Proof. We define by induction

$$S_0 = S, \qquad S_{n+1} = \bigcup S_n$$

and

$$T = \mathrm{TC}(S) = \bigcup_{n=0}^{\infty} S_n$$

Clearly, T is transitive and $T \supseteq S$. ■

Since every transitive set must satisfy $\bigcup T \subseteq T$, it follows that the set $\mathrm{TC}(S)$ is the smallest transitive $T \supseteq S$, i.e.,

$$\mathrm{TC}(S) = \bigcap \{T : T \supseteq S \text{ and } T \text{ is transitive}\}$$

We call $\mathrm{TC}(S)$ the *transitive closure* of S.

Lemma 9.2. *Every nonempty class* **C** *has an* $\in$*-minimal element.*

Proof. Let $S \in \mathbf{C}$ be arbitrary. If $S \cap \mathbf{C} = \emptyset$, then S is a minimal element of $\mathbf{C}$; if $S \cap \mathbf{C} \neq \emptyset$, we let $X = T \cap \mathbf{C}$ where $T = \mathrm{TC}(S)$. X is a nonempty set and by the axiom of regularity, there is $x \in X$ such that $x \cap X = \emptyset$. It follows that $x \cap \mathbf{C} = \emptyset$: otherwise, if $y \in x$ and $y \in \mathbf{C}$, then $y \in T$ since T is transitive, and so $y \in x \cap T \cap \mathbf{C} = x \cap X$. Hence x is a minimal element of $\mathbf{C}$. ■

The Cumulative Hierarchy of Sets

We define, by transfinite induction,

$$\begin{aligned} V_0 &= \emptyset \\ V_\alpha &= \bigcup_{\beta<\alpha} V_\beta \qquad \text{if } \alpha \text{ is a limit ordinal} \\ V_{\alpha+1} &= P(V_\alpha) \end{aligned}$$

The sets V_α have the following properties (by induction):

(a) Each V_α is transitive.
(b) If $\alpha < \beta$, then $V_\alpha \subset V_\beta$.
(c) $\alpha \subseteq V_\alpha$.

The axiom of regularity implies that every set is in some V_α:

Lemma 9.3. *For every x there is α such that $x \in V_\alpha$:*

$$\bigcup_{\alpha \in Ord} V_\alpha = V \tag{9.1}$$

Proof. Let **C** be the class of all x that are not in any V_α. If **C** is nonempty, then **C** has an $\in$-minimal element x. That is, $x \in \mathbf{C}$, and $z \in \bigcup_\alpha V_\alpha$ for every $z \in x$. Hence $x \subseteq \bigcup_{\alpha \in Ord} V_\alpha$.

By Replacement, there exists an ordinal γ such that $x \subseteq \bigcup_{\alpha<\gamma} V_\alpha$. Hence $x \subseteq V_\gamma$ and so $x \in V_{\gamma+1}$. Thus **C** is empty and we have (9.1). ■

Since every x is in some V_α, we may define the *rank* of x:

$$\text{rank}(x) = \text{least } \alpha \text{ such that } x \in V_{\alpha+1} \tag{9.2}$$

Thus each V_α is the collection of all sets of rank less than α, and we have

(a) If $x \in y$, then $\text{rank}(x) < \text{rank}(y)$.
(b) $\text{rank}(\alpha) = \alpha$.

Exercise 9.1. $\text{rank } x = \sup\{\text{rank}(z) + 1 : z \in x\}$.

Exercise 9.2. The transfinite sequence $\beth_\alpha = |V_{\omega+\alpha}|$ is a normal sequence:

$$\beth_0 = \aleph_0, \qquad \beth_1 = 2^{\aleph_0}, \qquad \beth_2 = 2^{2^{\aleph_0}}, \qquad \ldots$$

and so there are κ such that $\kappa = \beth_\kappa = |V_\kappa|$.

Exercise 9.3. If $\beth_\kappa = \kappa$, then κ is a strong limit and $\aleph_\kappa = \kappa$.

Exercise 9.4. If κ is inaccessible, then $|V_\kappa| = \kappa$.

One of the uses of the rank function is a definition of equivalence classes in case of an equivalence relation over a proper class. The basic trick is the following:

Given a class **C**, let

$$\hat{\mathbf{C}} = \{x \in \mathbf{C} : (\forall z \in \mathbf{C}) \text{ rank } x \leq \text{rank } z\} \tag{9.3}$$

$\hat{\mathbf{C}}$ is always a set, and if **C** is nonempty, then $\hat{\mathbf{C}}$ is nonempty. Moreover, (9.3) can be applied uniformly.

Thus, for example, if $\equiv$ is an equivalence over a proper class **C**, we apply (9.3) to each (ordinary) equivalence class of $\equiv$, or define outright

$$[x] = \{y \in \mathbf{C} : y \equiv x \text{ and } \forall z \in \mathbf{C}\ (z \equiv x \rightarrow \text{rank } y \leq \text{rank } z)\}$$

and

$$\mathbf{C}/\!\equiv\ = \{[x] : x \in \mathbf{C}\}$$

In particular, this trick enables us to define isomorphism types for a given isomorphism. For instance, one can define order-types of linearly ordered sets, or cardinal numbers (even without AC).

We use the same argument to prove the following.

Collection Principle

$$\forall X\ \exists Y (\forall u \in X)\ [\exists v\ \varphi(u, v, p) \rightarrow (\exists v \in Y)\ \varphi(u, v, p)] \tag{9.4}$$

(p is a parameter). The collection principle is a schema of formulas. We can formulate it as follows:

Given a "collection of classes" $\mathbf{C}_u$, $u \in X$ (X is a set), then there is a set Y such that for every $u \in X$,

$$\text{if} \quad \mathbf{C}_u \neq \emptyset, \qquad \text{then } \mathbf{C}_u \cap Y \neq \emptyset$$

To prove (9.4), we let

$$Y = \bigcup_{u \in X} \hat{\mathbf{C}}_u$$

(where $\mathbf{C}_u = \{v : \varphi(u, v, p)\}$); i.e.,

$$v \in Y \leftrightarrow (\exists u \in X)\,[\varphi(u, v, p) \text{ and } \forall z(\varphi(u, z, p) \rightarrow \text{rank } v \leq \text{rank } z)]$$

That Y is a set follows from the replacement schema.

Note that the collection principle implies the replacement schema: Given a function $\mathbf{F}$, then for every set X we let Y be such that

$$(\forall u \in X)\,(\exists v \in Y)\,\mathbf{F}(u) = v$$

Then

$$\mathbf{F} | X = \mathbf{F} \cap (X \times Y)$$

is a set by the separation schema.

$\in$-induction

The method of transfinite induction can be extended to an arbitrary transitive class (instead of *Ord*), for both proof and definition by induction:

Theorem 25 (A) (*Proof by $\in$-induction*). *Let* $\mathbf{T}$ *be a transitive class, let* Φ *be a property. Assume that*

(i) $\Phi(\emptyset)$;
(ii) *if* $x \in \mathbf{T}$ *and* $\Phi(z)$ *holds for every* $z \in x$, *then* $\Phi(x)$.

Then every $x \in \mathbf{T}$ *has the property* Φ.

(B) (*Definition by $\in$-induction*). *Let* $\mathbf{T}$ *be a transitive class and let* $\mathbf{G}$ *be a function (defined for all* x*). Then there is a function* $\mathbf{F}$ *on* $\mathbf{T}$ *such that*

$$\mathbf{F}(x) = \mathbf{G}(\mathbf{F}|x) \tag{9.5}$$

for every $x \in \mathbf{T}$.

Moreover, $\mathbf{F}$ *is the unique function that satisfies* (9.5).

Proof. (A) Let $\mathbf{C}$ be the class of all $x \in \mathbf{T}$ that do not have the property Φ. If $\mathbf{C}$ is nonempty, then it has an $\in$-minimal element x; apply (i) or (ii). ■

(B) We let, for every $x \in \mathbf{T}$,

$\mathbf{F}(x) = y \quad \leftrightarrow \quad$ there exists a function f such that dom(f) is a transitive subset of T and:

(a) $(\forall z \in \text{dom}(f))[f(z) = \mathbf{G}(f | z)]$
(b) $f(x) = y$

That $\mathbf{F}$ is a (unique) function on $\mathbf{T}$ satisfying (9.5) is proved by $\in$-induction. ■

Example. Let $\mathbf{A}$ be a class. There is a unique class $\mathbf{B}$ such that

$$\mathbf{B} = \{x \in \mathbf{A} : x \subseteq \mathbf{B}\} \tag{9.6}$$

Proof. Let

$$\mathbf{F}(x) = \begin{cases} 1 & \text{if } x \in \mathbf{A} \text{ and } \mathbf{F}(z) = 1 \text{ for all } z \in x \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{B} = \{x : \mathbf{F}(x) = 1\}$. The uniqueness of $\mathbf{B}$ is proved by $\in$-induction. ■

We denote the class in (9.6) by

$$\mathbf{B} = \mathscr{H}\mathbf{A}$$

and say that each $x \in \mathbf{B}$ is *hereditarily* in $\mathbf{A}$.

Exercise 9.5. $\mathscr{H}\mathbf{A} = \{x \in \mathbf{A} : \mathrm{TC}(x) \subseteq \mathbf{A}\}$.

Exercise 9.6. $\mathscr{H}\mathbf{A}$ is the largest transitive class $\mathbf{B}$ such that $\mathbf{B} \subseteq \mathbf{A}$.

One consequence of the axiom of regularity is that the universe does not admit nontrivial $\in$-automorphisms. More generally:

Theorem 26 (*Isomorphism Theorem*). *No two distinct transitive classes are isomorphic. More specifically: Let $\mathbf{T}_1$, $\mathbf{T}_2$ be transitive classes and let π be an $\in$-isomorphism of T_1 onto T_2; i.e., π is one-to-one, and*

$$u \in v \quad \leftrightarrow \quad \pi u \in \pi v \tag{9.7}$$

Then $\mathbf{T}_1 = \mathbf{T}_2$ and $\pi u = u$ for every $u \in \mathbf{T}_1$.

Proof. We show, by $\in$-induction, that $\pi x = x$ for every $x \in \mathbf{T}_1$. Assume that $\pi z = z$ for each $z \in x$ and let $y = \pi x$.

We have $x \subseteq y$ because if $z \in x$, then $z = \pi z \in \pi x = y$.

We also have $y \subseteq x$: Let $t \in y$. Since $y \subseteq \mathbf{T}_2$, there is $z \in \mathbf{T}_1$ such that $\pi z = t$. Since $\pi z \in y$, we have $z \in x$, and so $t = \pi z = z$. Thus $t \in x$.

Therefore $\pi x = x$ for all $x \in \mathbf{T}_1$, and $\mathbf{T}_2 = \mathbf{T}_1$. ■

Well-Founded Relations

The notion of well-founded relations that was introduced in Section 2 can be generalized to relations over proper classes, and one can extend the method of induction to well-founded relations.

Let $\mathbf{E}$ be a binary relation over a class $\mathbf{P}$. For each $x \in \mathbf{P}$, we let

$$\mathrm{ext}_{\mathbf{E}}(x) = \{z \in \mathbf{P} : z \mathrel{\mathbf{E}} x\}$$

the *extension* of x.

A relation **E** over **P** is *well-founded*, if:

(9.8) (i) every nonempty set $X \subseteq \mathbf{P}$ has an **E**-minimal element;
(ii) $\mathrm{ext}_{\mathbf{E}}(x)$ is a set, for every $x \in \mathbf{P}$.

(Condition (ii) is vacuous if **P** is a set.) Note that the relation $\in$ is well-founded over any class, by the axiom of regularity.

Lemma 9.4. *If* **E** *is a well-founded relation over* **P**, *then every nonempty* $\mathbf{C} \subseteq \mathbf{P}$ *has an* **E**-*minimal element.*

Proof. We follow the proof of Lemma 9.2; we are looking for $x \in \mathbf{C}$ such that $\mathrm{ext}_{\mathbf{E}}(x) \cap \mathbf{C} = \emptyset$. Let $S \in \mathbf{C}$ be arbitrary and assume that $\mathrm{ext}(S) \cap \mathbf{C} \neq \emptyset$. We let $X = T \cap \mathbf{C}$ where

$$T = \bigcup_{n=0}^{\infty} S_n$$

and

$$S_0 = \mathrm{ext}\, S, \qquad S_{n+1} = \bigcup \{\mathrm{ext}(z) : z \in S_n\}$$

(See Fig. 9.1.) As in Lemma 9.2, it follows that an **E**-minimal element x of X is **E**-minimal in **C**. ■

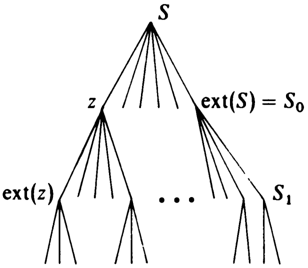

FIGURE 9.1

Theorem 25′ (*Well-founded Induction*). *Let* **E** *be a well-founded relation over* **P**.

(A) *Let* Φ *be a property. Assume that:*

(i) *every* **E**-*minimal element* x *has the property* Φ;
(ii) *if* $x \in \mathbf{P}$ *and* $\Phi(z)$ *holds for every* z *such that* $z \mathrel{\mathbf{E}} x$, *then* $\Phi(x)$.

Then every $x \in \mathbf{P}$ *has the property* Φ.

(B) *Let* **G** *be a function* (*on* $V \times V$). *Then there is a* (*unique*) *function* **F** *on* **P** *such that*

$$\mathbf{F}(x) = \mathbf{G}(x, \mathbf{F} \mid \mathrm{ext}(x)) \tag{9.9}$$

for every $x \in \mathbf{P}$.

Proof. A modification of the proof of Theorem 25. ■

(Note that if **F** ~~satisfies~~ (9.9), then $\mathbf{F}(x) = \mathbf{F}(y)$ whenever $\mathrm{ext}(x) = \mathrm{ext}(y)$; in particular, $\mathbf{F}(x)$ is the same for all minimal elements.)

Examples.

1 (*The rank function*). We define, by induction, for all $x \in \mathbf{P}$:

$$\rho(x) = \sup\{\rho(z) + 1 : z \mathbin{\mathbf{E}} x\}$$

(compare with (2.10)). The range of ρ is either an ordinal or the class *Ord*. For all $x, y \in \mathbf{P}$,

$$x \mathbin{\mathbf{E}} y \rightarrow \rho(x) < \rho(y)$$

2 (*The transitive collapse*). By induction, let

$$\pi(x) = \pi[\mathrm{ext}(x)] = \{\pi(z) : z \mathbin{\mathbf{E}} x\}$$

for every $x \in \mathbf{P}$. The range of π is a transitive class, and for all $x, y \in \mathbf{P}$,

$$x \mathbin{\mathbf{E}} y \quad \rightarrow \quad \pi(x) \in \pi(y)$$

APPENDIX: BERNAYS–GÖDEL AXIOMATIC SET THEORY

There is an alternative axiomatization of set theory. We consider two types of objects: *sets* and *classes*.

Axioms:

A. 1. Extensionality (for classes): $\forall u(u \in \mathbf{X} \leftrightarrow u \in \mathbf{Y}) \rightarrow \mathbf{X} = \mathbf{Y}$
   2. Every set is a class.
   3. If $\mathbf{X} \in \mathbf{Y}$, then $\mathbf{X}$ is a set.
   4. Pairing: For any sets x and y there exists a set $\{x, y\}$.

B. Comprehension

$$\forall \mathbf{X}_1, \ldots, \forall \mathbf{X}_n \; \exists \mathbf{Y} \; \mathbf{Y} = \{x : \varphi(x, \mathbf{X}_1, \ldots, \mathbf{X}_n)\}$$

where φ is a formula in which only set variables are quantified (no class variables).

C. 1. Infinity: There is an infinite set.

$$\left.\begin{array}{l}\text{2. Union}\\ \text{3. Power set}\end{array}\right\} : \text{For every set } X \text{ there exists a set } \left\{\begin{array}{l}\bigcup X.\\ P(X).\end{array}\right.$$

   4. Replacement: If a class $\mathbf{F}$ is a function and X is a set, then $\mathbf{F}[X]$ is a set.

D. Regularity.

E. Choice: There is a function $\mathbf{F}$ such that $\mathbf{F}(X) \in X$ for every nonempty set X.

Let BG denote the axiomatic theory A–D, and let BGC denote BG + choice.

Exercise. If a set-theoretical statement is provable in ZF (ZFC), then it is provable in BG (BGC).

On the other hand, a theorem of Shoenfield (using proof-theoretic methods) states that if a sentence involving only set variables is provable in BG, then it is provable in ZF. This result can be extended to BGC/ZFC using the method of forcing; cf. the remark at the end of Section 20.

CHAPTER 2

TRANSITIVE MODELS OF SET THEORY

10. MODELS OF SET THEORY

Let $\mathbf{E}$ be a binary relation over a class $\mathbf{M}$. We call $\mathfrak{M} = (\mathbf{M}, \mathbf{E})$* a *model* (of the language of set theory), and define, for every formula $\varphi(x_1, \ldots, x_n)$,

$$(10.1) \qquad \mathfrak{M} \models \varphi(a_1, \ldots, a_n) \qquad (a_1, \ldots, a_n \in \mathbf{M})$$

($\mathfrak{M}$ *satisfies* $\varphi(a_1, \ldots, a_n)$, φ *holds* in $\mathfrak{M}$), as follows:

(a) If φ is an atomic formula:

$$\mathfrak{M} \models a \in b \quad \text{if} \quad a\,\mathbf{E}\,b$$

$$\mathfrak{M} \models a = b \quad \text{if} \quad a = b$$

(b) If φ is a negation, conjunction, etc.:

$$\mathfrak{M} \models \neg\psi(a_1, \ldots, a_n) \quad \text{if not } \mathfrak{M} \models \psi(a_1, \ldots, a_n)$$

$$\mathfrak{M} \models \psi \wedge \chi(a_1, \ldots, a_n) \quad \text{if } \mathfrak{M} \models \psi(a_1, \ldots, a_n) \text{ and } \mathfrak{M} \models \chi(a_1, \ldots, a_n)$$

$$\mathfrak{M} \models \psi \vee \chi(a_1, \ldots, a_n) \quad \text{if } \mathfrak{M} \models \psi(a_1, \ldots, a_n) \text{ or } \mathfrak{M} \models \chi(a_1, \ldots, a_n)$$

$$\mathfrak{M} \models \psi \rightarrow \chi(a_1, \ldots, a_n) \quad \text{if } \mathfrak{M} \models \psi(a_1, \ldots, a_n) \text{ implies } \mathfrak{M} \models \chi(a_1, \ldots, a_n)$$

$$\mathfrak{M} \models \psi \leftrightarrow \chi(a_1, \ldots, a_n) \quad \text{if } \mathfrak{M} \models \psi(a_1, \ldots, a_n) \text{ iff } \mathfrak{M} \models \chi(a_1, \ldots, a_n)$$

(c) If φ is $\exists x\psi$ or $\forall x\psi$

$$\mathfrak{M} \models \exists x\psi(x, a_1, \ldots, a_n) \quad \text{if} \quad (\exists a \in \mathbf{M})\, \mathfrak{M} \models \psi(a, a_1, \ldots, a_n)$$

$$\mathfrak{M} \models \forall x\psi(x, a_1, \ldots, a_n) \quad \text{if} \quad (\forall a \in \mathbf{M})\, \mathfrak{M} \models \psi(a, a_1, \ldots, a_n)$$

The above definition ("by induction on complexity of φ") gives us instructions on how to "decode" (10.1) for any particular formula φ. For instance,

$$\mathfrak{M} \models \forall u(u \in X \leftrightarrow u \in Y) \rightarrow X = Y$$

---

* The sophisticated reader will not object to our use of a pair of classes.

is the formula

$$\forall u \in \mathbf{M}(u \mathrel{\mathbf{E}} X \leftrightarrow u \mathrel{\mathbf{E}} Y) \quad \rightarrow \quad X = Y$$

If σ is a sentence and $\mathfrak{M} \vDash \sigma$, then we say that $\mathfrak{M}$ is a *model of* σ. A *model of* ZF (ZFC) is an $\mathfrak{M}$ that satisfies every axiom of ZF (ZFC).

Relative Consistency

An axiomatic theory is *consistent* if the axioms are not contradictory—if there is no proof of a contradiction in the theory. Gödel's *second incompleteness theorem* states that no axiomatic theory at least as powerful as the axiomatic arithmetic can be proven consistent by methods that are formalizable in that theory itself. In particular, it is impossible to show consistency of ZF (or related theories) by means limited to ZF alone.

On the other hand, once we assume that ZF (or ZFC) is consistent, we may ask whether the theory remains consistent if we add an additional axiom A.

Let T be a mathematical theory (in our case, T is either ZF or ZFC), and let A be an additional axiom. We say that T + A is *consistent relative to* T (or that A is *consistent with* T) if the following implication holds:

if T is consistent, then so is T + A

If both A and $\neg$A are consistent with T, we say that A is *independent* of T.

The question whether A is consistent with T is equivalent to the question whether the negation of A is provable in T (provided T is consistent); this is because T + A is consistent if and only if $\neg$A is not provable in T.

The way to show that an axiom A is consistent with ZF (ZFC) is to use models. For assume that we have a model $\mathfrak{M}$ of ZF such that $\mathfrak{M} \vDash$ A. Then A is consistent with ZF: If it were not, then $\neg$A would be provable in ZF, and since $\mathfrak{M}$ is a model of ZF, $\mathfrak{M}$ would satisfy $\neg$A. (Here we tacitly use the fact that if $\mathfrak{M}$ is a model of a theory T and σ is a sentence provable in T, then $\mathfrak{M}$ is a model of σ.) However, $\mathfrak{M} \vDash \neg$A contradicts $\mathfrak{M} \vDash$ A since $\mathfrak{M} \vDash \neg$A iff $\mathfrak{M} \nvDash$ A.

The models introduced here are closely related to models as defined in model theory. They are in a sense more general since they are classes rather than just sets, but also more special since proper classes cannot be manipulated as freely as sets.

There is another remark that is in order. By Gödel's second incompleteness theorem, one cannot show consistency of ZFC within ZFC. Thus if Con(ZFC) is the statement that ZFC is consistent, we have

Con(ZFC) is not provable in ZFC

By Gödel's completeness theorem, an axiomatic theory is consistent if and only if it has a model (in the sense of model theory—thus a set). Therefore the sentence

$$\exists \mathfrak{A}(\mathfrak{A} \text{ is a model of ZFC})$$

is not provable in ZFC.

When dealing with models, we use standard terminology of model theory. Also, some results to be presented are related to standard theorems of model theory. For this reason, we insert a brief review of basic notions and facts from model theory.

Review of Model Theory

A *language* is a set of symbols: relation symbols, function symbols, and constant symbols:

$$\mathscr{L} = \{P, \ldots, F, \ldots, c, \ldots\}$$

Each P is assumed to be an n-placed relation for some integer $n \geq 1$; each F is an m-place function symbol for some $m \geq 1$.

Terms and *formulas* of a language $\mathscr{L}$ are certain finite sequences of symbols of $\mathscr{L}$, and of *logical symbols* (identity symbol, parentheses, variables, connectives, and quantifiers). The set of all terms and the set of all formulas are defined by induction. If the language is countable (i.e., if $|\mathscr{L}| \leq \aleph_0$), then we may identify the symbols of $\mathscr{L}$, as well as the logical symbols, with some hereditarily finite sets (elements of V_ω); then formulas of $\mathscr{L}$ are also hereditarily finite (*arithmetization* of language).

A *model* for a given language $\mathscr{L}$ is a pair $\mathfrak{A} = (A, \mathscr{I})$, where A is the universe of $\mathfrak{A}$ and $\mathscr{I}$ is the *interpretation* function which maps the symbols of $\mathscr{L}$ to appropriate relations, functions, and constants in A. A model for $\mathscr{L}$ is usually written in displayed form as

$$\mathfrak{A} = \langle A, P^{\mathfrak{A}}, \ldots, F^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots\rangle$$

or just

$$\mathfrak{A} = \langle A, P, \ldots, F, \ldots, c, \ldots\rangle$$

By induction on length of terms and formulas one defines the *value* of a term

$$t^{\mathfrak{A}}[a_1, \ldots, a_n]$$

and *satisfaction*

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$$

(t is a term, φ is a formula, $n \in N$, and $\langle a_1, \ldots, a_n\rangle$ is an n-termed sequence in A).

The relation of these models to the models defined in (10.1) is the following: Let $\mathscr{L}$ be a language consisting of one binary predicate symbol $\in$:

$$\mathscr{L} = \{\in\}$$

and let $Form \subset V_\omega$ be the set of all formulas of $\mathscr{L}$. As with any actual (metamathematical) natural number we can associate the corresponding element of N, we can similarly associate with any given formula of set theory the corresponding element of the set *Form*. To make the distinction, if φ is a formula, let $\ulcorner\varphi\urcorner$ denote the corresponding element of *Form*.

If M is a set and E is a binary relation over M and if $a_1, \ldots, a_n$ are elements of M, then

$$(10.2) \qquad \mathfrak{M} \models \varphi(a_1, \ldots, a_n) \quad \text{iff} \quad \langle M, E\rangle \models \ulcorner\varphi\urcorner[a_1, \ldots, a_n]$$

as can easily be verified. Thus in the case when M is a set and φ a particular (metamathematical) formula, we shall not make a distinction between the two meanings of the symbol $\models$. We note however that the left-hand side of (10.2) is *not* defined for $\varphi \in Form$, and the right-hand side is *not* defined if M is a proper class.

To proceed with our review of model theory, let us recall Gödel's *completeness theorem*: A set Σ of sentences of $\mathscr{L}$ is consistent if and only if it has a model. Construction of a model of Σ requires in general the axiom of choice; in fact, the ultrafilter theorem (Theorem 21) suffices.

The *compactness theorem* states that a set of sentences Σ has a model if and only if every finite $S \subseteq \Sigma$ has a model. The *Löwenheim–Skolem theorem* says that if Σ has a model, then it has a model $\mathfrak{A}$ such that $|A| \le \max(|\Sigma|, \aleph_0)$. In particular, if $\mathscr{L}$ is countable, then every consistent set of sentences has an at most countable model. These theorems again use the axiom of choice.

Two models $\mathfrak{A} = \langle A, P, \ldots, F, \ldots, c, \ldots\rangle$ and $\mathfrak{A}' = \langle A', P', \ldots, F', \ldots, c', \ldots\rangle$ are *isomorphic* if there is an *isomorphism* between $\mathfrak{A}$ and $\mathfrak{A}'$, that is a one-to-one function f of A onto A' such that:

(i) $P(x_1, \ldots, x_n)$ iff $P'(f(x_1), \ldots, f(x_n))$,
(ii) $f(F(x_1, \ldots, x_n)) = F'(f(x_1), \ldots, f(x_n))$,
(iii) $f(c) = c'$,

for all relations, functions, and constants of $\mathfrak{A}$. If f is an isomorphism, then $f(t^{\mathfrak{A}}[a_1, \ldots, a_n]) = t^{\mathfrak{A}'}[f(a_1), \ldots, f(a_n)]$ for each term, and

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad \mathfrak{A}' \models \varphi[f(a_1), \ldots, f(a_n)]$$

A *submodel* of $\mathfrak{A}$ is a subset $B \subseteq A$ endowed with the relations $P^{\mathfrak{A}} \cap B^n, \ldots$, functions $F^{\mathfrak{A}} | B^m, \ldots$, and constants $c^{\mathfrak{A}}, \ldots$; B has to be such that all $c^{\mathfrak{A}}$ belong to B, and that B is closed under all $F^{\mathfrak{A}}$ (if $\langle x_1, \ldots, x_m\rangle \in B^m$, then $F^{\mathfrak{A}}(x_1, \ldots, x_m) \in B$).

An *embedding* of $\mathfrak{B}$ into $\mathfrak{A}$ is an isomorphism between $\mathfrak{B}$ and a submodel $\mathfrak{B}' \subseteq \mathfrak{A}$.

A submodel $\mathfrak{B} \subseteq \mathfrak{A}$ is an *elementary submodel*

$$\mathfrak{B} \prec \mathfrak{A}$$

if for every formula φ, and every $a_1, \ldots, a_n \in B$,

$$(10.3) \qquad \mathfrak{B} \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad \mathfrak{A} \models \varphi[a_1, \ldots, a_n]$$

The key lemma in construction of elementary submodels is: A subset $B \subseteq A$ forms an elementary submodel of $\mathfrak{A}$ if and only if for every formula φ, and every $a_1, \ldots, a_n \in B$,

$$\text{(10.4)} \qquad \text{if } \exists a \in A \text{ such that } \mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n], \text{ then } \exists a \in B \text{ such that } \mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n]$$

A function $h: {}^nA \to A$ is a *Skolem function* for φ if*

$$\exists a \in A\ \mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n] \quad \text{implies} \quad \mathfrak{A} \vDash \varphi[h(a_1, \ldots, a_n), a_1, \ldots, a_n]$$

for every $a_1, \ldots, a_n$. Using the axiom of choice, one can construct a Skolem function for every φ. If a subset $B \subseteq A$ is closed under (some) Skolem functions for all formulas, then B satisfies (10.4) and hence forms an elementary submodel of $\mathfrak{A}$.

An *elementary embedding* is an embedding whose range is an elementary submodel.

A set $X \subseteq A$ is *definable over* $\mathfrak{A}$ if there is a formula φ and $a_1, \ldots, a_n \in A$ such that

$$X = \{x \in A : \mathfrak{A} \vDash \varphi[x, a_1, \ldots, a_n]\}$$

(X is *definable in* $\mathfrak{A}$ *from* $a_1, \ldots, a_n$). If φ is a formula of x only, without parameters $a_1, \ldots, a_n$, then X is *definable* in $\mathfrak{A}$. An element $a \in A$ is *definable* (*from* $a_1, \ldots, a_n$) if the set $\{a\} \subseteq A$ is definable (from $a_1, \ldots, a_n$).

$\in$-Models and Transitive Models

A model $\mathfrak{M} = (\mathbf{M}, \mathbf{E})$ is an $\in$-model if the relation $\mathbf{E}$ is just the $\in$-relation, i.e., if $\mathbf{E} = \in \cap \mathbf{M}^2$. If $\mathfrak{M}$ is an $\in$-model, then we call the class $\mathbf{M}$ a model and write

$$(\mathbf{M}, \in) \vDash \varphi \qquad \text{or just} \qquad \mathbf{M} \vDash \varphi$$

Note that the formula $\mathbf{M} \vDash \varphi$ is obtained from φ by replacing all $\exists x$ and $\forall x$ by $\exists x \in \mathbf{M}$ and $\forall x \in \mathbf{M}$. In particular, if φ is quantifier-free, $\mathbf{M} \vDash \varphi$ is the same as φ.

If $\mathbf{M}$ is a transitive class, then the $\in$-model $\mathbf{M}$ is called a *transitive model.* An important property of transitive models is the following:

Call $\exists u \in X$ and $\forall u \in X$ *restricted quantifiers*, and say that a formula φ is a *restricted formula* if it has no other quantifiers than restricted quantifiers.

Lemma 10.1. *If* $\mathbf{M}$ *is a transitive class and* φ *a restricted formula, then for all* $x_1, \ldots, x_n \in \mathbf{M}$,

$$\text{(10.5)} \qquad \mathbf{M} \vDash \varphi(x_1, \ldots, x_n) \quad \textit{iff} \quad \varphi(x_1, \ldots, x_n)$$

---

* If $n = 0$ then a Skolem function for φ is an element $h \in A$ such that $\exists a \in A\ \mathfrak{A} \vDash \varphi[a]$ implies $\mathfrak{A} \vDash \varphi[h]$.

Proof. If φ is an atomic formula, then (10.5) holds. If (10.5) holds for φ and ψ, then it holds for $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$.

Let φ be the formula $(\exists u \in x)\psi(u, x, \ldots)$ and assume that (10.5) is true for ψ. We show that (10.5) is true for φ (the proof for $\forall u \in x$ is similar).

If $\mathbf{M} \models \varphi$, then

$$\mathbf{M} \models \exists u(u \in x \ \wedge \ \psi)$$

i.e.,

$$(\exists u \in \mathbf{M})\,(u \in x \ \wedge \ \mathbf{M} \models \psi)$$

Since $\mathbf{M} \models \psi$ if and only if ψ, we have

$$(\exists u \in x)\,\psi$$

Conversely, if $(\exists u \in x)\,\psi$, then since $\mathbf{M}$ is transitive, u belongs to $\mathbf{M}$, and since $\psi(u, x, \ldots) \leftrightarrow \mathbf{M} \models \psi(u, x, \ldots)$, we have

$$\exists u(u \in \mathbf{M} \ \wedge \ u \in x \ \wedge \ \mathbf{M} \models \psi)$$

and so

$$\mathbf{M} \models (\exists u \in x)\psi \quad \blacksquare$$

Example. Every transitive model satisfies the axiom of extensionality:

Proof. $[(\forall u \in X)\,(u \in Y) \wedge (\forall u \in Y)\,(u \in X)] \to X = Y$ is a restricted formula.

Consistency of the Axiom of Regularity

As the first application of models we show that the axiom of regularity is consistent with ZF minus Regularity (axioms I–VII). We recall the construction of the sets V_α. The construction is performed without using Regularity; the axiom of regularity is used to show that $V = \bigcup_{Ord} V_\alpha$.

We shall show, using only axioms I–VII, that the class $\mathbf{P} = \bigcup_{\alpha \in Ord} V_\alpha$ is a model of ZF. This proves consistency of the axiom of regularity relative to the theory ZF minus Regularity. We shall make use of the fact that if X is a subset of $\mathbf{P}$, then $X \in \mathbf{P}$.

I. $\mathbf{P} \models$ Extensionality

This is so because $\mathbf{P}$ is transitive.

II. $\mathbf{P} \models$ Pairing

$$\mathbf{P} \models \forall a\, \forall b\, \exists c[a \in c \wedge b \in c \wedge (\forall x \in c)\,(x = a \vee x = b)] \tag{10.6}$$

Given $a, b \in \mathbf{P}$, let $c = \{a, b\}$. Clearly, $c \in \mathbf{P}$ and $P \models [\cdots]$ since the formula $[\cdots]$ is restricted and holds (in the universe; it says: $c = \{a, b\}$).

III. $\mathbf{P} \models$ Separation

$$\mathbf{P} \models \forall X\, \forall p\, \exists Y\, \forall u(u \in Y \leftrightarrow u \in X \wedge \varphi(u, p))$$

Given $X, p \in \mathbf{P}$, we let

$$Y = \{u \in X : \mathbf{P} \models \varphi(u, p)\}$$

Since $Y \subseteq X$ and $X \in \mathbf{P}$, we have $Y \in \mathbf{P}$, and

$$\mathbf{P} \models \forall u(u \in Y \leftrightarrow u \in X \wedge \varphi(u, p))$$

IV. $\mathbf{P} \models$ Union

$$(10.7) \quad \mathbf{P} \models \forall X\ \exists Y[(\forall z \in X)\,(\forall u \in z)\, u \in Y \wedge (\forall u \in Y)\,(\exists z \in X)\, u \in z]$$

Given $X \in \mathbf{P}$, we let $Y = \bigcup X$. It is easy to see that $Y \in \mathbf{P}$ (if $X \subseteq V_\alpha$, then $\bigcup X \subseteq V_\alpha$). Moreover, $\mathbf{P} \models [\cdots]$ since $[\cdots]$ is a restricted formula, saying $Y = \bigcup X$.

V. $\mathbf{P} \models$ Power set

$$(10.8) \qquad \mathbf{P} \models \forall X\ \exists Y\ \forall u[u \in Y \leftrightarrow u \subseteq X]$$

Given $X \in \mathbf{P}$, we let $Y = P(X)$. Again, $Y \in \mathbf{P}$. However, the formula $\forall u[\cdots]$ is not restricted; but we argue as follows: $[u \in Y \leftrightarrow (\forall z \in u) z \in X]$ is restricted, and for every $u \in \mathbf{P}$, $\mathbf{P} \models [\cdots]$ since $[\cdots]$ is true for all u.

VI. $\mathbf{P} \models$ Infinity

$$(10.9) \qquad \mathbf{P} \models \exists S[\varnothing \in S\ \wedge\ (\forall x \in S) x \cup \{x\} \in S]$$

The formula in (10.9) contains defined notions, $\{\ \}$, $\cup$, and $\varnothing$; and strictly speaking, we should first eliminate these symbols and use a formula in which they are replaced by their definitions, using only $\in$ and $=$. However, we have already proved that both pairing and union are the same in the universe as in $\mathbf{P}$, and similarly one shows that $X \in \mathbf{P}$ is empty iff $\mathbf{P} \models X$ is empty. In other words,

$$\{a, b\}^{\mathbf{P}} = \{a, b\}, \qquad \textstyle\bigcup^{\mathbf{P}} X = X, \qquad \varnothing^{\mathbf{P}} = \varnothing$$

where $\{a, b\}^{\mathbf{P}}$, $\bigcup^{\mathbf{P}}$, and $\varnothing^{\mathbf{P}}$ denote pairing, union, and empty set in the model $\mathbf{P}$.

Since $\omega \in \mathbf{P}$, we let $S = \omega$ and easily verify that (10.9) holds for this S.

VII. $\mathbf{P} \models$ Replacement

$$(10.10) \qquad \mathbf{P} \models \forall x\ \forall y\ \forall z[\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y = z] \\ \rightarrow \forall X\ \exists Y\ \forall y[y \in Y \leftrightarrow (\exists x \in X)\ \varphi(x, y, p)]$$

Given $p \in \mathbf{P}$, assume that $\mathbf{P} \models \forall x\ \forall y\ \forall z[\cdots]$. Thus

$$\mathbf{F} = \{(x, y) \in \mathbf{P} : \mathbf{P} \models \varphi(x, y, p)\}$$

is a function, and we let $Y = \mathbf{F}[X]$. Since $Y \subset \mathbf{P}$, we have $Y \in \mathbf{P}$, and one verifies that for every $y \in \mathbf{P}$,

$$\mathbf{P} \models y \in Y \leftrightarrow (\exists x \in X)\ \varphi(x, y, p)$$

VIII. $\mathbf{P} \models$ Regularity

$$(10.11) \qquad \mathbf{P} \models \forall S[S \neq \varnothing \rightarrow (\exists x \in S)\ S \cap x = \varnothing]$$

If $S \in \mathbf{P}$ is nonempty, then let $x \in S$ be of least rank; then $S \cap x = \varnothing$. Hence $[\cdots]$ is true for any S; moreover, $(S \cap x)^{\mathbf{P}} = S \cap x$, and $[\cdots]$ has only a restricted quantifier. Thus $\mathbf{P} \models [\cdots]$.

Exercise 10.1. If the axiom of choice holds, then $\mathbf{P} \models$ axiom of choice.

Exercise 10.2. If α is a limit ordinal, then V_α is a model of: Extensionality, Pairing, Separation, Union, Power set, and Regularity. If AC holds, then V_α is a model of AC.

Exercise 10.3. If $\alpha > \omega$, then V_α is a model of Infinity.

Exercise 10.4. V_ω, the set of all hereditarily finite sets, is a model of ZFC minus Infinity.

Inaccessibility of Inaccessible Cardinals

Theorem 27. *Existence of inaccessible cardinals is not provable in* ZFC. *Moreover, it cannot be shown that the existence of inaccessible cardinals is consistent with* ZFC.

We shall prove the first assertion and invoke Gödel's second incompleteness theorem to obtain the second part.

First we prove (in ZFC):

Lemma 10.2. *If κ is an inaccessible cardinal, then V_κ is a model of* ZFC.

Proof. The proof of all axioms of ZFC except Replacement is as in the proof of consistency of the axiom of regularity (see Exercises 10.1–10.3). To show that $V_\kappa \models$ Replacement, it is enough to show:

(10.12) If F is a function from some $X \in V_\kappa$ into V_κ, then $F \in V_\kappa$.

Since κ is inaccessible, we have $|V_\kappa| = \kappa$ and $|X| < \kappa$ for every $X \in V_\kappa$. If F is a function from $X \in V_\kappa$ into V_κ, then $|F[X]| \le |X| < \kappa$ and (since κ is regular) $F[X] \subseteq V_\alpha$ for some $\alpha < \kappa$. It follows that $F \in V_\kappa$. ■

Proof of Theorem 27. If κ is an inaccessible cardinal, then not only is V_κ a model of ZFC, but in addition

$V_\kappa \models \alpha$ is an ordinal iff α is an ordinal.
$V_\kappa \models \alpha$ is a cardinal iff α is a cardinal.
$V_\kappa \models \alpha$ is a regular cardinal iff α is a regular cardinal.
$V_\kappa \models \alpha$ is an inaccessible cardinal iff α is an inaccessible cardinal.

(We leave the details to the reader.)

In particular, if κ is the least inaccessible cardinal, then

$$V_\kappa \models \text{there is no inaccessible cardinal}$$

Thus we have a model of ZFC + "there is no inaccessible cardinal" (if there is no inaccessible cardinal, we take the universe as the model). Hence it cannot be proved in ZFC that inaccessible cardinals exist.

To prove the second part, assume that it can be shown that the existence of inaccessible cardinals is consistent with ZFC; in other words, we assume

if ZFC is consistent, then so is ZFC + I

where I is the statement "there is an inaccessible cardinal."

We naturally assume that ZFC is consistent. Since I is consistent with ZFC, we conclude that ZFC + I is consistent. It is provable in ZFC + I that there is a model of ZFC (Lemma 10.2). Thus the sentence "ZFC is consistent" is provable in ZFC + I. However, we have assumed that "I is consistent with ZFC" is provable, and so "ZFC + I is consistent" is provable in ZFC + I. This contradicts Gödel's second incompleteness theorem. ■

The wording of the second part of Theorem 27 (and its proof) is somewhat vague; "it cannot be shown" means: It cannot be shown by methods formalizable in ZFC.

Exercise 10.5. If κ is an inaccessible cardinal then $V_\kappa \models$ there is a countable model of ZFC.

[Since $\langle V_\kappa, \in\rangle$ is a model of ZFC, there is a countable model (by the Löwenheim–Skolem theorem). Thus there is $E \subset \omega \times \omega$ such that $\mathfrak{A} = \langle \omega, E\rangle$ is a model of ZFC. Verify that $V_\kappa \models$ ($\mathfrak{A}$ is a countable model of ZFC).]

Exercise 10.6. If κ is an inaccessible cardinal, then there is $\alpha < \kappa$ such that $\langle V_\alpha, \in\rangle \prec \langle V_\kappa, \in\rangle$. Moreover, the set $\{\alpha < \kappa : \langle V_\alpha, \in\rangle \prec \langle V_\kappa, \in\rangle\}$ is closed unbounded.

[Construct Skolem functions h for V_κ, and let $\alpha = \lim_n \alpha_n$, where $\alpha_{n+1} < \kappa$ is such that $h \mid V_{\alpha_n} \subseteq V_{\alpha_{n+1}}$ for each h.]

We recall the definition (9.6):

$$\mathscr{H}\mathbf{A} = \{x \in \mathbf{A} : x \subset \mathscr{H}\mathbf{A}\}$$

(the class of all x hereditarily in $\mathbf{A}$).

If $\mathbf{A}$ is the class of all finite sets, then $\mathscr{H}\mathbf{A}$ is the set of all hereditarily finite sets, denoted by HF. In fact, $\mathrm{HF} = V_\omega$.

Let HC denote the class of all *hereditarily countable* sets:

$$\begin{aligned}\mathrm{HC} &= \mathscr{H}(\{X : |X| \le \aleph_0\}) \\ &= \{X : \text{every } y \in \mathrm{TC}(\{X\}) \text{ is at most countable}\}\end{aligned}$$

Exercise 10.7. For each α, let $H_\alpha = \mathrm{HC} \cap V_\alpha$. Show that $H_0 = \emptyset$, $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$ if α is a limit ordinal, and $H_{\alpha+1} = \{X \subseteq H_\alpha : |X| \le \aleph_0\}$.

HC is a set:

Exercise 10.8. (In ZFC) $\mathrm{HC} = H_{\omega_1}$.

Exercise 10.9.* (In ZF) $\mathrm{HC} = H_{\omega_2}$.

[Show that $|\mathrm{rank}(A)| \le \aleph_1$ for every $A \in \mathrm{HC}$. Since $\mathrm{rank}(A) = \bigcup_{n=0}^\infty \{\mathrm{rank}(x) : x \in \bigcup^n A\}$, it suffices to find, uniformly for all $A \in \mathrm{HC}$, functions F_n^A ($n \in \omega$) such that each F_n^A maps ω_1^{n+1} onto the set $\{\mathrm{rank}(x) : x \in \bigcup^n A\}$. Let $F_0^A(\alpha) =$ the

αth element of the set $\{\text{rank}(x) : x \in A\}$, and $F^A_{n+1}(\alpha_0, \ldots, \alpha_{n+1})$ = the α_{n+1}th element of the set $\{F^X_n(\alpha_0, \ldots, \alpha_n) : X \in A\}$.]

Exercise 10.10. (In ZFC) HC is a transitive model of ZFC minus Power set.

[AC is needed to show that $\bigcup X \in \text{HC}$ if $X \in \text{HC}$ (the countable axiom of choice suffices).]

11. TRANSITIVE MODELS OF ZF

In this section we shall investigate the transitive classes that satisfy the axioms of ZF. In view of Gödel's second incompleteness theorem, we shall concentrate on models of ZF that are proper classes. First we prove the Mostowski collapsing theorem, the import of which is that every *well-founded* model of ZF is isomorphic to a *transitive* model. (We leave the subject of non-well-founded models for a later time.) The main theorem of this section (Theorem 31) gives a complete description of transitive models of ZF that are proper classes.

Throughout the section (and in fact throughout most of Chapter II) we work in ZF; that is, we do not use the axiom of choice.

Isomorphism of Models

Let $\mathfrak{M} = (\mathbf{M}, \mathbf{E})$ and $\bar{\mathfrak{M}} = (\bar{\mathbf{M}}, \bar{\mathbf{E}})$ be models. $\mathfrak{M}$ and $\bar{\mathfrak{M}}$ are *isomorphic* if there is an *isomorphism* between $\mathfrak{M}$ and $\bar{\mathfrak{M}}$, that is, a one-to-one function π of $\mathbf{M}$ onto $\bar{\mathbf{M}}$ such that

$$(11.1) \qquad x \mathbf{E} y \quad \text{iff} \quad \pi(x) \bar{\mathbf{E}} \pi(y) \qquad \text{for all } x, y \in \mathbf{M}$$

It follows easily from the definition (11.1) that if φ is a formula, then

$$(11.2) \qquad \mathfrak{M} \models \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad \bar{\mathfrak{M}} \models \varphi(\pi x_1, \ldots, \pi x_n)$$

We say that two classes $\mathbf{M}$ and $\bar{\mathbf{M}}$ are isomorphic (or $\in$-isomorphic) if the $\in$-models $\mathbf{M}$ and $\bar{\mathbf{M}}$ are isomorphic.

Extensional Classes and Relations

We say that a class $\mathbf{M}$ is *extensional* if for any distinct $X, Y \in \mathbf{M}$,

$$(11.3) \qquad X \cap \mathbf{M} \neq Y \cap \mathbf{M}$$

That is, if $X, Y \in \mathbf{M}$ and

$$u \in X \quad \leftrightarrow \quad u \in Y$$

holds for every $u \in \mathbf{M}$, then $X = Y$.

A relation $\mathbf{E}$ over a class $\mathbf{M}$ is called *extensional* if

$$(11.4) \qquad \text{ext}_{\mathbf{E}}(X) \neq \text{ext}_{\mathbf{E}}(Y)$$

whenever $X, Y \in \mathbf{M}$ and $X \neq Y$.

Thus a class is extensional iff the relation $\in$ over $\mathbf{M}$ is extensional.

Clearly, (11.4) is equivalent to

$$\forall u \in \mathbf{M}(u \mathbf{E} X \leftrightarrow u \mathbf{E} Y) \quad \to \quad X = Y$$

which is just the statement that $(\mathbf{M}, \mathbf{E})$ satisfies the axiom of extensionality. Thus we have:

Lemma 11.1. *A relation* $\mathbf{E}$ *over* $\mathbf{M}$ *is extensional iff* $(\mathbf{M}, \mathbf{E})$ *is a model of Extensionality. A class* $\mathbf{M}$ *is extensional iff it is a model of Extensionality.* ■

Note that every transitive class is extensional: If $X \in \mathbf{M}$, then $X \cap \mathbf{M} = X$ and (11.3) is trivially satisfied. The following theorem shows that every extensional class is isomorphic to a transitive class and the isomorphism is unique (the *transitive collapse*).

Theorem 28 (*Mostowski's Collapsing Theorem*).

(A) *If* $\mathbf{E}$ *is a well-founded and extensional relation over a class* $\mathbf{P}$, *then there is a transitive class* $\mathbf{M}$ *and an isomorphism* π *between* $(\mathbf{P}, \mathbf{E})$ *and* $(\mathbf{M}, \in)$. *The transitive class* $\mathbf{M}$ *and the isomorphism* π *are unique.*

(B) *In particular, every extensional class* $\mathbf{P}$ *is isomorphic to a transitive class* $\mathbf{M}$. *The transitive class* $\mathbf{M}$ *and the isomorphism* π *are unique.*

(C) *In case* (B), *if* $\mathbf{T} \subseteq \mathbf{P}$ *is transitive, then* $\pi x = x$ *for every* $x \in \mathbf{T}$.

Proof. Since (B) is a special case of (A) ($\mathbf{E} = \in$ in case (B)), we shall prove the existence of an isomorphism in the general case.

Since $\mathbf{E}$ is a well-founded relation, we can define π by well-founded induction (Theorem 25′), i.e., $\pi(x)$ can be defined in terms of the $\pi(z)$'s, where $z \mathbf{E} x$. We let, for each $x \in \mathbf{P}$,

$$\pi(x) = \pi[\text{ext}_E(x)] = \{\pi(z) : z \mathbf{E} x\} \tag{11.5}$$

In particular, in the case $\mathbf{E} = \in$, (11.5) becomes

$$\pi(x) = \pi[x \cap \mathbf{P}] \tag{11.6}$$

The function π maps $\mathbf{P}$ onto a class $\mathbf{M} = \pi[\mathbf{P}]$, and it is immediate from the definition (11.5) that $\mathbf{M}$ is transitive.

We use extensionality of $\mathbf{E}$ to show that π is one-to-one. Let $z \in \mathbf{M}$ be of least rank such that $z = \pi(x) = \pi(y)$ for some $x \neq y$. Then $\text{ext}(x) \neq \text{ext}(y)$ and there is, e.g., $u \in \text{ext}(x)$ such that $u \notin \text{ext}(y)$. Let $t = \pi(u)$. Since $t \in z = \pi(y)$, there is $v \in \text{ext}(y)$ such that $t = \pi(v)$. Thus we have $t = \pi(u) = \pi(v)$, $u \neq v$, and t is of lesser rank than z (since $t \in z$). A contradiction.

Now it follows easily that

$$x \mathbf{E} y \quad \leftrightarrow \quad \pi(x) \in \pi y \tag{11.7}$$

If $x \mathbf{E} y$, then $\pi(x) \in \pi(y)$ by definition (11.5). On the other hand, if $\pi(x) \in \pi(y)$, then by (11.5), $\pi(x) = \pi(z)$ for some $z \mathbf{E} y$. Since π is one-to-one, we have $x = z$ and so $x \mathbf{E} y$.

Uniqueness of the isomorphism π (and the transitive class $\mathbf{M} = \pi[\mathbf{P}]$) follows from the isomorphism theorem (Theorem 26). If π_1 and π_2 are two isomorphisms of $\mathbf{P}$ and $\mathbf{M}_1$, $\mathbf{M}_2$, respectively, then $\pi_2\pi_1^{-1}$ is an isomorphism between $\mathbf{M}_1$ and $\mathbf{M}_2$, and therefore the identity mapping. Hence $\pi_1 = \pi_2$.

It remains to prove (C). If $\mathbf{T} \subseteq \mathbf{P}$ is transitive, then we first observe that $x \subset \mathbf{P}$ for every $x \in \mathbf{T}$ and so $x \cap \mathbf{P} = x$, and we have

$$\pi(x) = \pi[x]$$

for all $x \in \mathbf{T}$. It follows easily by $\in$-induction that $\pi x = x$ for all $x \in \mathbf{T}$. ■

It is worth noting that Theorem 28 is a generalization of Theorem 2 (every well-ordered set is isomorphic to a unique ordinal number).

While dealing with transitive models (most of Chapter II), we shall use the letter $\mathfrak{M}$ to denote a transitive class (possibly a proper class).

Reflection Principle

The theorem to be proved below is the counterpart of the Löwenheim–Skolem theorem. While that theorem states that every model has a small elementary submodel, the reflection principle provides, for any *finite* number of formulas, a set M that is like an "elementary submodel" of the universe, with respect to the given formulas. The theorem is proved without the use of the axiom of choice, but using AC, one can obtain a *countable* M.

Theorem 29 (*Reflection Principle*).

(A) *Let $\varphi(x_1, \ldots, x_n)$ be a formula. For each M_0 there exists a set $M \supseteq M_0$ such that*

$$M \models \varphi(x_1, \ldots, x_n) \quad \textit{iff} \quad \varphi(x_1, \ldots, x_n) \tag{11.8}$$

for every $x_1, \ldots, x_n \in M$. (We say that M reflects *φ.)*

(B) *Moreover, there is a transitive $M \supseteq M_0$ that reflects φ; moreover, there is a limit ordinal α such that $M_0 \subseteq V_\alpha$ and V_α reflects φ.*

(C) *Assuming* AC, *there is $M \supseteq M_0$ such that M reflects φ and* $|M| \leq \max(|M_0|, \aleph_0)$. *In particular, there is a countable M that reflects φ.*

Remarks. 1. We may require either that M be transitive or that $|M| \leq \max(|M_0|, \aleph_0)$ but not both.

2. The proof works equally for any finite number of formulas, not just one. Thus if $\varphi_1, \ldots, \varphi_n$ are formulas, then there exists a set M that reflects each of $\varphi_1, \ldots, \varphi_n$.

3. If σ is a true sentence, then the reflection principle yields a set M that is a model of σ; using AC, one can get a countable transitive model of σ.

4. As a consequence of the reflection principle, and Gödel's second incompleteness theorem, it follows that the theory ZF is not finitely axiomatizable. For, any finite number of theorems of ZF have a model (a set) by the reflection principle, while existence of a model of ZF is not provable. (By the same argument, no consistent extension of ZF is finitely axiomatizable.)

The key step in the proof of Theorem 29 is the following lemma, which we prove first.

Lemma 11.2.

(i) *Let* $\varphi(u_1, \ldots, u_n, x)$ *be a formula. For each set* M_0 *there exists a set* $M \supseteq M_0$ *such that*

$$\text{(11.9)} \qquad \textit{if} \quad \exists x\, \varphi(u_1, \ldots, u_n, x) \qquad \textit{then} \quad (\exists x \in M)\varphi(u_1, \ldots, u_n, x)$$

for every $u_1, \ldots, u_n \in M$.

Assuming AC, *there is* $M' \supseteq M_0$ *such that* (11.9) *holds for* M' *and* $|M'| \leq \max(|M_0|, \aleph_0)$.

(ii) *If* $\varphi_1, \ldots, \varphi_k$ *are formulas, then for each* M_0 *there is* $M \supseteq M_0$ *such that* (11.9) *holds for each* $\varphi_1, \ldots, \varphi_k$.

Proof. We shall give a detailed proof of (i). An obvious modification of the proof gives (ii); we leave that to the reader.

Note that the operation $\mathscr{H}(u_1, \ldots, u_n)$ defined below has the same role as Skolem functions in the Löwenheim–Skolem theorem.

Let us recall the definition (9.3):

$$\text{(11.10)} \qquad \hat{\mathbf{C}} = \{x \in \mathbf{C} : (\forall z \in \mathbf{C}) \text{ rank } x \leq \text{rank } z\}$$

For every $u_1, \ldots, u_n$, let

$$\text{(11.11)} \qquad \mathscr{H}(u_1, \ldots, u_n) = \hat{\mathbf{C}}$$

where

$$\text{(11.12)} \qquad \mathbf{C} = \{x : \varphi(u_1, \ldots, u_n, x)\}$$

Thus $\mathscr{H}(u_1, \ldots, u_n)$ is a set with the property

$$\text{(11.13)} \quad \text{if} \quad \exists x \varphi(u_1, \ldots, u_n, x), \quad \text{then} \quad \exists x \in \mathscr{H}(u, \ldots, u_n) \varphi(u, \ldots, u_n, x)$$

We construct the set M by induction. We let

$$\text{(11.14)} \qquad M_{i+1} = M_i \cup \bigcup \{\mathscr{H}(u_1, \ldots, u_n) : u_1, \ldots, u_n \in M_i\}$$

and

$$M = \bigcup_{i=0}^{\infty} M_i$$

Now, if $u_1, \ldots, u_n \in M$, then there is $i \in N$ such that $u_1, \ldots, u_n \in M_i$ and if $\varphi(u, \ldots, u_n, x)$ holds for some x, then it holds for some $x \in M_{i+1}$ by (11.13) and (11.14).

Assuming AC, let F be a choice function on $P(M)$. For every $u_1, \ldots, u_n \in M$, let

$$h(u_1, \ldots, u_n) = F(\mathscr{H}(u_1, \ldots, u_n))$$

and let us define

$$M'_0 = M_0$$

$$M'_{i+1} = M'_i \cup \{h(u_1, \ldots, u_n) : u_1, \ldots, u_n \in M'_i\}$$

$$M' = \bigcup_{i=0}^{\infty} M'_i$$

(11.9) can be verified for M' in the same way as for M. Moreover, each M'_i has cardinality at most $\max(|M_0|, \aleph_0)$, and so does M'.

Proof of Theorem 29. Let $\varphi(x_1, \ldots, x_n)$ be a formula. We may assume that the universal quantifier does not occur in φ ($\forall x \cdots$ can be replaced by $\neg \exists x \neg \cdots$). Let $\varphi_1, \ldots, \varphi_k$ be all the subformulas of the formula φ.

Given a set M_0, there exists, by Lemma 11.2(ii), a set $M \supseteq M_0$, such that

$$\exists x\, \varphi_j(u, \ldots, x) \rightarrow (\exists x \in M)\, \varphi_j(u, \ldots, x), \qquad j = 1, \ldots, k \tag{11.15}$$

for all $u, \ldots \in M$. We claim that M reflects each $\varphi_j, j = 1, \ldots, k$, and in particular M reflects φ. This is proved by induction on the complexity of φ_j.

It is easy to see that (every) M reflects atomic formulas, and that if M reflects formulas ψ and χ, then M reflects $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \rightarrow \chi$, and $\psi \leftrightarrow \chi$. Thus assume that M reflects $\varphi_j(u_1, \ldots, u_m, x)$ and let us prove that M reflects $\exists x \varphi_j$.

If $u_1, \ldots, u_m \in M$, then

$$\begin{aligned} M \vDash \exists x\, \varphi_j(u_1, \ldots, u_m, x) &\leftrightarrow (\exists x \in M)\, M \vDash \varphi_j(u_1, \ldots, u_m, x) \\ &\leftrightarrow (\exists x \in M)\, \varphi_j(u_1, \ldots, u_m, x) \\ &\leftrightarrow \exists x\, \varphi_j(u_1, \ldots, u_m, x) \end{aligned}$$

The last equivalence holds by (11.15).

This proves part (A) of the theorem. Part (C) is proved by taking M of size $\leq \max(|M_0|, \aleph_0)$. To prove (B), one has to modify the proof of Lemma 11.2 so that the set M used in (11.15) is transitive (or $M = V_\alpha$). This is done as follows: In (11.14), we replace M_{i+1} by its transitive closure (or by the least $V_\gamma \supset M_{i+1}$). Then M is transitive (or $M = V_\alpha$). ■

Exercise 11.1. Find a formula $\varphi(x)$ such that every nonempty transitive set that reflects $\varphi(x)$ is uncountable.

Exercise 11.2. For every formula φ, there is a closed unbounded class $\mathbf{C}_\varphi \subseteq Ord$ such that for each $\alpha \in \mathbf{C}_\varphi$, V_α reflects φ.

[$\mathbf{C}_{\varphi \wedge \psi} = \mathbf{C}_\varphi \cap \mathbf{C}_\psi$, $\mathbf{C}_{\exists x \varphi} = \mathbf{C}_\varphi \cap \mathbf{K}_\varphi$, where $\mathbf{K}_\varphi$ is the closed unbounded class $\mathbf{K}_\varphi = \{\alpha \in Ord : \forall x_1 \cdots x_n \in V_\alpha[\exists x\, \varphi(x, x_1, \ldots, x_n) \rightarrow (\exists x \in V_\alpha)\, \varphi(x, x_1, \ldots, x_n)]\}$.]

Exercise 11.3. Let $\mathfrak{M}$ be a transitive class and let φ be a formula. For each $M_0 \subseteq \mathfrak{M}$ there exists $M \subseteq \mathfrak{M}$ such that $M_0 \subseteq M$ and that

$$\mathfrak{M} \vDash \varphi(x_1, \ldots, x_n) \quad \leftrightarrow \quad M \vDash \varphi(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in M$.

A transfinite sequence $\langle H_\alpha : \alpha \in Ord \rangle$ is called a *cumulative hierarchy* if

(11.16) (i) $H_\alpha \subseteq H_{\alpha+1} \subseteq P(H_\alpha)$
(ii) if α is a limit, then $H_\alpha = \bigcup_{\beta < \alpha} H_\beta$

(Note that each H_α is transitive and $H_\alpha \subseteq V_\alpha$.)

Exercise 11.4. Let $\langle H_\alpha : \alpha \in Ord \rangle$ be a cumulative hierarchy, and let $\mathbf{H} = \bigcup_{\alpha \in Ord} H_\alpha$. Let φ be a formula. Show that there are arbitrary large limit ordinals α such that

$$\mathbf{H} \vDash \varphi(x_1, \ldots, x_n) \quad \leftrightarrow \quad H_\alpha \vDash \varphi(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in H_\alpha$.

Gödel Operations

The axiom schema of separation states that, given a formula $\varphi(x)$, then for every X there exists a set

$$Y = \{u \in X : \varphi(u)\} \tag{11.17}$$

It turns out that for restricted formulas, the construction of Y from X in (11.17) can be described uniformly for all restricted formulas, by means of a finite number of elementary operations.

Theorem 30 (*Normal Form Theorem*). *There exist operations $\mathfrak{F}_1, \ldots, \mathfrak{F}_{10}$ such that if $\varphi(u_1, \ldots, u_n)$ is a restricted formula, then there is an operation $\mathfrak{F}$, a composition of $\mathfrak{F}_1, \ldots, \mathfrak{F}_{10}$ such that for all $X_1, \ldots, X_n$,*

(11.18)

$$\mathfrak{F}(X_1, \ldots, X_n) = \{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \quad \text{and } \varphi(u_1, \ldots, u_n)\}$$

In case $n = 1$, we define $(u) = u$, and we have in particular

$$\mathfrak{F}(X) = \{u \in X : \varphi(u)\}$$

The operations $\mathfrak{F}_1, \ldots, \mathfrak{F}_{10}$ will be defined below. Compositions of $\mathfrak{F}_1, \ldots, \mathfrak{F}_{10}$ are called *Gödel operations*.

We call the following sentence an instance of *restricted separation*:

$$\forall p_1 \cdots \forall p_n\, \forall X\, \exists Y\, \forall u (u \in Y \leftrightarrow u \in X \wedge \varphi(u, p_1, \ldots, p_n)) \tag{11.19}$$

where φ is a restricted formula. We say that a transitive class $\mathfrak{M}$ satisfies *Restricted Separation* if for every restricted φ, $\mathfrak{M}$ satisfies (11.19).

A class $\mathbf{C}$ is *closed under an operation* $\mathfrak{F}$ if $\mathfrak{F}(x_1, \dots, x_n) \in \mathbf{C}$ whenever $x_1, \dots, x_n \in \mathbf{C}$. Let us call a class $\mathfrak{M}$ (*Gödel*) *closed* if it is closed under the operations $\mathfrak{F}_1, \dots, \mathfrak{F}_{10}$; equivalently, if $\mathfrak{M}$ is closed under all Gödel operations.

Corollary. *If* $\mathfrak{M}$ *is a closed transitive class, then* $\mathfrak{M}$ *satisfies Restricted Separation.*

Proof. Let $\varphi(u, p_1, \dots, p_n)$ be a restricted formula, and let $X, p_1, \dots, p_n \in \mathfrak{M}$. Let

$$Y = \{u \in X : \varphi(u, p_1, \dots, p_n)\}$$

By Lemma 10.1, it suffices to show that $Y \in \mathfrak{M}$, in order that $\mathfrak{M}$ satisfy (11.19). By the normal form theorem, there is a Gödel operation $\mathfrak{F}$ such that

$$\mathfrak{F}(X, \{p_1\}, \dots, \{p_n\}) = \{(u, p_1, \dots, p_n) : u \in X \wedge \varphi(u, p_1, \dots, p_n)\}$$

It follows that

$$\begin{aligned} Y &= \{u : \exists u_1 \cdots \exists u_n (u, u_1, \dots, u_n) \in \mathfrak{F}(X, \{p_1\}, \dots, \{p_n\})\} \\ &= \underbrace{\mathrm{dom} \cdots \mathrm{dom}}_{n \text{ times}} \mathfrak{F}(X, \{p_1\}, \dots, \{p_n\}) \end{aligned}$$

Since both $\{x, y\}$ and $\mathrm{dom}(x)$ are Gödel operations (see below) and since $\mathfrak{M}$ is closed, we have $Y \in \mathfrak{M}$. ■

Gödel Operations:

$$\begin{aligned} \mathfrak{F}_1(X, Y) &= \{X, Y\} \\ \mathfrak{F}_2(X, Y) &= X \times Y \\ \mathfrak{F}_3(X, Y) &= \varepsilon(X, Y) = \{(u, v) : u \in X \wedge v \in Y \wedge u \in v\} \\ \mathfrak{F}_4(X, Y) &= X - Y \\ \mathfrak{F}_5(X, Y) &= X \cap Y \\ \mathfrak{F}_6(X) &= \bigcup X \\ \mathfrak{F}_7(X) &= \mathrm{dom}(X) \\ \mathfrak{F}_8(X) &= \{(u, v) : (v, u) \in X\} \\ \mathfrak{F}_9(X) &= \{(u, v, w) : (u, w, v) \in X\} \\ \mathfrak{F}_{10}(X) &= \{(u, v, w) : (v, w, u) \in X\} \end{aligned}$$

Proof of Theorem 30. The theorem is proved by induction on the complexity of restricted formulas. To simplify matters, we consider only *normal formulas*: A formula $\varphi(u_1, \dots, u_n)$ is normal if:

(i) the only logical symbols in φ are $\neg$, $\wedge$, and restricted $\exists$;
(ii) $=$ does not occur;
(iii) the only occurrence of $\in$ is $u_i \in u_j$ where $i \neq j$;
(iv) the only occurrence of $\exists$ is

$$(\exists u_{m+1} \in u_i)\psi(u_1, \dots, u_{m+1})$$

where $i \leq m$.

Every restricted formula can be rewritten as a normal formula: The use of logical symbols can be restricted to $\neg$, $\wedge$, and $\exists$; $x = y$ can be replaced by a restricted formula

$$(\forall u \in x)\, u \in y \wedge (\forall v \in y)\, v \in x$$

$x \in x$ can be replaced by

$$(\exists u \in x)\, u = x$$

and the bound variables in $\varphi(u_1, \dots, u_n)$ can be renamed so that the variable with the highest index is quantified.

Note that we allow "dummy" variables, so that for instance $\varphi(u_1, \dots, u_5) = u_3 \in u_2$ and $\varphi(u_1, \dots, u_6) = u_3 \in u_2$ are considered separately.

Thus let $\varphi(u_1, \dots, u_n)$ be a normal formula and let us assume that the theorem holds for all subformulas of φ.

Case I. $\varphi(u_1, \dots, u_n)$ is an atomic formula, $u_i \in u_j$ $(i \neq j)$. We prove this case by induction on n.

Case Ia. $n = 2$. Here we have

$$\{(u_1, u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_1 \in u_2\} = \varepsilon(X_1, X_2)$$

and

$$\{(u_1, u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_2 \in u_1\} = \mathfrak{F}_8(\varepsilon(X_2, X_1))$$

Case Ib. $n > 2$ and $i, j \neq n$. By the induction hypothesis, there is $\mathfrak{F}$ such that

$$\{(u_1, \dots, u_{n-1}) : u_1 \in X_1, \dots, u_{n-1} \in X_{n-1} \wedge u_i \in u_j\} = \mathfrak{F}(X_1, \dots, X_{n-1})$$

Obviously,

$$\{(u_1, \dots, u_n) : u_1 \in X_1, \dots, u_n \in X_n \wedge u_i \in u_j\} = \mathfrak{F}(X_1, \dots, X_{n-1}) \times X_n$$

Case Ic. $n > 2$ and $i, j \neq n - 1$. By the induction hypothesis (Case Ib) there is $\mathfrak{F}$ such that

$$\{(u_1, \dots, u_{n-2}, u_n, u_{n-1}) : u_1 \in X_1, \dots, u_n \in X_n \text{ and } u_i \in u_j\} = \mathfrak{F}(X_1, \dots, X_n)$$

Noting that

$$(u_1, \dots, u_{n-2}, u_n, u_{n-1}) = ((u_1, \dots, u_{n-2}), u_n, u_{n-1})$$

we get

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_i \in u_j\} = \mathfrak{F}_9(\mathfrak{F}(X_1, \ldots, X_n))$$

Case Id. $i = n - 1, j = n$. By Ia, we have

$$\{(u_{n-1}, u_n) : u_{n-1} \in X_{n-1} \wedge u_n \in X_n \wedge u_{n-1} \in u_n\} = \varepsilon(X_{n-1}, X_n)$$

and so

$$\begin{aligned}\{((u_{n-1}, u_n), (u_1, \ldots, u_{n-2})) &: u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_{n-1} \in u_n\} \\ &= \varepsilon(X_{n-1}, X_n) \times (X_1 \times \cdots \times X_{n-2}) \\ &= \mathfrak{F}(X_1, \ldots, X_n)\end{aligned}$$

Now we note that

$$((u_{n-1}, u_n), (u_1, \ldots, u_{n-2})) = (u_{n-1}, u_n, (u_1, \ldots, u_{n-2}))$$

and

$$(u_1, \ldots, u_n) = ((u_1, \ldots, u_{n-2}), u_{n-1}, u_n)$$

and thus

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_{n-1} \in u_n\} = \mathfrak{F}_{10}(\mathfrak{F}(X_1, \ldots, X_n))$$

Case Ie. $i = n, j = n - 1$. Similar to Id.

Case II. $\varphi(u_1, \ldots, u_n)$ is a negation, $\neg\psi(u_1, \ldots, u_n)$. By the induction hypothesis, there is $\mathfrak{F}$ such that

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \psi(u_1, \ldots, u_n)\} = \mathfrak{F}(X_1, \ldots, X_n)$$

Clearly,

$$\begin{aligned}\{(u_1, \ldots, u_n) &: u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\} \\ &= (X_1 \times \cdots \times X_n) - \mathfrak{F}(X_1, \ldots, X_n)\end{aligned}$$

Case III. φ is a conjunction, $\psi_1 \wedge \psi_2$. By the induction hypothesis,

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \psi_i(u_1, \ldots, u_n)\} = \mathfrak{G}_i(X_1, \ldots, X_n) \qquad (i = 1, 2)$$

Hence

$$\begin{aligned}\{(u_1, \ldots, u_n) &: u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\} \\ &= \mathfrak{G}_1(X_1, \ldots, X_n) \cap \mathfrak{G}_2(X_1, \ldots, X_n)\end{aligned}$$

Case IV. $\varphi(u_1, \ldots, u_n)$ is the formula $(\exists u_{n+1} \in u_i)\psi(u_1, \ldots, u_{n+1})$. Let $\chi(u_1, \ldots, u_{n+1})$ be the formula $\psi(u_1, \ldots, u_{n+1}) \wedge u_{n+1} \in u_i$. By the induction hypothesis (we consider χ less complex than φ), there is $\mathfrak{F}$ such that

$$\begin{aligned}\{(u_1, \ldots, u_{n+1}) &: u_1 \in X_1, \ldots, u_{n+1} \in X_{n+1} \text{ and } \chi(u_1, \ldots, u_{n+1})\} \\ &= \mathfrak{F}(X_1, \ldots, X_{n+1})\end{aligned} \tag{11.20}$$

for all $X_1, \ldots, X_{n+1}$. We claim that

$$(11.21) \qquad \{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\} \\ = (X_1 \times \cdots \times X_n) \cap \operatorname{dom}(\mathfrak{F}(X_1, \ldots, X_n, \bigcup X_i))$$

Let us denote $u = (u_1, \ldots, u_n)$ and $X = X_1 \times \cdots \times X_n$. For all $u \in X$, we have

$$\begin{aligned} \varphi(u) &\leftrightarrow \exists v \in u_i \psi(u, v) \\ &\leftrightarrow \exists v(v \in u_i \wedge \psi(u, v) \wedge v \in \bigcup X_i) \\ &\leftrightarrow u \in \operatorname{dom}\{(u, v) \in X \times \bigcup X_i : \chi(u, v)\} \end{aligned}$$

and (11.21) follows. This completes the proof of Theorem 30. ■

We recall that a set W is called *Gödel closed* if it is closed under all Gödel operations. For every set M, there is a smallest closed $W \supseteq M$: Let $W = \bigcup_{n=0}^{\infty} W_n$, where $W_0 = M$ and†

$$W_{n+1} = W_n \cup \{\mathfrak{F}_i(X, Y) : X \in W_n, Y \in W_n, i = 1, \ldots, 10\}$$

Let us denote

$$W = \operatorname{cl}(M)$$

and call W the *(Gödel) closure* of M.

Exercise 11.5. If M is a transitive set, then $\operatorname{cl}(M)$ is transitive.

[Show (by induction on W_n) that if $X \in W$, then $\operatorname{TC}(X) \subseteq W$.]

Exercise 11.6. If M is closed and extensional and if $X \in M$ is finite, then $X \subseteq M$. In particular, if $(x, y) \in M$, then $x \in M$ and $y \in M$.

Exercise 11.7.* If M is closed and extensional, and if π is the transitive collapse of M, then

$$\pi(\mathfrak{F}_i(X, Y)) = \mathfrak{F}_i(\pi X, \pi Y), \qquad i = 1, \ldots, 10$$

for all $X, Y \in M$.

[Use the normal form theorem.]

Exercise 11.8. The operations $\mathfrak{F}_5$ and $\mathfrak{F}_8$ are compositions of the remaining $\mathfrak{F}_i$. [$\mathfrak{F}_8(X) = \operatorname{dom}(\mathfrak{F}_{10}(\mathfrak{F}_{10}(\mathfrak{F}_9(\mathfrak{F}_{10}(X \times X)))))$.]

Exercise 11.9.* The axioms of comprehension in Bernays–Gödel set theory can be proved from a finite number of axioms of the form

$$\forall X\, \forall Y\, \exists Z\; Z = \mathfrak{G}_i(X, Y)$$

where the $\mathfrak{G}_i$ are like the operations $\mathfrak{F}_i$. Thus Bernays–Gödel set theory is finitely axiomatizable.

[Formulate and prove an analog of the normal form theorem.]

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† We shall treat the operations $\mathfrak{F}_6, \ldots, \mathfrak{F}_{10}$ as binary operations; we let $\mathfrak{F}_i(X, Y) = \mathfrak{F}_i(X)$ for $i = 6, \ldots, 10$.

Transitive Models of ZF

The main theorem of Section 11 gives sufficient conditions for a transitive class to be a model of ZF. In fact, the conditions are necessary for proper classes, and we have a characterization of transitive models of ZF that contain all ordinals.

We say that a class $\mathfrak{M}$ is *almost universal* if every subset $X \subseteq \mathfrak{M}$ is included in some $Y \in \mathfrak{M}$. (Note that $\mathfrak{M}$ has to be a proper class.)

Theorem 31. *Let $\mathfrak{M}$ be a transitive, closed, almost universal class. Then $\mathfrak{M}$ is a model of* ZF.

Remark. The conditions are necessary for a transitive model $\mathfrak{M}$ of ZF that is a proper class: We shall show in Section 12 that the notion of ordinals and rank, as well as Gödel operations are absolute for transitive models of ZF, that is $Ord^{\mathfrak{M}} = Ord \cap \mathfrak{M}$, $\operatorname{rank}^{\mathfrak{M}}(x) = \operatorname{rank}(x)$, and $\mathfrak{F}_i^{\mathfrak{M}}(X, Y) = \mathfrak{F}_i(X, Y)$. Hence $\mathfrak{M}$ is closed under the $\mathfrak{F}_i$; and if it is a proper class, then $\mathfrak{M} \supset Ord$; and if $X \subseteq \mathfrak{M}$ then $X \subset Y \in \mathfrak{M}$, where $Y = V_\alpha^{\mathfrak{M}}$ for some α.

The only difficulty in the proof of Theorem 31 is the proof of the separation axioms. Half of the proof has been provided by Theorem 30; the other part is contained in the following lemma.

Lemma 11.3 (Reduction Lemma). *For any formula $\varphi(u_1, \ldots, u_n)$ with k quantifiers, let $\bar{\varphi}(u_1, \ldots, u_n, Y_1, \ldots, Y_k)$ denote the restricted formula that results if we replace each $\exists x$, $\forall x$ in φ by $\exists x \in Y_j$, $\forall x \in Y_j$, $j = 1, \ldots, k$.*

Let $\mathfrak{M}$ be a transitive almost universal class. Then for every $X \in \mathfrak{M}$ there exist $Y_1, \ldots, Y_k \in \mathfrak{M}$ such that

$$(11.22) \qquad \mathfrak{M} \models \varphi(u_1, \ldots, u_n) \quad \textit{iff} \quad \bar{\varphi}(u_1, \ldots, u_n, Y_1, \ldots, Y_k)$$

for all $u_1, \ldots, u_n \in X$.

Proof. By induction on the complexity of φ. We may assume that φ has only existential quantifiers. If φ is quantifier-free, then $\bar{\varphi} = \varphi$ and (11.22) holds. If φ is $\neg\psi$, $\psi \wedge \chi$, $\psi \to \chi$, etc., and if (11.22) holds for ψ and χ, then it also holds for φ. Thus let $\varphi(u)$ be $\exists v\, \psi(u, v)$ and assume that (11.22) holds for ψ. Assuming that ψ has k quantifiers, $\bar{\varphi}$ is $(\exists v \in Y_{k+1})\, \bar{\psi}(u, v, Y_1, \ldots, Y_k)$.

Let $X \in \mathfrak{M}$. We look for $Y_1, \ldots, Y_k, Y_{k+1} \in \mathfrak{M}$ such that for every $u \in X$,

$$(11.23) \qquad \mathfrak{M} \models \exists v \psi(u, v) \quad \text{iff} \quad (\exists v \in Y_{k+1})\, \bar{\psi}(u, v, Y_1, \ldots, Y_k)$$

By the collection principle (9.4) (applied to the formula $v \in \mathfrak{M} \wedge \mathfrak{M} \models \psi(u, v)$), there exists a set M such that $X \subseteq M \subset \mathfrak{M}$ and that for every $u \in X$,

$$(11.24) \qquad (\exists v \in \mathfrak{M})\, \mathfrak{M} \models \psi(u, v) \quad \text{iff} \quad (\exists v \in M)\, \mathfrak{M} \models \psi(u, v)$$

Since $\mathfrak{M}$ is almost universal, there exists $Y \in \mathfrak{M}$ such that $M \subseteq Y$. It follows from (11.24) that for every $u \in X$,

$$(11.25) \qquad (\exists v \in \mathfrak{M})\, \mathfrak{M} \vDash \psi(u, v) \quad \text{iff} \quad (\exists v \in Y)\, \mathfrak{M} \vDash \psi(u, v)$$

By the induction hypothesis, given $Y \in \mathfrak{M}$, there exist $Y_1, \ldots, Y_k \in \mathfrak{M}$ such that for all $u, v \in Y$,

$$(11.26) \qquad \mathfrak{M} \vDash \psi(u, v) \quad \text{iff} \quad \bar{\psi}(u, v, Y_1, \ldots, Y_k)$$

Thus we let $Y_{k+1} = Y$, and since $X \subseteq Y$, we have for all $u \in X$,

$$\begin{aligned} \mathfrak{M} \vDash \exists v \psi(u, v) \quad &\text{iff} \quad (\exists v \in \mathfrak{M})\, \mathfrak{M} \vDash \psi(u, v) \\ &\text{iff} \quad (\exists v \in Y)\, \mathfrak{M} \vDash \psi(u, v) \\ &\text{iff} \quad (\exists v \in Y)\, \bar{\psi}(u, v, Y_1, \ldots, Y_k) \quad \blacksquare \end{aligned}$$

Proof of Theorem 31. Let $\mathfrak{M}$ be transitive, closed and almost universal. We shall show that every axiom of ZF holds in $\mathfrak{M}$.

I. Extensionality

Every transitive class satisfies Extensionality.

II. Pairing

"$z = \{x, y\}$" can be expressed by a restricted formula, see (10.6). $\mathfrak{M}$ is closed under $\{\ ,\ \}$ and so it satisfies Pairing.

III. Separation

If $\varphi(u, p)$ is a formula, we wish to show that for every $X, p \in \mathfrak{M}$, the set $Y = \{u \in X : \mathfrak{M} \vDash \varphi(u, p)\}$ is in $\mathfrak{M}$.

By the reduction lemma, there are $Y_1, \ldots, Y_k \in \mathfrak{M}$ such that

$$Y = \{u \in X : \bar{\varphi}(u, p, Y_1, \ldots, Y_k)\}$$

(use a "parametric" version of Lemma 11.3 or replace X by $X \cup \{p\}$). Since $\bar{\varphi}$ is a restricted formula, Y belongs to $\mathfrak{M}$ by the corollary to Theorem 30.

IV. Union

"$Y = \bigcup X$" is a restricted formula, see (10.7). Since $\mathfrak{M}$ is closed under $\bigcup$, it satisfies Union.

V. Power set

Since $\mathfrak{M}$ is a model of Separation, it suffices to prove

$$(11.27) \qquad \mathfrak{M} \vDash \forall X\, \exists Y\, \forall u (u \subseteq X \rightarrow u \in Y)$$

Since "$u \subseteq X$" is a restricted formula, (11.27) is equivalent to

$$(\forall X \in \mathfrak{M})\, (\exists Y \in \mathfrak{M})\, P(X) \cap \mathfrak{M} \subseteq Y$$

This follows from almost universality of $\mathfrak{M}$.

VI. Infinity

It suffices to show that $\omega \in \mathfrak{M}$ (see the proof of consistency of Regularity in Section 10, (10.9)). Since $\mathfrak{M}$ is closed, it can be shown by induction that $\mathfrak{M}$

contains all natural numbers; and since $\mathfrak{M}$ is almost universal, we have $\omega \subseteq X$ for some $X \in \mathfrak{M}$. Thus it is enough to show that

"u is a natural number"

is a restricted formula, for then

$$\omega = \{u \in X : \mathfrak{M} \models u \text{ is a natural number}\}$$

By the axiom of regularity, $\in$ is well-founded and so we have

(11.28) u is an ordinal iff u is transitive and u is linearly ordered by $\in$

It is fairly easy to verify that the right-hand side of (11.28) is a restricted formula. Moreover, we have $\alpha \in \omega$ iff $\alpha \in Ord$ and $\alpha \neq 0 \rightarrow (\exists\beta \in \alpha)\, \alpha = \beta + 1$, and

$$(\forall\gamma \in \alpha)\, [\gamma \neq 0 \rightarrow (\exists\beta \in \gamma)\, \gamma = \beta + 1]$$

and $\alpha = \beta + 1$ is a restricted formula:

$$\beta \in \alpha \wedge \beta \subseteq \alpha \wedge (\forall\xi \in \alpha)\, [\xi \in \beta \vee \xi = \beta]$$

VII. Replacement

Let φ be a formula and assume that

$$\mathfrak{M} \models \forall x, y, z[\varphi(x, y) \wedge \varphi(x, z) \rightarrow y = z]$$

Let **F** be the function

$$\mathbf{F} = \{(x, y) \in \mathfrak{M} : \mathfrak{M} \models \varphi(x, y)\}$$

It is enough to show that $\mathbf{F}[X] \in \mathfrak{M}$ for every $X \in \mathfrak{M}$. In fact, since $\mathfrak{M}$ satisfies Separation, it is enough to show that $\mathbf{F}[X] \subseteq Y$ for some $Y \in \mathfrak{M}$. Such Y exists by almost universality of $\mathfrak{M}$.

VIII. Regularity

As in (10.11), every nonempty $S \in \mathfrak{M}$ satisfies (in $\mathfrak{M}$)

$$(\exists x \in S)\, S \cap x = \emptyset$$

since $(S \cap x)^{\mathfrak{M}} = S \cap x$. ■

Exercise 11.10. If $\mathfrak{M}$ is transitive, almost universal, and closed under $\mathfrak{F}_1$, then $\mathfrak{M}$ satisfies the axioms of Extensionality, Regularity, Pairing and weak Union, Power set, and Replacement ((1.8)–(1.10)).

12. CONSTRUCTIBLE SETS

We shall now give an example of a transitive model of ZF. This model, the *constructible universe* of Gödel, is of particular importance: (i) It is the least transitive model containing all ordinals; (ii) it satisfies the axiom of choice and the generalized continuum hypothesis, thus providing a proof of consistency of AC and GCH with ZF.

We recall that $\mathrm{cl}(X)$ denotes the Gödel closure of X. Let us denote

$$\mathrm{def}(U) = \mathrm{cl}(U \cup \{U\}) \cap P(U) \tag{12.1}$$

i.e., the family of all $X \subseteq U$ obtained by a finite number of successive applications of Gödel operations to U and elements of U.

Exercise 12.1. $U \models \varphi(x_1, \ldots, x_n)$ is a restricted formula (with variables $U, x_1, \ldots, x_n$).

Exercise 12.2. Let φ be a formula. There is a Gödel operation $\mathfrak{F}$ such that for every transitive U and all $x_1, \ldots, x_n \in U$,

$$\{u \in U : U \models \varphi(u, x_1, \ldots, x_n)\} = \mathfrak{F}(U, x_1, \ldots, x_n).$$

[Use the normal form theorem.]

Exercise 12.3.* If U is a transitive set and $X \subseteq U$ is definable over $\mathfrak{U} = \langle U, \in \cap U^2 \rangle$ then $X \in \mathrm{def}(U)$.

[Use induction on $\varphi \in$ *Form*.]

We shall show in Section 14 that if $\mathfrak{F}$ is a Gödel operation, then $u \in \mathfrak{F}(x_1, \ldots, x_n)$ can be written as a restricted formula. Use this fact in the next exercise.

Exercise 12.4. If $\mathfrak{F}$ is a Gödel operation, then there is a formula φ such that for every transitive set U and all $x_1, \ldots, x_n \in U$, if $X = \mathfrak{F}(U, x_1, \ldots, x_n) \subseteq U$, then $X = \{u \in U : U \models \varphi(x_1, \ldots, x_n)\}$.

Exercise 12.5.* If U is a transitive set and $X \in \mathrm{cl}(U \cup \{U\})$ is a subset of U, then X is definable over $\mathfrak{U} = \langle U, \in \cap U^2 \rangle$.

In view of Exercises 12.3 and 12.5 we call the set $\mathrm{def}(U)$ (for any transitive set U) the *set of all subsets definable over* U.

Exercise 12.6. If U is a finite transitive set, then $\mathrm{def}(U) = P(U)$.

We now define the *constructible hierarchy:*

$$\begin{aligned} L_0 &= \emptyset \\ L_\alpha &= \bigcup_{\beta<\alpha} L_\beta \qquad \text{if } \alpha \text{ is a limit ordinal} \\ L_{\alpha+1} &= \mathrm{def}(L_\alpha) \\ L &= \bigcup_{\alpha \in Ord} L_\alpha \end{aligned} \tag{12.2}$$

Note that $U \subseteq \mathrm{def}(U) \subseteq P(U)$ for every U and so the L_α form a commulative hierarchy (see (11.16)). Thus each L_α is transitive and $L_\alpha \subseteq V_\alpha$. Using the fact that "$\alpha$ is an ordinal" is a restricted formula (see (11.28)), and Exercise 12.2, we can easily prove by induction that $\alpha \subseteq L_\alpha$ and so

$$L_\alpha \cap Ord = \alpha$$

Elements of the class L are called *constructible sets.*

Axiom of Constructibility *($V = L$). Every set is constructible.*

Theorem 32 (Gödel).

(i) *L is a model of* ZF.
(ii) *L satisfies the axiom of constructibility.*
(iii) *If $\mathfrak{M}$ is a transitive model of* ZF *containing all ordinals, then $L \subseteq \mathfrak{M}$.*

Consequently, the axiom $V = L$ is consistent with ZF. It will be shown in Section 13 that the axiom of constructibility implies the axiom of choice and the generalized continuum hypothesis.

Proof of (i). We show that L is transitive, closed, and almost universal. L is transitive since $\langle L_\alpha : \alpha \in Ord\rangle$ is a cumulative hierarchy. To show that L is closed, it suffices to verify that for each limit α, L_α is closed. We leave the verification to the reader. And finally, L is obviously almost universal since any subset of L is included in some L_α and $L_\alpha \in L$. ■

Exercise 12.7. $L_\omega = V_\omega$.

Exercise 12.8. Let $\langle H_\alpha : \alpha \in Ord\rangle$ be a cummulative hierarchy and assume that for each α, $H_{\alpha+1} \supseteq \operatorname{def}(H_\alpha)$. Then $\mathbf{H} = \bigcup_{\alpha \in Ord} H_\alpha$ is a model of ZF.

To prove the rest of Theorem 32, we shall introduce the notion of *absoluteness* of set theoretical concepts.

Absoluteness

Let $\mathfrak{M}$ be a transitive class and let $\varphi(x_1, \ldots, x_n)$ be a formula. We say that φ is *absolute for* $\mathfrak{M}$ if

$$\mathfrak{M} \models \varphi(x_1, \ldots, x_n) \qquad \text{iff} \qquad \varphi(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in \mathfrak{M}$.

We have proved in Lemma 10.1 that if φ is a restricted formula and $\mathfrak{M}$ is transitive, then φ is absolute for $\mathfrak{M}$. In fact, the proof of Lemma 10.1 gives a little more general result: If φ and ψ are absolute for a transitive $\mathfrak{M}$, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, and $(\forall x \in y)\, \varphi$, $(\exists x \in y)\, \varphi$ are absolute for $\mathfrak{M}$.

In the following lemma we present a number of restricted formulas, which are therefore absolute for any transitive class $\mathfrak{M}$.

Lemma 12.1. *The following expressions can be written as restricted formulas and thus are absolute for all transitive classes.*

(a) $x = \{u, v\}$, $x = (u, v)$, *x is empty*, $x \subseteq y$, *x is transitive*, *x is an ordinal*, *x is a limit ordinal*, *x is a natural number*, $x = \omega$.
(b) $Z = \mathfrak{F}_i(X, Y)$, $i = 1, \ldots, 10$.
(c) *X is a relation*, $y = \operatorname{dom}(x)$, $y = \operatorname{ran}(x)$, *$f$ is a function*, $y = f(x)$, $Y = f[X]$, $g = f \restriction X$.

Proof.

(a) $x = \{u, v\} \leftrightarrow u \in x \wedge v \in x \wedge (\forall w \in x)(w = u \vee w = v)$
$x = (u, v) \leftrightarrow (\exists w \in x)(\exists z \in x)[w = \{u\} \ \wedge \ z = \{u, v\}] \ \wedge$
$(\forall w \in x)(w = \{u\} \ \vee \ w = \{u, v\})$
x is empty $\leftrightarrow (\forall u \in x) u \neq u$
$x \subseteq y \leftrightarrow (\forall u \in x) u \in y$
x is transitive $\leftrightarrow (\forall u \in x) u \subseteq x$
x is an ordinal $\leftrightarrow x$ is transitive $\wedge \ (\forall u \in x)(\forall v \in x)[u \in v \ \vee \ v \in u \ \vee \ u = v) \ \wedge$
$(\forall u \in x)(\forall v \in x)(\forall w \in x)[u \in v \in w \rightarrow u \in w]$
x is a limit ordinal $\leftrightarrow x$ is an ordinal $\wedge (\forall u \in x)(\exists v \in x) u \in v$
x is a natural number $\leftrightarrow x$ is an ordinal $\wedge$ (x is not a limit $\vee \ x = 0) \wedge$
$(\forall u \in x)[u = 0 \ \vee \ u$ is not a limit]
$x = \omega \leftrightarrow x$ is a limit ordinal $\wedge \ x \neq 0 \wedge (\forall u \in x) \ x$ is a natural number

(b) $Z = \{X, Y\} \cdots$ see above
$Z = X \times Y \leftrightarrow (\forall z \in Z)(\exists x \in X)(\exists y \in Y) \quad z = (x, y) \ \wedge$
$(\forall x \in X)(\forall y \in Y)(\exists z \in Z) \quad z = (x, y)$
$Z = \varepsilon(X, Y) \cdots$ similarly
$Z = X - Y \leftrightarrow (\forall z \in Z)[z \in X \wedge z \notin Y] \wedge (\forall z \in X)[z \notin Y \rightarrow z \in Z]$
$Z = X \cap Y \cdots$ similarly
$Z = \bigcup X \leftrightarrow (\forall z \in Z)(\exists x \in X) z \in x \wedge (\forall x \in X)(\forall z \in x) z \in Z$
$Z = \operatorname{dom}(X) \leftrightarrow (\forall z \in Z) z \in \operatorname{dom} X \wedge (\forall z \in \operatorname{dom} X) z \in X$

and we show that:

(12.3) (i) $z \in \operatorname{dom} X$ is a restricted formula;
(ii) if φ is restricted, then $(\forall z \in \operatorname{dom} X)\varphi$ is restricted.

(i) $z \in \operatorname{dom} X \leftrightarrow (\exists x \in X)(\exists u \in x)(\exists v \in u) x = (z, v)$.
(ii) $(\forall z \in \operatorname{dom} X) \ \varphi \leftrightarrow (\forall x \in X) \ (\forall u \in x) \ (\forall z, v \in u) \ [x = (z, v) \rightarrow \varphi]$.

An assertion similar to (12.3) holds for $\operatorname{ran}(X)$, and for $\exists$.

$Z = \mathfrak{F}_8(X) \leftrightarrow (\forall z \in Z)(\exists u \in \operatorname{ran} X)(\exists v \in \operatorname{dom} X) z = (u, v) \ \wedge$
$(\forall u \in \operatorname{ran} X)(\forall v \in \operatorname{dom} X)(\exists z \in Z) z = (u, v)$
$Z = \mathfrak{F}_9(X) \cdots$ similarly
$Z = \mathfrak{F}_{10}(X) \cdots$ similarly

(c) X is a relation $\leftrightarrow (\forall x \in X)(\exists u \in \operatorname{dom} X)(\exists v \in \operatorname{ran} X) x = (u, v)$
f is a function $\leftrightarrow f$ is a relation $\wedge$
$(\forall x \in \operatorname{dom} f)(\forall y, z \in \operatorname{ran} f)[(x, y) \in f \ \wedge \ (x, z) \in f \rightarrow y = z]$
where
$$(x, y) \in f \leftrightarrow (\exists u \in f) u = (x, y)$$

$g = f \mid X \leftrightarrow g$ is a function $\wedge \ g \subseteq f \ \wedge \ (\forall x \in \operatorname{dom} g) x \in X \ \wedge$
$(\forall x \in X)([x \in \operatorname{dom} f \rightarrow x \in \operatorname{dom} g]$ ■

Exercise 12.9. The following can be written as restricted formulas: x is an ordered pair, x is a partial (linear) ordering of y, x and y are disjoint, $z = x \cup y$, $y = x \cup \{x\}$, x is an inductive set, $z \in \mathfrak{F}_i(X, Y)$ $(i = 1, \ldots, 10)$, f is a one-to-one function of X into (onto) Y, f is an increasing ordinal function, f is a normal function.

It should be emphasized that cardinal concepts are generally not absolute. In particular, the following expressions are known not to be absolute:

$$Y = P(X), \quad |Y| = |X|, \quad \alpha \text{ is a cardinal}, \quad \beta = \mathrm{cf}(\alpha), \quad \alpha \text{ is regular}$$

Exercise 12.10. Let $\mathfrak{M}$ be a transitive class.

(a) If $\mathfrak{M} \models |X| \leq |Y|$, then $|X| \leq |Y|$.
(b) If $\alpha \in \mathfrak{M}$ and if α is a cardinal, then $\mathfrak{M} \models \alpha$ is a cardinal.

[$|X| \leq |Y| \leftrightarrow \exists f \varphi(f, X, Y)$; α is a cardinal $\leftrightarrow \neg \exists f\,(\exists \beta \in \alpha)\,\psi(\alpha, \beta, f)$, where φ and ψ are absolute formulas.]

Absolute Operations

Let $\mathfrak{M}$ be a model (of the language of set theory). Strictly speaking, we defined $\mathfrak{M} \models \varphi$ only for formulas whose only nonlogical symbols are $\in$ and $=$. However, we shall frequently use familiar set theoretical concepts such as classes and operations in the context of a model $\mathfrak{M}$, and use $\mathfrak{M} \models \varphi$ even if φ is an expression that contains other set theoretical symbols.

We have introduced *classes* in order to use the notation

$$\mathbf{C} = \{x : \varphi(x)\}$$

We shall thus use the symbol $\mathbf{C}^{\mathfrak{M}}$ to denote corresponding class in $\mathfrak{M}$:

$$\mathbf{C}^{\mathfrak{M}} = \{x \in \mathfrak{M} : \mathfrak{M} \models \varphi(x)\}$$

For instance, $V^{\mathfrak{M}} = \mathfrak{M}$ and $\emptyset^{\mathfrak{M}} = \emptyset$. Using the absoluteness of ordinals, one can easily see that if $\mathfrak{M}$ is a transitive class, then

$$Ord^{\mathfrak{M}} = Ord \cap \mathfrak{M} = \begin{cases} \text{either } Ord \\ \text{or the least } \alpha \notin \mathfrak{M} \end{cases} \tag{12.4}$$

Let $\mathfrak{M}$ be a transitive class, and let $\mathbf{F}$ be a *function*:

$$y = \mathbf{F}(x) \quad \leftrightarrow \quad \varphi(x, y)$$

Then $\{(x, y) \in \mathfrak{M} : \mathfrak{M} \models \varphi(x, y)\}$ is a relation in $\mathfrak{M}$ (the notion of an ordered pair is absolute). If, moreover,

$$\mathfrak{M} \models \forall x\, \forall y\, \forall z (\varphi(x, y) \wedge \varphi(x, z) \rightarrow y = z) \tag{12.5}$$

then the relation is a function in $\mathfrak{M}$ and we denote it by $\mathbf{F}^{\mathfrak{M}}$:

$$y = \mathbf{F}^{\mathfrak{M}}(x) \quad \leftrightarrow \quad \mathfrak{M} \models \varphi(x, y) \tag{12.6}$$

An *operation* is usually defined by a defining formula

$$\mathfrak{F}(x) = \{u : \psi(x, u)\}$$

(Our restriction to one variable x is, of course, inessential.) Thus we let, for $x \in \mathfrak{M}$,

$$\mathfrak{F}^{\mathfrak{M}}(x) = \{u \in \mathfrak{M} : \mathfrak{M} \models \psi(x, u)\} \tag{12.7}$$

Since an operation is a function and since

$$y = \mathfrak{F}(x) \quad \leftrightarrow \quad y = \{u : \psi(x, u)\}$$

(12.7) and (12.6) (with $\varphi(x, y)$ being the formula $y = \{u : \psi(x, u)\}$) give the same definition of $\mathfrak{F}^{\mathfrak{M}}(x)$, provided $\mathfrak{F}^{\mathfrak{M}}(x) \in \mathfrak{M}$. However, we define (12.7) even if $\mathfrak{F}^{\mathfrak{M}}(x) \notin \mathfrak{M}$.

A function **F** is said to be *absolute* for $\mathfrak{M}$ if (i) (12.5) holds, i.e., if $\mathbf{F}^{\mathfrak{M}}$ is a function in $\mathfrak{M}$, and (ii) $y = \mathbf{F}(x)$ is an absolute formula, i.e.,

$$\mathfrak{M} \models \varphi(x, y) \quad \text{iff} \quad \varphi(x, y) \tag{12.8}$$

for all $x, y \in \mathfrak{M}$. An operation $\mathfrak{F}$ is *absolute* if $\mathfrak{F}^{\mathfrak{M}}(x) = \mathfrak{F}(x)$ for all $x \in \mathfrak{M}$. Thus for an operation $\mathfrak{F}$ defined by

$$\mathfrak{F}(x) = \{u : \psi(x, u)\}$$

to be absolute for $\mathfrak{M}$ it suffices that (i) $\mathfrak{F}(x) \subseteq \mathfrak{M}$ for all $x \in \mathfrak{M}$, and (ii) ψ is absolute for $\mathfrak{M}$.

In general, if the defining formula ψ is absolute, then we have

$$\mathfrak{F}^{\mathfrak{M}}(x) = \mathfrak{F}(x) \cap \mathfrak{M}$$

An example is the operation $P(X)$:

$$P^{\mathfrak{M}}(X) = P(X) \cap \mathfrak{M}$$

We say that an operation $\mathfrak{F}$ is *defined* in $\mathfrak{M}$ if $\mathfrak{F}^{\mathfrak{M}}(x) \in \mathfrak{M}$ for all $x \in \mathfrak{M}$.

Lemma 12.2.

(a) *If $\mathfrak{F}$ is an absolute operation (for $\mathfrak{M}$) and defined in $\mathfrak{M}$, and if φ is absolute, then $\varphi(\mathfrak{F}(x))$ is absolute.*

(b) *If $\mathfrak{F}$ is absolute and defined in $\mathfrak{M}$, and if $\mathfrak{G}$ is absolute, then $\mathfrak{G} \circ \mathfrak{F}$ is absolute.*

Proof. (a) Given $x \in \mathfrak{M}$, we have $y = \mathfrak{F}(x) = \mathfrak{F}^{\mathfrak{M}}(x) \in \mathfrak{M}$, and

$$\mathfrak{M} \models \varphi(y) \quad \text{iff} \quad \varphi(y)$$

(b) Apply (a) to the formula $z \neq \mathfrak{G}(y)$. ∎

Lemma 12.3.

(a) *The operations $\mathfrak{F}_i(X, Y)$, $i = 1, \ldots, 10$, are absolute for all transitive classes.*

(b) *If $\mathfrak{M}$ is transitive and closed and $\mathfrak{F}$ is a Gödel operation, then $\mathfrak{F}$ is absolute for $\mathfrak{M}$.*

Proof. (a) follows from Lemma 12.1b, and (b) follows from (a) and Lemma 12.2. (In Section 14 we shall prove a stronger result.) ∎

Absoluteness for Models of ZF

Lemma 12.4. *Let* $\mathfrak{M}$ *be a transitive model of* ZF, *let* $\mathbf{G}$ *be a function on* V, *such that* $\mathbf{G}$ *is absolute for* $\mathfrak{M}$ *and* $\operatorname{dom}(\mathbf{G}^{\mathfrak{M}}) = \mathfrak{M}$. *Let* $\mathbf{F}$ *be defined by induction:*

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F} \mid \alpha) \tag{12.9}$$

Then $\mathbf{F}$ *is absolute for* $\mathfrak{M}$ *and* $\operatorname{dom}(\mathbf{F}^{\mathfrak{M}}) = Ord^{\mathfrak{M}}$.

Proof. The function $\mathbf{F}$ is defined as follows:

$$y = \mathbf{F}(\alpha) \quad \leftrightarrow \quad \alpha \text{ is an ordinal} \wedge \exists f\, \varphi(f, \alpha) \tag{12.10}$$

where $\varphi(f, \alpha)$ is the formula

$$f \text{ is a function} \wedge \operatorname{dom} f = \alpha \wedge (\forall \xi \in \alpha)\, f(\xi) = \mathbf{G}(f \mid \xi) \tag{12.11}$$

The operation $\mathfrak{F}(f, \xi) = f \mid \xi$ is absolute for $\mathfrak{M}$; and since $\mathfrak{M}$ is a model of ZF, $\mathfrak{F}$ is defined in $\mathfrak{M}$, and so $\varphi(f, \alpha)$ is absolute for $\mathfrak{M}$.

Therefore, if $(y, \alpha) \in \mathbf{F}^{\mathfrak{M}}$, then $\exists f \in \mathfrak{M}\ \varphi(f, \alpha)$, and so $y = \mathbf{F}(\alpha)$. This also implies that $\mathbf{F}^{\mathfrak{M}}$ is a function in $\mathfrak{M}$. It remains to show that if $\alpha \in \mathfrak{M}$ and $y = \mathbf{F}(\alpha)$, then $y \in \mathfrak{M}$ and $y = \mathbf{F}^{\mathfrak{M}}(\alpha)$. We have proved (in ZF) that (given $\mathbf{G}$), for every α there exists a function f on α such that $f(\xi) = \mathbf{G}(f \mid \xi)$ for all $\xi < \alpha$. Since $\mathfrak{M}$ is a model of ZF, we have

$$\mathfrak{M} \models \exists f\, \varphi(f, \alpha)$$

and if $y = \mathbf{G}(f)$, then $y = \mathbf{F}^{\mathfrak{M}}(\alpha) = \mathbf{F}(\alpha)$. ∎

In particular, if $\mathfrak{M}$ is a transitive model of ZF and $\mathbf{G}$ is absolute and $\mathbf{G}[\mathfrak{M}] \subseteq \mathfrak{M}$, then the function f on ω defined by

$$f(0) = a_0, \qquad f(n + 1) = \mathbf{G}(f(n))$$

is in $\mathfrak{M}$.

Exercise 12.11. The function Γ: $Ord^2 \to Ord$ defined in Section 2 is absolute for transitive models of ZF.

[Define Γ^{-1} by induction.]

Exercise 12.12. The operations $\alpha + \beta$ and $\alpha \cdot \beta$ are absolute for transitive models of ZF.

Exercise 12.13. $\mathrm{TC}(S)$ is an absolute operation for transitive models of ZF.

I have already mentioned that the operation $P(X)$ is not absolute: $P^{\mathfrak{M}}(X) = P(X) \cap \mathfrak{M}$. If $\mathfrak{M}$ is a transitive model of ZF, then the function V_α is defined in $\mathfrak{M}$, and one can prove by induction

$$V_\alpha^{\mathfrak{M}} = V_\alpha \cap \mathfrak{M} \tag{12.12}$$

(because $\bigcup$ is an absolute operation).

We shall now prove absoluteness of the notion of constructibility and complete the proof of Theorem 32.

Lemma 12.5.

(i) *The function* $\alpha \mapsto L_\alpha$ *is absolute for transitive models of* ZF.

(ii) *If* $\mathfrak{M}$ *is a transitive model of* ZF *containing all ordinals, then* "*x is constructible*" *is absolute for* $\mathfrak{M}$ *and in fact* $L^{\mathfrak{M}} = L$.

As a consequence of (ii), $L^{\mathfrak{M}} = L$ and thus $L \subseteq \mathfrak{M}$, for any transitive model of ZF containing all ordinals. Also by (ii), if $x \in L$, then

$$L \vDash x \text{ is constructible}$$

and so L satisfies the axiom of constructibility. Thus the statements (ii) and (iii) of Theorem 32 follow from Lemma 12.5.

Proof. (ii) follows from (i); since $\mathfrak{M}$ is a model of ZF, the function $\alpha \mapsto L_\alpha^{\mathfrak{M}}$ is defined in $\mathfrak{M}$; and by absoluteness, we have for all x,

$$x \in L \leftrightarrow \exists\alpha(x \in L_\alpha) \leftrightarrow \exists\alpha(x \in L_\alpha^{\mathfrak{M}}) \leftrightarrow x \in L^{\mathfrak{M}}$$

To prove (i), let $\mathfrak{M}$ be a transitive model of ZF. Throughout the proof, we tacitly use the fact that each operation in question is defined in $\mathfrak{M}$ (since $\mathfrak{M}$ is a model of ZF).

The function $\alpha \mapsto L_\alpha$ is defined by induction:

$$(12.2) \qquad L_0 = \varnothing, \qquad L_\alpha = \bigcup_{\beta<\alpha} L_\beta \quad (\text{if } \alpha \text{ is a limit}), \qquad L_{\alpha+1} = \operatorname{def}(L_\alpha)$$

where

$$(12.1) \qquad u \in \operatorname{def}(U) \leftrightarrow u \in \operatorname{cl}(U \cup \{U\}) \wedge u \subseteq U$$

and $\operatorname{cl}(M)$ is defined by induction:

$$(12.13) \qquad \operatorname{cl}(M) = \bigcup \operatorname{ran}(W)$$

where

$$(12.14) \qquad W(0) = M,$$

$$W(n+1) = W(n) \cup \{\mathfrak{F}_i(X, Y) : X \in W(n), Y \in W(n), i = 1, \ldots, 10\}$$

To start, we show that in (12.14) $W(n+1)$ is obtained from $W(n)$ by an absolute operation. For all U, Z, we have

$$(12.15) \qquad Z = \{\mathfrak{F}_i(x, y) : x \in U, y \in U, i = 1, \ldots, 10\} \leftrightarrow$$

$$(\forall z \in Z)(\exists x, y \in U)[z = \mathfrak{F}_1(x, y) \vee \cdots \vee z = \mathfrak{F}_{10}(x, y)] \wedge$$

$$(\forall x, y \in U)(\exists z_1, \ldots, z_{10} \in Z)[z_1 = \mathfrak{F}_1(x, y) \wedge \cdots \wedge z_{10} = \mathfrak{F}_{10}(x, y)]$$

Since $z = \mathfrak{F}_i(x, y)$ is a restricted formula ($i = 1, \ldots, 10$), the right-hand side of (12.15) is also restricted and thus absolute. By Lemma 12.4, the function W in (12.14) is absolute and so, by (12.13),

$$\text{cl}^{\mathfrak{M}}(M) = \text{cl}(M) \tag{12.16}$$

for all $M \in \mathfrak{M}$. Using (12.1), we see that the formula

$$u \in \text{def}(U)$$

is absolute for $\mathfrak{M}$, and moreover, $\text{def}(U) \subseteq \mathfrak{M}$ for all $U \in \mathfrak{M}$. Thus the operation $\text{def}(U)$ is absolute for $\mathfrak{M}$:

$$\text{def}^{\mathfrak{M}}(U) = \text{def}(U) \tag{12.17}$$

for all $U \in \mathfrak{M}$.

Now we apply Lemma 12.4 to the inductive definition of L_α (12.2) and since both $\cup$ and def are absolute, we conclude that

$$L_\alpha^{\mathfrak{M}} = L_\alpha$$

for all ordinals $\alpha \in \mathfrak{M}$. ■

Exercise 12.14. If $\mathfrak{M}$ is a transitive model of ZF and $\mathfrak{M}$ is a set, then $L^{\mathfrak{M}} = L_\alpha$ where α is the least ordinal not in $\mathfrak{M}$.

Exercise 12.15. Let us define a sequence $\langle J_\alpha : \alpha \in Ord \rangle$ as follows: $J_0 = \emptyset$, $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ if α is limit, and $J_{\alpha+1} = \text{cl}(J_\alpha \cup \{J_\alpha\})$. Prove:

(i) Each J_α is transitive.
(ii) $\bigcup_{\alpha \in Ord} J_\alpha$ is a model of ZF (and so $L \subseteq \bigcup_{\alpha \in Ord} J_\alpha$).
(iii) The function $\alpha \mapsto J_\alpha$ is absolute for L and thus $L = \bigcup_{\alpha \in Ord} J_\alpha$.
(iv) The class $\mathbf{C} = \{\alpha : J_\alpha = L_\alpha\}$ is closed and unbounded.

Using the fact that $L \models$ AC (Section 13) one can show that for every regular $\kappa > \aleph_0$, $\mathbf{C} \cap \kappa$ is closed unbounded in κ.

Exercise 12.16. If $\mathfrak{M}$ is a transitive model of ZF, then $V_\omega \subset \mathfrak{M}$ and so "x is finite" is absolute for $\mathfrak{M}$.

We conclude this section by showing that the notion of well-foundedness is absolute for transitive models of ZF.

Lemma 12.6. *If* $\mathfrak{M}$ *is a transitive model of* ZF, *then the formula*

(12.18) *E is a well-founded relation over P*

is absolute for $\mathfrak{M}$.

Proof. On the one hand, we can write (12.18) as

(12.19) E is a relation over P and
$(\forall X)\,[\emptyset \neq X \subseteq P \rightarrow (\exists a \in X)\, a$ is E-minimal in $X]$.

If (12.19) holds (in the universe), then it holds in $\mathfrak{M}$ since both "E is a relation over P" and $[\cdots]$ are restricted formulas, and so $[\cdots]$ holds in $\mathfrak{M}$ for every $X \in \mathfrak{M}$.

On the other hand, E is well-founded if and only if there exists a function on P into *Ord* such that $f(x) < f(y)$ whenever $x\ E\ y$. Thus (12.18) is equivalent to

(12.20) E is a relation over P and $\exists f[f$ is a function $\wedge\ (\forall u \in \operatorname{ran}(f))\ u$ is an ordinal $\wedge\ (\forall x, y \in P)(x\ E\ y \rightarrow f(x) < f(y))]$

Since $[\cdots]$ is a restricted formula, if $[\cdots]$ holds in $\mathfrak{M}$ for some $f \in \mathfrak{M}$, then $\exists f[\cdots]$ holds; and so if (12.20) holds in $\mathfrak{M}$, it holds in V. ■

13. CONSISTENCY OF THE AXIOM OF CHOICE AND THE GENERALIZED CONTINUUM HYPOTHESIS

In the present section we show that the constructible universe L satisfies the axiom of choice and the generalized continuum hypothesis. Consequently, both AC and GCH are consistent with ZF.

Theorem 33 (Gödel). *There exists a well-ordering of the class L. Thus if $V = L$, we have a well-ordering of the universe, and so the axiom of constructibility implies the axiom of choice.*

Proof. We shall in fact show that the class L has a *definable* well ordering: we shall define a well-ordering $<$ of the constructible universe. In turn, we can define a function $\mathbf{F}$ on $L - \{\emptyset\}$ by

$$\mathbf{F}(X) = \text{the } <\text{-least element of } X$$

and have a definable choice function on L.

By induction, we construct for each α a well-ordering $<_\alpha$ of L_α. We do it in such a way that if $\alpha < \beta$, then $<_\beta$ is an *end-extension* of $<_\alpha$, i.e.,

(13.1) (a) if $x <_\alpha y$, then $x <_\beta y$;
(b) if $x \in L_\alpha$ and $y \in L_\beta - L_\alpha$, then $x <_\beta y$.

Notice that (13.1) implies

$$\text{(13.2)} \qquad \text{if} \quad x \in y \in L_\alpha, \qquad \text{then} \quad x <_\alpha y$$

First let us assume that α is a limit ordinal and that we have constructed $<_\beta$ for all $\beta < \alpha$ and that if $\beta_1 < \beta_2 < \alpha$, then $<_{\beta_2}$ is an end extension of $<_{\beta_1}$. In this case we simply let

$$\text{(13.3)} \qquad <_\alpha = \bigcup_{\beta<\alpha} <_\beta$$

i.e., if $x, y \in L_\alpha$, we let

$$\text{(13.4)} \qquad x <_\alpha y \quad \text{iff} \quad (\exists\beta < \alpha)\ x <_\beta y$$

Thus assume that we have defined $<_\alpha$ and let us construct $<_{\alpha+1}$, a well-ordering of $L_{\alpha+1}$. We recall the definition of $L_{\alpha+1}$:

$$L_{\alpha+1} = P(L_\alpha) \cap \operatorname{cl}(L_\alpha \cup \{L_\alpha\}) = P(L_\alpha) \cap \bigcup_{n=0}^{\infty} W_n^\alpha \tag{13.5}$$

where

$$W_0^\alpha = L_\alpha \cup \{L_\alpha\}, \tag{13.6}$$

$$W_{n+1}^\alpha = \{\mathfrak{F}_i(X, Y) : X, Y \in L_\alpha \cup \{L_\alpha\},\ i = 1, \ldots, 10\}$$

The idea of the construction of $<_{\alpha+1}$ is now as follows: First we take the elements of L_α, then L_α, then the remaining elements of W_1, then the remaining elements of W_2, etc. To order the elements of W_{n+1}, we use the already defined well-ordering of W_n since every $x \in W_{n+1}$ is equal to $\mathfrak{F}_i(u, v)$ for some $i = 1, \ldots, 10$ and some $u, v \in W_n$. We let

(13.7) (i) $<_{\alpha+1}^0$ is the well-ordering of $L_\alpha \cup \{L_\alpha\}$ that extends $<_\alpha$ and such that L_α is the last element

(ii) $<_{\alpha+1}^{n+1}$ is the following well-ordering of W_{n+1}:

$x <_{\alpha+1}^{n+1} y$ iff either: $x <_{\alpha+1}^{n} y$

or: $x \in W_n$ and $y \notin W_n$

or: $x \notin W_n$ and $y \notin W_n$ and

(a) the least i such that $\exists u, v \in W_n (x = \mathfrak{F}_i(u, v)) <$ the least j such that $\exists s, t \in W_n\ (y = \mathfrak{F}_j(s, t))$

or (b) the least $i =$ the least j and [the $<^n$-least $u \in W_n$ such that $\exists v \in W_n (x = \mathfrak{F}_i(u, v))] <^n$ [the $<^n$-least $s \in W_n$ such that $\exists t \in W_n\ (y = \mathfrak{F}_i(s, t))]$

or (c) the least $i =$ the least j and the least $u =$ the least s and [the $<^n$-least $v \in W_n$ such that $x = \mathfrak{F}_i(u, v)] <^n$ [the $<^n$-least $t \in W_n$ such that $y = \mathfrak{F}_i(u, t)]$

Now we let

$$<_{\alpha+1} = \bigcup_{n=0}^{\infty} <_{\alpha+1}^n \cap (P(L_\alpha) \times P(L_\alpha)) \tag{13.8}$$

and it is clear that $<_{\alpha+1}$ is an end-extension of $<_\alpha$ and is a well-ordering of $L_{\alpha+1}$.

Having defined $<_\alpha$ for all α, we let

$$x <_L y \quad \text{iff} \quad \exists \alpha\ x <_\alpha y$$

The relation $<_L$ is a well-ordering of L. ■

We call $<_L$ the *canonical well-ordering* of L.

Exercise 13.1. The sequence $\langle <_\alpha : \alpha \in Ord\rangle$ is absolute for models of ZF; consequently, the well-ordering $<$ of L is absolute.

Exercise 13.2. For all $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$.

The proof of the generalized continuum hypothesis in L is a combination of an absoluteness argument and the reflection principle.

Theorem 34 (Gödel). *If $V = L$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every α.*

The key element of the proof is the following lemma:

Lemma 13.1. *Assume that $V = L$. If $X \subseteq \omega_\alpha$, then there exists $\gamma < \omega_{\alpha+1}$ such that $X \in L_\gamma$.*

Proof of Theorem 34. Assume that $V = L$. By Lemma 13.1, every subset of ω_α is constructible before the stage $\omega_{\alpha+1}$, i.e., $P(\omega_\alpha) \subset L_{\omega_{\alpha+1}}$. It is easy to show by induction that for every $\gamma \geq \omega$, $|L_\gamma| = |\gamma|$, and so $|L_{\omega_{\alpha+1}}| = \aleph_{\alpha+1}$. Therefore, $|P(\omega_\alpha)| = \aleph_{\alpha+1}$. ■

To prove Lemma 13.1 we need a more subtle absoluteness argument than Lemma 12.5.

We say that a transitive set M is *adequate* if

(13.9) (a) M is closed;
(b) if $U \in M$, then $\mathrm{cl}(U) \in M$;
(c) if $\alpha \in M$, then $\langle L_\beta : \beta < \alpha\rangle \in M$.

Lemma 13.2.

(i) *The function $\alpha \mapsto L_\alpha$ is absolute for every adequate transitive set M.*
(ii) *If M is an adequate transitive set, then M satisfies the axiom of constructibility if and only if $M = L_\alpha$ for some α.*

Proof. (i) We follow closely the proof of Lemma 12.5. Since we do not assume that M is a model of ZF, we have to prove not only that certain operations are absolute for M but also that these operations are defined in M.

To start, for each $i = 1, \ldots, 10$, the formula

$$z = \mathfrak{F}_i(x, y) \tag{13.10}$$

is restricted; since M is closed, every Gödel operation is absolute for M and defined in M (see Lemma 12.3b).

Furthermore, the formula

$$Z = \{\mathfrak{F}_i(x, y) : x \in U, y \in U, i = 1, \ldots, 10\} \tag{13.11}$$

is restricted and thus absolute for M. Also, if $U \in M$, then $Z \in M$: If we denote $C = \mathrm{cl}(U)$, then

$$Z = \{u \in C : \exists x, y \in U[u = \mathfrak{F}_1(x, y) \vee \cdots \vee u = \mathfrak{F}_{10}(x)]\}$$

and so, by the normal form theorem, we have $Z = \mathfrak{F}(U, C)$ where $\mathfrak{F}$ is a Gödel operation.

Next we show that

$$x \in \mathrm{cl}(U) \tag{13.12}$$

is absolute for M. Given $U \in M$, let W be the function on ω defined as follows:

$$W(0) = U, \tag{13.13}$$
$$W(n+1) = W(n) \cup \{\mathfrak{F}_i(x, y) : x \in W(n), y \in W(n), i = 1, \ldots, 10\}$$

Since M is closed, it contains all natural numbers, and one can see (using induction) that $W \mid n \in M$ for all n. Thus the function W is absolute for M and defined in M (although W itself does not have to be in M), and since

$$x \in \mathrm{cl}(U) \quad \text{iff} \quad \exists n\, x \in W_n$$

the formula (13.12) is absolute for M.

Consequently, the operation $\mathrm{def}(U)$ is absolute for M, and also it is defined in M since

$$\mathrm{def}(U) = \{X \in C : X \subseteq U\} = \mathfrak{F}(C, U)$$

where $C = \mathrm{cl}(U \cup \{U\})$ and $\mathfrak{F}$ is a Gödel operation.

It remains to show that the function $\alpha \mapsto L_\alpha$ is absolute for M. Since the function is defined by induction, we follow Lemma 12.14. We have

(13.14) $y = L_\alpha$ iff α is an ordinal and $\exists f$ such that:
f is a function on α
and $(\forall \xi < \alpha)\, [\xi + 1 < \alpha \rightarrow f(\xi + 1) = \mathrm{def}(f(\xi))]$
and $(\forall \xi < \alpha)\, [\xi \text{ is a limit} \rightarrow f(\xi) = \bigcup \mathrm{ran}(f \mid \xi)]$

The operations $\mathrm{def}(X)$, $\bigcup X$, $\mathrm{ran}(f)$, and $f \mid \xi$ are absolute for M and defined in M (because $\mathrm{ran}(f)$ and $f \mid \xi$ are Gödel operations, as one can easily verify). Thus, using (13.9c), it follows that the formula $y = L_\alpha$ is absolute for M.

We conclude the proof of (i) with two remarks. First, notice that if $\alpha \in M$ then $L_\alpha \in M$; this is because $\alpha + 1 \in M$ and L_α is in the transitive closure of $\langle L_\beta : \beta < \alpha + 1 \rangle$, and M is transitive. Second, the formula $x \in L_\alpha$ is also absolute for M:

$$x \in L_\alpha \quad \text{iff} \quad \exists y (x \in y \wedge y = L_\alpha)$$

(ii) Let M be an adequate transitive set. If $M = L_\alpha$ for some α, then α is a limit ordinal since M is closed and $Ord \cap L_\alpha = \alpha$. Thus we have

$$(\forall x \in M)\, (\exists \beta \in M)\, x \in L_\beta \tag{13.15}$$

and since $x \in L_\beta$ is absolute for M,

$$M \models \forall x\, \exists \beta\, x \in L_\beta \tag{13.16}$$

i.e., M satisfies $V = L$.

On the other hand, if M satisfies $V = L$, then by absoluteness of $x \in L_\beta$ we get (13.15) and so $M = \bigcup \{L_\beta : \beta \in Ord^M\} = L_\alpha$ where $\alpha = Ord^M$. ■

An almost immediate consequence of the proof of part (i) of the preceding lemma is the following:

Lemma 13.3. *There is a sentence Θ (provable in* ZF) *such that for every transitive set M, M is adequate if and only if $M \vDash \Theta$.*

Proof. Let Θ be the conjunction of the sentences

$$\begin{aligned} (13.17)\quad &(a)\ \forall X, Y\ \exists Z\ \ Z = \mathfrak{F}_i(X, Y), \qquad i = 1, \ldots, 10;\\ &(b)\ \forall U\ \exists C\ \ C = \operatorname{cl}(U);\\ &(c)\ \forall \alpha\ \exists f\ \ f = \langle L_\beta : \beta < \alpha \rangle. \end{aligned}$$

If M is an adequate transitive set, then the above operations are absolute for M and defined in M and so $M \vDash \Theta$.

On the other hand, if M is transitive and satisfies Θ, then we proceed exactly as in the proof of Lemma 13.2(i) and show that the operations in (a), (b), (c) above are absolute for M and defined in M. It follows that M is adequate. ■

Exercise 13.3. The canonical well-ordering of L is absolute for all adequate transitive sets M.

Exercise 13.4. If $\{M_i : i \in I\}$ is a chain of adequate transitive sets (i.e., $M_i \subseteq M_j$ or $M_j \subseteq M_i$), then $\bigcup_{i \in I} M_i$ is transitive and adequate.

The following two exercises show that many L_α, in particular all L_κ where κ is an uncountable cardinal, are adequate. Later we shall show (Lemma 30.1) that by carefully writing the definition of L_α, one can prove that the function $\alpha \mapsto L_\alpha$ is absolute for every L_λ where λ is a limit ordinal. Similarly, Exercise 13.13 holds for every limit ordinal α.

Exercise 13.5. For every $\beta \geq \omega$, there is $\alpha \geq \beta$ such that $|\alpha| = |\beta|$ and L_α is adequate.

[By the reflection principle, let $S \supseteq \beta$ be such that S satisfies Extensionality, $V = L$ and Θ, and that $|S| = |\beta|$. Let M be the transitive collapse of S. Then M is adequate and $M = L_\alpha$.]

Exercise 13.6. If κ is a regular uncountable cardinal, then $\{\alpha < \kappa : L_\alpha \text{ is adequate}\}$ is closed unbounded in κ.

*Exercise 13.7.** Let M be a transitive model of ZF$^-$, set theory without the power-set axiom. Then M is adequate.

[The theorem on transfinite induction, Lemmas 12.4 and 12.5 work as well for the theory ZF$^-$.]

Now we are ready to prove the crucial lemma for Theorem 34:

Proof of Lemma 13.1. We assume that $V = L$ and let $X \subseteq \omega_\alpha$. Since X is constructible, there exists β such that $X \in L_\beta$.

By the reflection principle, there exists a set S such that:

(13.18) ( ) $S \vDash$ Extensionality;
(i) $S \vDash \Theta$;
(ii) $\omega_\alpha \subseteq S$, $X \in S$, $\beta \in S$;
(iii) $|S| = \aleph_\alpha$;
(iv) $S \vDash \beta$ is an ordinal and $X \in L_\beta$.

(We can find S satisfying (iii) since the axiom of choice holds.) Since S is extensional, there exists an isomorphism π of S onto a transitive set $M = \pi[S]$ and we have

(13.19) (i) M is adequate;
(ii) $\pi(\xi) = \xi$ for all $\xi < \omega_\alpha$, $\pi(X) = X$;
(iii) $|M| = \aleph_\alpha$;
(iv) $M \vDash \pi(\beta)$ is an ordinal and $\pi(X) \in L_{\pi(\beta)}$.

Ordinals are absolute and so $\gamma = \pi(\beta)$ is an ordinal and we have

$$M \vDash X \in L_\gamma$$

Since M is adequate, the formula $X \in L_\gamma$ is absolute for M and it follows that $X \in L_\gamma$. However, γ belongs to M and M is transitive and $|M| = \aleph_\alpha$. Thus necessarily $\gamma < \omega_{\alpha+1}$ and we are done. ■

Exercise 13.8. If $\alpha \geq \omega$ and X is a constructible subset of α, then $X \in L_\beta$, where β is the least cardinal in L greater than α.

[Do the construction (13.18) in L, and choose $S \supseteq \alpha$.]

Exercise 13.9. If X is a constructible set, then $X \in L_\alpha$ where α is the least infinite cardinal in L greater than $|\mathrm{TC}(X)|^L$.

[In (13.18) choose $S \supseteq \mathrm{TC}(X)$.]

Exercise 13.10. The canonical well-ordering of L, restricted to the set $\boldsymbol{R}^L = \boldsymbol{R} \cap L$ of all constructible reals, has order type ω_1^L.

[$\boldsymbol{R} \cap L \subset L_{\omega_1^L}$.]

Exercise 13.11. If κ is a regular uncountable cardinal in L, then L_κ is a model of ZF$^-$.

[Prove it in L. Replacement: (i) If $X \in L_\kappa$, then $|X| < \kappa$; (ii) if $Y \subseteq L_\kappa$ and $|Y| < \kappa$, then $Y \in L_\kappa$.]

Exercise 13.12. If κ is inaccessible in L, then $L_\kappa = V_\kappa^L = V_\kappa \cap L$ and L_κ is a model of ZFC + ($V = L$).

Exercise 13.13. If L_α is adequate and $X \prec \langle L_\alpha, \in\rangle$, then X is isomorphic to some L_β, $\beta \leq \alpha$.

Exercise 13.14.* If L_α is adequate, then the model $\langle L_\alpha, \in\rangle$ has definable Skolem functions. Therefore, for every $X \subset L_\alpha$, there exists a smallest $M \prec \langle L_\alpha, \in\rangle$ such that $X \subseteq M$.

[The well-ordering $<_\alpha$ is definable in $\langle L_\alpha, \in\rangle$. Let $h_\varphi(x)$ = the $<_\alpha$-least y such that $\langle L_\alpha, \in\rangle \vDash \varphi[x, y]$.]

Exercise 13.15.* Assume $V = L$. If $M \prec \langle L_{\omega_1}, \in\rangle$, then $M = L_\alpha$ for some α.

[Show that M is transitive. Let $X \in M$. Let f be the $<$-least mapping of ω onto X. Since f is definable in $\langle L_{\omega_1}, \in\rangle$ from X, f is in M. Hence $f(n) \in M$ for each n and we get $X \subseteq M$.]

Exercise 13.16.* Assume $V = L$. If $M \prec \langle L_{\omega_2}, \in\rangle$, then $\omega_1 \cap M = \alpha$ for some $\alpha \leq \omega_1$.

[Same argument as in Exercise 13.15: If $\gamma < \omega_1$ and $\gamma \in M$, then $\gamma \subset M$.]

14. THE Σ_n HIERARCHY OF CLASSES, RELATIONS, AND FUNCTIONS

We shall digress slightly from our investigation of transitive models and look more closely into the complexity of definition of the fundamental notions of set theory. Our starting point is the fact that restricted formulas are absolute for all transitive models, and the understanding that the more unrestricted quantifiers are needed to define a concept, the more complex is the concept.

A formula φ is a *Σ_0-formula* if it has only restricted quantifiers $\exists x \in y$ and $\forall x \in y$ (i.e., if φ is restricted). A *Σ_1-formula* is a formula of the form

$$\exists x \varphi(x, \ldots)$$

where $\varphi(x, \ldots)$ is a Σ_0-formula. A *Π_1-formula* is a formula of the form

$$\forall x \varphi(x, \ldots)$$

where $\varphi(x, \ldots)$ is a Σ_0-formula. Similarly, $\Sigma_2, \Sigma_3, \Sigma_4, \ldots, \Pi_2, \Pi_3, \Pi_4, \ldots,$ formulas are of the form

$$\begin{array}{llll} \exists x\, \forall y\, \varphi, & \exists x\, \forall y\, \exists z\, \varphi, & \exists x\, \forall y\, \exists z\, \forall u\, \varphi, & \ldots \\ \forall x\, \exists y\, \varphi, & \forall x\, \exists y\, \forall z\, \varphi, & \forall x\, \exists y\, \forall z\, \exists u\, \varphi, & \ldots \end{array}$$

where φ is a Σ_0-formula.

A class **C** (a property **P**, a relation **R**) is a *Σ_n-class* (*Σ_n-property*, *Σ_n-relation*) if $x \in \mathbf{C}$ ($\mathbf{P}(x)$, $\mathbf{R}(x, y)$) can be written as a Σ_n-formula. Similarly for Π_n. A function **F** is a *Σ_n-function* if the relation

$$y = \mathbf{F}(x)$$

is a Σ_n-relation. [Note that while Σ_n-ness of a formula is a purely syntactical notion, the Σ_n-ness of a class or a function depends on the axioms of ZF: Using the axioms, we may find an equivalent way of expressing $x \in \mathbf{C}$.]

A property **P** is a *Δ_n-property* if it is both Σ_n and Π_n (similarly for relations and functions). That is, there is a Σ_n-formula φ and a Π_n-formula ψ such that

$$\mathbf{P}(x) \leftrightarrow \varphi(x) \leftrightarrow \psi(x)$$

It should be noted that if a concept is Σ_n (or Π_n), then it is both Σ_{n+1} and Π_{n+1} (just add a "dummy" quantifier). Also, every definable concept is Σ_n for some n

(write the definition in the prenex normal form). Thus we talk about the "Σ_n-hierarchy."

We have already shown (Lemma 12.1, Exercise 12.9) that the following notions are Σ_0 : $\mathfrak{F}_i(X, Y)$, $i = 1, \ldots, 10$, $x \subseteq y$, x is empty, x is an ordered pair, x is transitive, x is an ordinal, x is a limit ordinal, x is a natural number, $x = \omega$, x is a relation, f is a function, $y = f(x)$, $g = f \mid X$, $Y = f[X]$, f is one-to-one, X is an ordering.

Lemma 14.1. *Every Gödel operation is a Σ_0-function.*

Proof. We already know that $Z = \mathfrak{F}_i(X, Y)$ is Σ_0 for $i = 1, \ldots, 10$. We want to show that $Z = \mathfrak{F}(X_1, \ldots, X_n)$ is Σ_0 whenever $\mathfrak{F}$ is a composition of $\mathfrak{F}_1, \ldots, \mathfrak{F}_{10}$.

We show it by induction on the complexity of $\mathfrak{F}$. In fact, we simultaneously show the following:

(14.1) $u \in \mathfrak{F}(X, \ldots)$ is Σ_0.

(14.2) If φ is Σ_0, then so are $(\forall u \in \mathfrak{F}(X, \ldots))\varphi$ and $(\exists u \in \mathfrak{F}(X, \ldots))\varphi$.

(14.3) $Z = \mathfrak{F}(X, \ldots)$ is Σ_0.

(14.4) If φ is Σ_0, then so is $\varphi(\mathfrak{F}(X, \ldots))$.

We shall prove (14.1) and (14.2) only for a typical example and entrust the execution of the full proof to the reader. In (14.1) consider the formula

$$u \in \mathfrak{F}(X, \ldots) \times \mathfrak{G}(X, \ldots)$$

This can be written as

$$\exists x \in \mathfrak{F}(X, \ldots)\ \exists y \in \mathfrak{G}(X, \ldots)\ u = (x, y)$$

In (14.2), consider the formula

$$\forall u \in \{\mathfrak{F}(X, \ldots), \mathfrak{G}(X, \ldots)\}\ \varphi(u)$$

This can be written as

$$\varphi(\mathfrak{F}(X, \ldots)) \wedge \varphi(\mathfrak{G}(X, \ldots))$$

(14.3) then follows from (14.1) and (14.2):

$$Z = \mathfrak{F}(X, \ldots) \quad \text{iff} \quad (\forall u \in Z)\ u \in \mathfrak{F}(X, \ldots) \wedge \forall u \in \mathfrak{F}(X, \ldots)\ u \in Z$$

To prove (14.4), let φ be a Σ_0-formula. Then in $\varphi(\mathfrak{F}(X, \ldots))$, $\mathfrak{F}(X, \ldots)$ occurs in the form

$$u \in \mathfrak{F}(X, \ldots), \qquad \mathfrak{F}(X, \ldots) \in u, \qquad Z = \mathfrak{F}(X, \ldots),$$
$$\forall u \in \mathfrak{F}(X, \ldots), \qquad \exists u \in \mathfrak{F}(X, \ldots)$$

Since $\mathfrak{F}(X, \ldots) \in u$ can be replaced by $(\exists v \in u)\ v = \mathfrak{F}(X, \ldots)$, we use (14.1)–(14.3) to show that $\varphi(\mathfrak{F}(X, \ldots))$ is a Σ_0-property. ■

The following lemma is of practical value—using it, one can tell when a relation or function is Σ_n, Π_n, or Δ_n.

Lemma 14.2. *Let* $n \geq 1$.

(i) *If* $\mathbf{P}$, $\mathbf{Q}$ *are* Σ_n*-properties, then so are* $\exists x \mathbf{P}$, $\mathbf{P} \wedge \mathbf{Q}$, $\mathbf{P} \vee \mathbf{Q}$, $(\exists u \in x)\, \mathbf{P}$, $(\forall u \in x)\, \mathbf{P}$.
(ii) *If* $\mathbf{P}$, $\mathbf{Q}$ *are* Π_n*-properties, then so are* $\forall x \mathbf{P}$, $\mathbf{P} \wedge \mathbf{Q}$, $\mathbf{P} \vee \mathbf{Q}$, $(\forall u \in x)\, \mathbf{P}$, $(\exists u \in x)\, \mathbf{P}$.
(iii) *If* $\mathbf{P}$ *is* Σ_n, *then* $\neg \mathbf{P}$ *is* Π_n *; if* $\mathbf{P}$ *is* Π_n, *then* $\neg \mathbf{P}$ *is* Σ_n.
(iv) *If* $\mathbf{P}$ *is* Π_n *and* $\mathbf{Q}$ *is* Σ_n, *then* $\mathbf{P} \rightarrow \mathbf{Q}$ *is* Σ_n *; if* $\mathbf{P}$ *is* Σ_n *and* $\mathbf{Q}$ *is* Π_n, *then* $\mathbf{P} \rightarrow \mathbf{Q}$ *is* Π_n.
(v) *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are* Δ_n, *then so are* $\neg \mathbf{P}$, $\mathbf{P} \wedge \mathbf{Q}$, $\mathbf{P} \vee \mathbf{Q}$, $\mathbf{P} \rightarrow \mathbf{Q}$, $\mathbf{P} \leftrightarrow \mathbf{Q}$, $(\forall u \in x)\, \mathbf{P}$, $(\exists u \in x)\, \mathbf{P}$.
(vi) *If* $\mathbf{F}$ *is a* Σ_n*-function, then* dom(**F**) *is a* Σ_n*-class.*
(vii) *If* $\mathbf{F}$ *is a* Σ_n*-function and* dom(**F**) *is* Δ_n, *then* $\mathbf{F}$ *is* Δ_n.
(viii) *If* $\mathbf{F}$ *and* $\mathbf{G}$ are Σ_n*-functions, then so is* $\mathbf{F} \circ \mathbf{G}$.
(ix) *If* $\mathbf{F}$ is a Σ_n*-function and if* $\mathbf{P}$ *is a* Σ_n*-property, then* $\mathbf{P}(\mathbf{F}(x))$ *is* Σ_n.

Proof. Let us prove the lemma for $n = 1$. The general case follows easily by induction.

(i) Let

$$\begin{aligned} \mathbf{P}(x, \ldots) &\leftrightarrow \exists z\, \varphi(z, x, \ldots) \\ \mathbf{Q}(x, \ldots) &\leftrightarrow \exists u\, \psi(u, x, \ldots) \end{aligned}$$

where φ and ψ are Σ_0-formulas. We have

$$\begin{aligned} \exists x\, \mathbf{P}(x, \ldots) &\leftrightarrow \exists x\, \exists z\, \varphi(z, x, \ldots) \\ &\leftrightarrow \exists v\, \exists w \in v\, \exists x \in w,\, \exists z \in w[v = (x, z) \wedge \varphi(z, x, \ldots)] \end{aligned} \tag{14.5}$$

The right-hand side of (14.5) is a Σ_1-formula. Furthermore,

$$\begin{aligned} \mathbf{P}(x, \ldots) \wedge \mathbf{Q}(x, \ldots) &\leftrightarrow \exists z\, \exists u[\varphi(z, x, \ldots) \wedge \psi(u, x, \ldots)] \\ \mathbf{P}(x, \ldots) \vee \mathbf{Q}(x, \ldots) &\leftrightarrow \exists z\, \exists u[\varphi(z, x, \ldots) \vee \psi(u, x, \ldots)] \\ (\exists u \in x)\, \mathbf{P}(x, \ldots) &\leftrightarrow \exists z\, \exists u[u \in x \wedge \varphi(z, x, \ldots)] \end{aligned}$$

To show that $(\forall u \in x)\, \mathbf{P}$ is a Σ_1-property, we use the collection principle (9.4):

$$\begin{aligned} (\forall u \in x)\mathbf{P}(x, \ldots) &\leftrightarrow (\forall u \in x)\, \exists z\, \varphi(z, x, \ldots) \\ &\leftrightarrow \exists y (\forall u \in x)\, (\exists v \in y)\, \varphi(z, x, \ldots) \end{aligned}$$

(ii) follows from (i) and (iii).

(iii)

$$\begin{aligned} \neg \exists z\, \varphi(z, x, \ldots) &\leftrightarrow \forall z\, \neg \varphi(z, x, \ldots) \\ \neg \forall z\, \varphi(z, x, \ldots) &\leftrightarrow \exists z\, \neg \varphi(z, x, \ldots) \end{aligned}$$

(iv)

$$(\mathbf{P} \to \mathbf{Q}) \quad \leftrightarrow \quad (\neg \mathbf{P} \vee \mathbf{Q})$$

(v) follows from (i)–(iv).

(vi)

$$x \in \operatorname{dom}(\mathbf{F}) \leftrightarrow \exists y\ y = \mathbf{F}(x)$$

(vii) Since **F** is a function, we have

$$y = \mathbf{F}(x) \quad \leftrightarrow \quad x \in \operatorname{dom}(\mathbf{F}) \wedge \forall z[z = \mathbf{F}(x) \to y = z] \tag{14.6}$$

If $z = \mathbf{F}(x)$ is Σ_n and $x \in \operatorname{dom}(\mathbf{F})$ is Π_n, then the right-hand side of (14.6) is Π_n.

(viii)

$$y = \mathbf{F}(\mathbf{G}(x)) \quad \leftrightarrow \quad \exists z[z = \mathbf{G}(x) \ \wedge \ y = \mathbf{F}(z)]$$

(ix)

$$\mathbf{P}(\mathbf{F}(x)) \quad \leftrightarrow \quad \exists y[y = \mathbf{F}(x) \wedge \mathbf{P}(y)] \quad \blacksquare$$

Since Σ_0-properties are absolute for all transitive models, it is clear that Σ_1-properties are *upward absolute*: If $\mathbf{P}(x)$ is Σ_1, if $\mathfrak{M}$ is a transitive model of ZF and if $\mathfrak{M} \vDash \mathbf{P}(x)$, then $\mathbf{P}(x)$. Similarly, the Π_1-properties are *downward absolute*, and consequently, Δ_1-properties are absolute for transitive models of ZF.

In Lemma 12.6, we showed that well-foundedness is absolute for transitive models ZF. In fact, the proof shows that the property

(12.18) $\quad E$ is a well-founded relation over P

is both Σ_1 (12.20) and Π_1 (12.19). Thus we have

Lemma 14.3. *"E is a well-founded relation over P" is* Δ_1. $\blacksquare$

Exercise 14.1. Show that "X is finite" is Δ_1.

[To get a Π_1-formulation, use T-finiteness from Section 2.]

The power-set operation $P(X)$ is obviously Π_1; since it is not absolute as we shall have the opportunity to see in Chapter 3, it is not Σ_1. Similarly, the cardinal concepts are Π_1 but not Σ_1:

Lemma 14.4. *"α is a cardinal," "α is a regular cardinal," and "α is a limit cardinal" are* Π_1.

Proof. (a) $\neg \exists f[f$ is a function and $\operatorname{dom}(f) \in \alpha$ and $\operatorname{ran}(f) = \alpha]$.

(b) $\alpha > 0$ is a limit ordinal and

$$\neg \exists f[f \text{ is a function and } \operatorname{dom}(f) \in \alpha \text{ and } \textstyle\bigcup \operatorname{ran}(f) = \alpha].$$

(c) $(\forall \beta < \alpha)\ (\exists \gamma < \alpha)\ [\beta < \gamma$ and γ is a cardinal$]$. $\blacksquare$

Exercise 14.2. "X is countable" is Σ_1.

Exercise 14.3. $|X| \leq |Y|$, $|X| = |Y|$ are Σ_1.

Lemma 14.5. *Let $n \geq 1$, let* $\mathbf{G}$ *be a Δ_n-function on* V, *and let* $\mathbf{F}$ *be defined by induction:*

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F} \mid \alpha)$$

Then $\mathbf{F}$ *is a Δ_n-function on Ord.*

Proof. Since *Ord* is a Σ_0 (and hence Π_n) class, it is enough to verify that the following expression is Σ_n:

$$(14.7)\quad y = \mathbf{F}(\alpha) \quad \text{iff}$$
$$\exists f[f \text{ is a function} \wedge \operatorname{dom}(f) = \alpha \wedge (\forall \xi < \alpha) f(\xi) = \mathbf{G}(f \mid \xi) \wedge y = \mathbf{G}(f)]$$

All the properties and operations in (14.7) are Σ_0 and $\mathbf{G}$ is Δ_n, and hence $y = \mathbf{F}(\alpha)$ is Σ_n. ∎

Exercise 14.4. The functions $\alpha + \beta$ and $\alpha \cdot \beta$ are Δ_1.

Exercise 14.5. The canonical and well-ordering of $Ord \times Ord$ is a Σ_0-relation. The function Γ is Δ_1.

Exercise 14.6. The function $S \mapsto \operatorname{TC}(S)$ is Δ_1.

Exercise 14.7. If $\mathbf{G}$ is a Σ_n-function ($n \geq 1$) and $\mathbf{F}$ is defined by $\in$-induction using $\mathbf{G}$, then $\mathbf{F}$ is a Σ_n-function.

Exercise 14.8. The function $x \mapsto \operatorname{rank}(x)$ is Δ_1.

Theorem 35.

(a) *The function $\alpha \mapsto L_\alpha$ is Δ_1.*
(b) *"x is constructible" is a Σ_1-property.*
(c) *The canonical well-ordering of L is a Σ_1-relation.*

Proof. (a) We intend to show that the relation

$$(14.8)\qquad y = L_\alpha$$

is Σ_1. Since (14.8) is defined by induction, it suffices to show that the induction step is Σ_1. Obviously, it suffices to verify that the function

$$(14.9)\qquad Y = \operatorname{cl}(M)$$

is Σ_1.

We recall that by (12.15), the relation

$$Z = \{\mathfrak{F}_i(x, y) : x \in U, y \in U, i = 1, \ldots, 10\}$$

is Σ_0. By (12.13) we have

$$(14.10)\quad Y = \mathrm{cl}(M) \leftrightarrow$$

$$\exists W[W \text{ is a function} \wedge \mathrm{dom}(W) = \omega \wedge W(0) = M$$
$$\wedge\ (\forall n \in \mathrm{dom}(W))(W(n+1) = W(n) \cup \{\mathfrak{F}_i(x, y) : x \in W(n), y \in W(n),$$
$$i = 1, \ldots, 10\})$$
$$\wedge\ Y = \bigcup \mathrm{ran}(W)]$$

and so $\mathrm{cl}(M)$ is a Δ_1-operation.

(b) $x \in L$ iff $\exists\alpha\ \exists y(x \in y \wedge y = L_\alpha)$.

(c) Since

$$x < y \qquad \text{iff} \qquad \exists\alpha(x <_\alpha y)$$

it suffices to prove that the function

$$\alpha \mapsto <_\alpha$$

which assigns to each α the canonical well-ordering of L_α is Δ_1.

The function $\alpha \mapsto <_\alpha$ is defined by induction and thus it suffices to show that the induction step is Σ_1. In fact, $<_{\alpha+1}$ is defined by induction from $<_\alpha$ (see (13.8), (13.7)). It suffices to verify that $<_{\alpha+1}$ is obtained from $<_\alpha$ by means of a Δ_1-operation (similar to the way in which $L_{\alpha+1}$ is obtained from L_α by $L_{\alpha+1} = \mathrm{def}(L_\alpha)$). The operation that yields $<_{\alpha+1}$ when applied to $<_\alpha$ is described in detail in (13.7). It can be written in a Σ_1-fashion in very much the same way as $\mathrm{cl}(M)$ is in (14.10). The only potential difficulty might be the use of the words "the $<$-least," and that can be overcome as follows: For example, in (13.7)(ii)(c)

$$\text{the } <^n\text{-least } v \in W_n \text{ such that } x = \mathfrak{F}_i(u, v) <^n$$
$$\text{the } <^n\text{-least } t \in W_n \text{ such that } y = \mathfrak{F}_i(u, t)$$

can be written as

$$(\exists v \in W_n)[x = \mathfrak{F}_i(u, v) \wedge (\forall t \in W_n)\ (y = \mathfrak{F}_i(u, t) \rightarrow v <^n t)] \quad \blacksquare$$

Exercise 14.9. Let σ be a theorem of ZF and let $\mathbf{P}(x)$ be a property that is upward absolute for transitive models of σ. Then $\mathbf{P}(x)$ is Σ_1.

[$\mathbf{P}(x) \leftrightarrow \exists M(M$ is transitive, $x \in M$, and $M \vDash \sigma \wedge \mathbf{P}(x))$.]

This gives another proof of the Σ_1-ness of "x is constructible."

A function $\mathbf{F}(X_1, \ldots, X_n)$ is *primitive recursive* if it is obtained by the following rules:

(i) Every Gödel operation is a primitive recursive function.
(ii) $\mathbf{F}(X_1, \ldots, X_n) = \omega$ is a primitive recursive function.

(iii) The composition of primitive recursive functions is primitive recursive.

(iv) Induction (Recursion Schema): If G is primitive recursive, then the function F defined by

$$\mathbf{F}(X, X_1, \dots, X_n) = \mathbf{G}(X, X_1, \dots, X_n, \mathbf{F} \mid X)$$

is primitive recursive.

Exercise 14.10. Every primitive recursive function is Δ_1.

Exercise 14.11. The following functions are primitive recursive: $\alpha + \beta$, $\alpha \cdot \beta$, $\mathrm{TC}(S)$, $\mathrm{rank}(x)$, $\alpha \mapsto L_\alpha$, $\alpha \mapsto <_\alpha$.

Exercise 14.12.* The following function is primitive recursive and hence Δ_1:

$\langle M, E\rangle \mapsto \{\langle \varphi, a_1, \dots, a_n\rangle : \varphi \in Form, \langle a_1, \dots, a_n\rangle \in {}^n M$ and $\langle M, E\rangle \vDash \varphi[a_1, \dots, a_n]\}$

The Lévy–Shoenfield Absoluteness Lemma

It follows easily from the definition that Σ_1-properties are upward absolute: If $\mathbf{P}(x)$ is a Σ_1-property and $\mathfrak{M}$ a transitive model of ZF, and if $\mathbf{P}(x)$ holds in $\mathfrak{M}$, then $\mathbf{P}(x)$ holds in the universe.

We shall now present a very powerful theorem which states that if a Σ_1-sentence σ is true in the universe, then it is true in L, and so in every transitive model containing all ordinals.

Theorem 36 (*Lévy–Shoenfield Absoluteness Lemma*). *Let σ be a Σ_1-sentence, $\sigma = \exists x \varphi(x)$, where $\varphi(x)$ is a Σ_0-formula of one free variable. If there is an x such that $\varphi(x)$, then there is a constructible x such that $\varphi(x)$:*

$$\exists x\, \varphi(x) \quad \rightarrow \quad (\exists x \in L)\, \varphi(x)$$

and so

$$\sigma \quad \leftrightarrow \quad L \vDash \sigma$$

In fact, if $\theta = \omega_1^L$, we have

$$\exists x\, \varphi(x) \quad \rightarrow \quad (\exists x \in L_\theta)\, \varphi(x)$$

Consequently, σ is absolute for all transitive M such that $M \supseteq L_\theta$.

Remark. It is important that σ is a Σ_1-*sentence*; an analogous theorem would be false for a Σ_1-formula $\psi(x)$: For example, "x is countable" can be written as a Σ_1-formula, and it is by no means provable that a countable constructible set is countable in L.

We shall use methods of model theory to prove Theorem 36. The theorem follows from Lemma 14.6.

We say that a model $\mathfrak{A} = \langle A, E, R_1, \dots, R_k\rangle$ is *well-founded* if E is a well-founded relation over A.

Lemma 14.6. *Let Θ be a sentence true in a well-founded model $\mathfrak{A} = \langle A, E, R_1, \ldots, R_k\rangle$. Then Θ has a well-founded* constructible *model $\mathfrak{B}$, and moreover $L \models \mathfrak{B}$ is countable.*

Proof of Theorem 36. Let $\varphi(x)$ be a Σ_0-formula and assume that

$$\exists x\ \varphi(x)$$

By the reflection principle, there exists a transitive set A such that

$$A \models \exists x\ \varphi(x)$$

Let Θ be the conjunction of the axiom of extensionality and $\exists x\ \varphi(x)$. Since $A \models \Theta$, there exists, by Lemma 14.6, a well-founded constructible model (B, E) of Θ, and B is countable in L. Applying the collapsing mapping, we obtain, in L, a countable transitive model M of $\exists x\ \varphi(x)$. If we denote $\theta = \omega_1^L$, then clearly $M \subseteq L_\theta$, and since φ is Σ_0, we have

$$L_\theta \models \exists x\ \varphi(x) \quad \blacksquare$$

Now the model theoretical methods. A sentence (of any language) is a $\forall\exists$-sentence if it has the form

$$\forall x_1 \cdots x_m\ \exists y_1 \cdots y_n\ \psi(x_1, \ldots, x_m, y_1, \ldots, y_n) \tag{14.11}$$

where ψ is quantifier-free.

Lemma 14.7. *Let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_k \subseteq \cdots$ be a chain of models, and let Θ be a $\forall\exists$-sentence* (14.11). *Assume that for each k, if $x_1, \ldots, x_m \in A_k$ then there are $y_1, \ldots, y_n \in A_{k+1}$ such that*

$$\mathfrak{A}_{k+1} \models \psi[x_1, \ldots, x_m, y_1, \ldots, y_n] \tag{14.12}$$

Then Θ is true in $\mathfrak{A} = \bigcup_{k=1}^\infty \mathfrak{A}_k$.

Proof. If $x_1, \ldots, x_m \in \bigcup_{k=0}^\infty A_k$, then there exists k such that $x_1, \ldots, x_m \in A_k$. Then we find $y_1, \ldots, y_n \in A_{k+1}$ such that (14.12) holds and since ψ is quantifier-free, we get $\mathfrak{A} \models \Theta$. $\blacksquare$

Lemma 14.8 (Skolem's Normal Form Theorem). *Let φ be a sentence of a language $\mathscr{L}$. Then there is a $\forall\exists$-sentence $\bar{\varphi}$ with additional predicates $R_1, \ldots, R_k$ such that:*

(a) *If $\mathfrak{A} \models \varphi$, then there are relations $R_1, \ldots, R_k$ such that $\langle\mathfrak{A}, R_1, \ldots, R_k\rangle \models \bar{\varphi}$.*
(b) *If $\langle\mathfrak{A}, R_1, \ldots, R_k\rangle \models \bar{\varphi}$, then $\mathfrak{A} \models \varphi$.*

Proof. We give the proof of a special case. (In the general case, first write φ in the prenex normal form, i.e., quantifiers first, and then proceed as in the following example.) Let φ be the sentence

$$\forall u\ \exists v\ \forall x\ \exists y\ \psi(u, v, x, y)$$

We replace $\exists y\,\psi$ by $R(u, v, x)$ and $\forall x\,\exists y\,\psi$ by $S(u, v)$, i.e., we replace φ by

$$(14.13)\quad \forall u\,\exists v\,S(u, v) \wedge \forall uv[S(u, v) \leftrightarrow \forall x\,R(u, v, x)] \wedge \forall uvx[R(u, v, x) \leftrightarrow \exists y\,\psi(u, v, x, y)]$$

To show that the sentence (14.13) can be written in a $\forall\exists$-way, it suffices to express it as a conjunction of $\forall\exists$-sentences. The second part of (14.13) is equivalent to

$$\forall uvx[S(u, v) \rightarrow R(u, v, x)] \wedge \forall uv\,\exists x[R(u, v, x) \rightarrow S(u, v)]$$

The third part of (14.13) is equivalent to

$$\forall uvx\,\exists y[R(u, v, x) \rightarrow \psi(u, v, x, y)] \wedge \forall uvxy[\psi(u, v, x, y) \rightarrow R(u, v, x)] \quad \blacksquare$$

Proof of Lemma 14.6. Let $\mathfrak{A} = \langle A, E, R_1, \ldots, R_k\rangle$ and let Θ be a sentence true in $\mathfrak{A}$. In view of Lemma 14.8, we may assume that Θ is a $\forall\exists$-sentence,

$$\forall x_1 \cdots x_m\,\exists y_1 \cdots y_n\,\psi(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

(ψ is quantifier-free). We also assume that Θ does not have a finite model (otherwise we are done). We shall construct $\mathfrak{B}$ as the union of a constructible ω-chain of finite models satisfying (14.12). Clearly, such $\mathfrak{B}$ is constructibly countable, and by Lemma 14.7, $\mathfrak{B}$ is a model of Θ. To be sure that $\mathfrak{B}$ is well-founded, we also construct a witness to it, a mapping of B into ordinals.

We say that a function $f\colon B \rightarrow Ord$ *demonstrates* that $\mathfrak{B} = \langle B, E^{\mathfrak{B}}, \ldots\rangle$ is well-founded if

$$u\,E^{\mathfrak{B}}\,v \qquad \text{implies} \qquad f(u) < f(v)$$

for all $u, v \in B$.

Let ρ be the function on A defined by

$$\rho(x) = \sup\{\rho(y) + 1 : yEx\}$$

and let α be the length of E (cf. (2.10), (2.11)).

Let P be the set of all pairs $(\mathfrak{B}, f)$ such that:

(14.14) (i) $\mathfrak{B}$ is a finite model of the language of $\mathfrak{A}$ and $B \subset \omega$;
(ii) $f\colon B \rightarrow \alpha$ demonstrates that $\mathfrak{B}$ is well-founded.

Let us define a partial ordering of P as follows:

$$(14.15)\qquad (\mathfrak{B}, f) < (\mathfrak{B}', f') \qquad \text{if:}$$

(i) $\mathfrak{B}'$ is a submodel of $\mathfrak{B}$;
(ii) $f' \subseteq f$;
(iii) for all $x_1, \ldots, x_m \in B'$, there exist $y_1, \ldots, y_n \in B$ such that

$$\mathfrak{B} \vDash \psi[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

It is obvious from the definition (14.14), (14.15) that the partially ordered set $(P, <)$ is constructible.

We shall show that $(P, <)$ is not well-founded. Let us first see how that gives a proof of the lemma. Since being well-founded is absolute, L satisfies that $(P, <)$ is not well-founded, and so there exists a constructible descending chain

$$(\mathfrak{B}_0, f_0) > (\mathfrak{B}_1, f_1) > \cdots > (\mathfrak{B}_n, f_n) > \cdots$$

If we let $\mathfrak{B} = \bigcup_{n=0}^{\infty} \mathfrak{B}_n$ and $f = \bigcup_{n=0}^{\infty} f_n$, then it is easy to see that $\mathfrak{B} \models \Theta$ and that f demonstrates that $\mathfrak{B}$ is well-founded.

Thus it suffices to show that $(P, <)$ is not well founded. This can be done by exhibiting a nonempty $X \subseteq P$ without a $<$-minimal element. Let X be the set of all $(\mathfrak{B}, f) \in P$ such that:

(a) $\mathfrak{B}$ is isomorphic to a finite submodel $\mathfrak{A}'$ of $\mathfrak{A}$;
(b) there is an isomorphism h of $\mathfrak{B}$ and $\mathfrak{A}'$ such that for all $b \in B$,

$$f(b) = \rho(h(b)) \tag{14.16}$$

It is clear that given an isomorphism h of $\mathfrak{B}$ and some $\mathfrak{A}' \subset \mathfrak{A}$, f is determined by (14.16). Therefore in order to show that X has no $<$-minimal element, it suffices to show that given a finite $\mathfrak{A}_0 \subset \mathfrak{A}$, there exists a finite $\mathfrak{A}_1$ such that $\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \mathfrak{A}$ and for all $x_1, \ldots, x_m \in A_0$, there exist $y_1, \ldots, y_n \in A_1$ such that

$$\mathfrak{A}_1 \models \psi[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

However, this is certainly true since ψ is quantifier-free and $\mathfrak{A}$ satisfies Θ. ■

The Σ_n-Hierarchy of Definable Sets in Models

In analogy with the Σ_n-hierarchy of formulas and classes, we can classify the elements of the set *Form* of all formulas of the language $\mathscr{L} = \{\in\}$, and the sets definable in models of the language $\mathscr{L}$. Thus for each $n \in \omega$ we denote by Σ_n (Π_n) the set of all $\varphi \in$ *Form* that have the appropriate form. Given a model $\mathfrak{A} = \langle A, E\rangle$ of $\mathscr{L}$, we say that a set $X \subseteq A$ is Σ_n (Π_n) *over* $\mathfrak{A}$ if there is $\varphi \in \Sigma_n$ $(\varphi \in \Pi_n)$ such that

$$X = \{x \in A : \mathfrak{A} \models \varphi[x]\}$$

In general, a relation R over A is Σ_n (Π_n) over $\mathfrak{A}$ if

$$R = \{\langle x_1, \ldots, x_m\rangle \in {}^mA : \mathfrak{A} \models \varphi[x_1, \ldots, x_m]\}$$

for some $\varphi \in \Sigma_n$ $(\varphi \in \Pi_n)$.

A relation is Δ_n over $\mathfrak{A}$ if it is both Σ_n and Π_n over $\mathfrak{A}$. A function F in $\mathfrak{A}$ is Σ_n (Π_n, Δ_n) over $\mathfrak{A}$ if the relation $y = F(x)$ is.

In general, the collection of Σ_n relations over a model $\mathfrak{A}$ does not have the closure properties proved in Lemma 14.2, unless the model $\mathfrak{A}$ satisfies some axioms of set theory. It can be easily verified that if M is a transitive set closed under the operation $\{\ \}$, then Σ_n and Π_n relations over $\langle M, \in\rangle$ have all the closure properties from Lemma 14.2, except closure under $(\forall u \in x)$ and $(\exists u \in x)$. Note that in the proof of Lemma 14.2 we used the collection principle in this case. For instance, to show that $(\forall u \in x)\, \mathbf{P}(x, \ldots)$ is Σ_1 if $\mathbf{P}$ is Σ_1, we used the following equivalence:

$$(\forall u \in x)\ \exists z\ \varphi(z, x, \ldots) \leftrightarrow \exists y (\forall u \in x)\ (\exists v \in y)\ \varphi(z, x, \ldots)$$

Similarly, a Δ_1-relation restricted to a transitive model $\langle M, \in\rangle$ is Δ_1 over M, provided M satisfies the theorem of ZF that is used to show that the relation in question is Δ_1. Thus we have:

Exercise 14.13. If M is a transitive model of Theorem 5, then "E is a well-founded relation over P" is Δ_1 over M.

Exercise 14.14. If M is a transitive set closed under the ~~Gödel operations and the function TC(S)~~, then $\mathrm{TC}(S)$ is a Δ_1-function over M.

Exercise 14.15. If M is a transitive ~~set closed under Gödel operations and the function rank(x)~~, then $\mathrm{rank}(x)$ is a Δ_1-function over M.

And most notably:

Exercise 14.16. If M is a transitive model of ZF^-, then the functions $\alpha \mapsto L_\alpha$ and $\alpha \mapsto <_\alpha$ are Δ_1 over M; and "x is constructible" and "$x <_L y$" are Σ_1 over M.

The result of Exercise 14.16 can be considerably improved. It has been shown that there is a sentence σ whose transitive models are exactly all the sets L_α, α an ordinal number. Consequently, the function $\alpha \mapsto L_\alpha$ is Σ_0.

We recall that assuming the countable axiom of choice, the set HC is a model of ZF^-. We include two more exercises: the first one is a variation of Exercise 14.9, and the second one is a consequence of the Lévy–Shoenfield absoluteness lemma:

Exercise 14.17. Assume AC. Let σ be a theorem of ZFC such that σ implies Extensionality, and let $\mathbf{P}(x)$ be a property upward absolute for transitive models of σ. Then $\mathbf{P}(x)$ is Σ_1 over HC.

[$\mathbf{P}(x)$ iff $\exists$ countable M such that $x \in M$ and $M \vDash \sigma \wedge \mathbf{P}(x)$ iff $\exists$ countable transitive $M \ni x$ such that $M \vDash \sigma \wedge \mathbf{P}(x)$.]

Exercise 14.18. If $\sigma \in \Sigma_1$ is a sentence true in $\langle \mathrm{HC}, \in\rangle$, then σ is true in $\langle \mathrm{HC}^L, \in\rangle$.

Truth Definition for Σ_n-Sentences

Closely related to Gödel's second incompleteness theorem is a theorem of Tarski, stating that the notion of truth in a theory containing arithmetic is not expressible in the theory. In particular, there is no set-theoretical property $\mathbf{T}(x)$

such that if σ is a sentence then $\mathbf{T}(\ulcorner\sigma\urcorner)$ holds if and only if σ holds. However, for every (metamathematical) natural number $n \geq 1$, truth for Σ_n-sentences is definable.

We shall sketch a proof of Tarski's theorem and present a truth definition for Σ_n-sentences.

As is customary, we arithmetize the syntax and consider some fixed effective enumeration of all expressions by natural numbers (*Gödel numbering*). In particular, if σ is a sentence, then $\#\sigma$ is the Gödel number of σ, a natural number.

We say that $\mathbf{T}(x)$ is a *truth definition* if:

(14.17) (i) $\forall x(\mathbf{T}(x) \rightarrow x \in \omega)$;
(ii) if σ is a sentence, then

$$\sigma \leftrightarrow \mathbf{T}(\#\sigma)$$

We shall show that a truth definition does not exist.

Let us assume that there is a formula $\mathbf{T}(x)$ satisfying (14.17). Let

$$\varphi_0, \quad \varphi_1, \quad \varphi_2, \quad \ldots$$

be an enumeration of all formulas with one free variable. Let $\psi(x)$ be the formula

$$x \in \omega \wedge \neg\mathbf{T}(\#(\varphi_x(x)))$$

There is a natural number k such that ψ is φ_k. Let σ be the sentence $\psi(k)$. Then we have

$$\sigma \leftrightarrow \psi(k) \leftrightarrow \neg\mathbf{T}(\#(\varphi_k(k))) \leftrightarrow \neg\mathbf{T}(\#\sigma)$$

which contradicts (14.17). ■

For every (metamathematical) natural number n, we define a binary predicate $\models_n$ as follows:

(14.18) $\models_0 \varphi[x_1, \ldots, x_k]$ iff
$\varphi \in \Sigma_0$, $\langle x_1, \ldots, x_k \rangle$ is a k-sequence and $\exists M[M$ is transitive, $\langle x_1, \ldots, x_k \rangle \in M$ and $\langle M, \in \rangle \models \varphi[x_1, \ldots, x_k]]$
$\models_1 \exists u \varphi[u, x_1, \ldots, x_k]$ iff
$\varphi \in \Sigma_0$, $\langle x_1, \ldots, x_k \rangle$ is a k-sequence and $\exists u$ such that $\models_0 \varphi[u, x_1, \ldots, x_k]$
$\models_2 \exists u\, \forall v \varphi[u, v, x_1, \ldots, x_k]$ iff
$\varphi \in \Sigma_0$, $\langle x_1, \ldots, x_k \rangle$ is a k-sequence and $\exists u\, \forall v \models_0 \varphi[u, v, x_1, \ldots, x_k]$
etc.

It is obvious that if $\varphi(x_1, \ldots, x_m)$ is a Σ_n-formula, then

$$\varphi(x_1, \ldots, x_m) \leftrightarrow \models_n \varphi[x_1, \ldots, x_m]$$

Thus if we define $\mathbf{T}_n$ as follows:

$\mathbf{T}_n(x)$ iff $x \in \omega$ and $\exists \sigma \in \Sigma_n[\sigma$ is a sentence, $x = \#\sigma$ and $\models_n \sigma]$

then $\mathbf{T}_n$ is a truth definition for Σ_n-sentences.

The predicates $\vDash_n$ are defined for all n, but in view of Tarski's theorem, it is impossible to define $\vDash_n$ uniformly for all $n \in \omega$, i.e., as a property of n.

Exercise 14.19. The relation $\vDash_0$ is Σ_1; for each $n \geq 1$, $\vDash_n$ is Σ_n.

If $\mathfrak{M}$ is a transitive class and n a natural number, then we can similarly define the predicate $\mathfrak{M} \vDash_n \varphi[x_1, \ldots, x_k]$ for $\varphi \in \Sigma_n$. We leave the details to the reader.

We can further use $\vDash_n$ to define the notion of a *Σ_n-elementary submodel of the universe.* For every n, we define

$$\begin{aligned} M \prec_n V \quad \text{iff} \quad & \forall\varphi \in \Sigma_n \, \forall\langle x_1, \ldots, x_k\rangle \in {}^kM \\ & (\langle M, \in\rangle \vDash \varphi[x_1, \ldots, x_k] \leftrightarrow \vDash_n \varphi[x_1, \ldots, x_k]) \end{aligned}$$

Exercise 14.20. $M \prec_0 V$ holds for every transitive set M.

Exercise 14.21. Let n be a natural number. For every M_0 there exists $M \supseteq M_0$ such that $M \prec_n V$.

[Reflection principle.]

One can similarly define $M \prec_n \mathfrak{M}$ for a transitive class $\mathfrak{M}$.

Exercise 14.22. If κ is an uncountable cardinal then $L_\kappa \prec_1 L$.

[Proceed as in the proof of GCH in L.]

There is of course no problem in defining $X \prec_n \langle M, \in\rangle$ for any set M and any $n \in \omega$.

Exercise 14.23. If L_α is a model of ZF$^-$ and if $X \prec_1 \langle L_\alpha, \in\rangle$, then X is isomorphic to some L_β, $\beta \leq \alpha$.

Exercise 14.24.* Let $\theta = \omega_1^L$. If $X \prec_1 \langle L_\theta, \in\rangle$, then $X = L_\alpha$ for some α.

[A stronger version of Exercise 13.15.]

15. RELATIVE CONSTRUCTIBILITY AND ORDINAL DEFINABILITY

A Few Remarks on Transitive Models

We have dealt quite extensively with the notion of absoluteness and shown that many fundamental concepts are absolute for transitive models of ZF. The cardinal concepts are in general not absolute. If $\mathfrak{M}$ is a transitive model of ZF, then we have:

(15.1) $P^{\mathfrak{M}}(X) = P(X) \cap \mathfrak{M}$, $V_\alpha^{\mathfrak{M}} = V_\alpha \cap \mathfrak{M}$.

(15.2) If $\mathfrak{M} \vDash |X| = |Y|$, then $|X| = |Y|$.

(15.3) If α is a cardinal, then α is a cardinal in $\mathfrak{M}$; if α is a limit cardinal, then α is a limit cardinal in $\mathfrak{M}$.

(15.4) $|\alpha| \leq |\alpha|^{\mathfrak{M}}$, $\mathrm{cf}(\alpha) \leq \mathrm{cf}^{\mathfrak{M}}(\alpha)$.

(15.5) If α is a regular cardinal, then α is a regular cardinal in $\mathfrak{M}$; if α is weakly inaccessible, then α is weakly inaccessible in $\mathfrak{M}$.

(15.6) If $\mathfrak{M}$ is a model of ZFC and κ is inaccessible, then κ is inaccessible in $\mathfrak{M}$.

Concerning the last statement, if $\alpha < \kappa$, then since $\mathfrak{M} \models$ AC, we must have either $(2^\alpha)^{\mathfrak{M}} < \kappa$ or $(2^\alpha)^{\mathfrak{M}} \geq \kappa$ and the latter is impossible since $2^\alpha < \kappa$.

If $\mathfrak{M}$ is a transitive model of ZFC, then the axiom of choice in $\mathfrak{M}$ enables us to code all sets in $\mathfrak{M}$ by sets of ordinals and the model is determined by its sets of ordinals. The precise statement of this fact is: If $\mathfrak{M}$ and $\mathfrak{N}$ are two transitive models of ZFC with the same sets of ordinals, then $\mathfrak{M} = \mathfrak{N}$. In fact, a slightly stronger assertion is true. (On the other hand, one cannot prove that $\mathfrak{M} = \mathfrak{N}$ if neither model satisfies AC.)

Lemma 15.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be transitive models of* ZF *and assume that the axiom of choice holds in $\mathfrak{M}$. If $\mathfrak{M}$ and $\mathfrak{N}$ have the same sets of ordinals, then $\mathfrak{M} = \mathfrak{N}$.*

Proof. We start with a rather trivial remark: $\mathfrak{M}$ and $\mathfrak{N}$ have the same sets of pairs of ordinals. To see this, use the absolute canonical one-to-one function Γ: *Ord* × *Ord* → *Ord*. If $X \subset Ord^2$ and $X \in \mathfrak{M}$, then $\Gamma[X]$ is both in $\mathfrak{M}$ and in $\mathfrak{N}$, and we have $X = \Gamma_{-1}[\Gamma[X]] \in \mathfrak{N}$.

First we prove that $\mathfrak{M} \subseteq \mathfrak{N}$. Let $X \in \mathfrak{M}$. Since $\mathfrak{M}$ satisfies AC, there is a one-to-one mapping $f \in \mathfrak{M}$ of some ordinal θ onto $\mathrm{TC}(\{X\})$. Let $E \in \mathfrak{M}$ be the following relation over θ:

$$\alpha \mathrel{E} \beta \quad \text{iff} \quad f(\alpha) \in f(\beta)$$

E is a set of pairs of ordinals and thus we have $E \in \mathfrak{N}$. In $\mathfrak{M}$, E is well-founded and extensional. However, these properties are absolute and so E is well-founded and extensional in $\mathfrak{N}$. Applying the collapsing theorem (in $\mathfrak{N}$), we get a transitive set $T \in \mathfrak{N}$ such that $(T, \in)$ is isomorphic to (θ, E). Hence T is isomorphic to $\mathrm{TC}(\{X\})$ and since both are transitive, we have $T = \mathrm{TC}(\{X\})$. It follows that $\mathrm{TC}(\{X\}) \in \mathfrak{N}$ and so $X \in \mathfrak{N}$.

Now we prove $\mathfrak{M} = \mathfrak{N}$ by $\in$-induction. Let $X \in \mathfrak{N}$ and assume that $X \subset \mathfrak{M}$; we prove that $X \in \mathfrak{M}$. Let $Y \in \mathfrak{M}$ be such that $X \subseteq Y$ (for instance let $Y = V_\alpha^{\mathfrak{M}}$ where $\alpha = \operatorname{rank} X$; the rank function is absolute). Let $f \in \mathfrak{M}$ be a one-to-one function of Y into the ordinals. Since $\mathfrak{M} \subseteq \mathfrak{N}$, f is in $\mathfrak{N}$ and so $f[X] \in \mathfrak{N}$. However, $f[X]$ is a set of ordinals and so $f[X] \in \mathfrak{M}$, and we have $X = f_{-1}[f[X]] \in \mathfrak{M}$. ■

Relative Constructibility

The notion of constructible sets will be generalized to define sets *constructible from a given set A*.

If A is an arbitrary set and U is a transitive set, let $\mathrm{def}_A(U)$ be the set

$$\mathrm{def}_A(U) = \mathrm{cl}(U \cup \{U\} \cup \{A \cap U\}) \cap P(U) \tag{15.7}$$

The set $\operatorname{def}_A(U)$ is transitive, and we have

$$U \subseteq \operatorname{def}_A(U) \subseteq P(U)$$

Note also that

$$\operatorname{def}_A(U) = \operatorname{def}_{A \cap U}(U) \tag{15.8}$$

Similarly to the case of $\operatorname{def}(U)$ (Exercises 12.3 and 12.5), the set $\operatorname{def}_A(U)$ consists of subsets of U definable over U with A considered a unary predicate on U:

Exercise 15.1.* If U is a transitive set and $A \subseteq U$, then $\operatorname{def}_A(U)$ is the set of all subsets of U definable over the model $\mathfrak{U} = \langle U, \in \cap U^2, A\rangle$.

The class of all sets *constructible from A* is defined as follows:

$$\begin{aligned} L_0[A] &= \emptyset \\ L_\alpha[A] &= \bigcup_{\beta<\alpha} L_\beta[A] \qquad \text{if } \alpha \text{ is a limit ordinal} \\ L_{\alpha+1}[A] &= \operatorname{def}_A(L_\alpha[A]) \\ L[A] &= \bigcup_{\alpha \in Ord} L_\alpha[A] \end{aligned} \tag{15.9}$$

The following theorem is the appropriate generalization of Theorems 32–34 on constructible sets:

Theorem 37. *Let A be an arbitrary set.*

(i) *$L[A]$ is a model of* ZFC.
(ii) *$L[A]$ satisfies the axiom $\exists X(V = L[X])$.*
(iii) *If $\mathfrak{M}$ is a transitive model of* ZF *containing all ordinals and such that $A \cap \mathfrak{M} \in \mathfrak{M}$, then $L[A] \subseteq \mathfrak{M}$.*
(iv) *There exists α_0 such that for all $\alpha \geq \alpha_0$,*

$$L[A] \models 2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

The proof that $L[A]$ is a model of ZF is exactly the same as for L: the class $L[A]$ is transitive, closed, and almost universal. The proof of the rest of Theorem 37 follows closely the corresponding proofs for L, but some additional arguments are needed.

Lemma 15.2. *Let $\bar{A} = A \cap L[A]$. Then $L[\bar{A}] = L[A]$ and moreover $\bar{A} \in L[\bar{A}]$.*

Proof. We show by induction on α that $L_\alpha[\bar{A}] = L_\alpha[A]$. The induction step is obvious if α is a limit ordinal; thus assume that $L_\alpha[\bar{A}] = L_\alpha[A]$ and let us prove $L_{\alpha+1}[\bar{A}] = L_{\alpha+1}[A]$.

If we denote $U = L_\alpha[A]$, then we have

$$A \cap U = A \cap U \cap L[A] = \bar{A} \cap U$$

and by (15.8)

$$L_{\alpha+1}[A] = \mathrm{def}_A(U) = \mathrm{def}_{A \cap U}(U) = \mathrm{def}_{\bar{A}}(U) = L_{\alpha+1}[\bar{A}]$$

Thus $L[\bar{A}] = L[A]$. Moreover, there is α such that $A \cap L[A] = A \cap L_\alpha[A]$ and thus $\bar{A} \in L_{\alpha+1}[A]$. ■

As was the case with constructibility, the key factor in the proof is the absoluteness of relative constructibility:

Lemma 15.3. *If $\mathfrak{M}$ is a transitive model of* ZF *and $A \in \mathfrak{M}$, then the function $\alpha \mapsto L_\alpha[A]$ is absolute for $\mathfrak{M}$.*

Proof. As in Lemma 12.5. ■

In particular, $L[A]$ satisfies $V = L[\bar{A}]$ and we have (ii).

Following Theorem 33, we observe that $V = L[X]$ implies the axiom of choice; in fact there is a well-ordering of the universe definable from X. Hence $L[A]$ is a model of ZFC. Part (iii) of Theorem 37 follows from the following lemma:

Lemma 15.4. *If $\mathfrak{M}$ is a transitive model of* ZF *containing all ordinals and if $A \cap \mathfrak{M} \in \mathfrak{M}$, then $L[A] = L[A \cap \mathfrak{M}]$.*

Proof. By induction on α,

$$L_\alpha[A] = L_\alpha[A \cap \mathfrak{M}]$$

The proof of the induction step is exactly as in Lemma 15.2, once we know that

$$A \cap L_\alpha[A] = A \cap \mathfrak{M} \cap L_\alpha[A] \tag{15.10}$$

(15.10) is true because by absoluteness, $L_\alpha[A] = L_\alpha^{\mathfrak{M}}[A \cap \mathfrak{M}]$ is a subset of $\mathfrak{M}$. ■

For part (iv) we need a stronger version of Lemma 15.3. Call a transitive set M *A-adequate* if:

(15.11) (a) M is closed, and if $X \in M$, then $A \cap X \in M$;
(b) if $U \in M$, then $\mathrm{cl}(U) \in M$;
(c) if $\alpha \in M$, then $\langle L_\beta[A] : \beta < \alpha \rangle \in M$.

Lemma 15.5. *The function $\alpha \mapsto L_\alpha[A]$ is absolute for every A-adequate transitive set M.*

Proof. As in Lemma 13.2. ■

We can similarly improve Lemma 13.3:

Lemma 15.6. *There is a sentence Θ^* (provable in* ZF*) such that for every transitive set M, $M \vDash \Theta^*$ if and only if M is A-adequate for all $A \in M$.* ■

Part (iv) of Theorem 37 now follows from

Lemma 15.7. *If $V = L[A]$ and $A \subseteq L_{\omega_\alpha}[A]$, then every $X \subseteq \omega_\alpha$ is in some $L_\gamma[A]$, $\gamma < \omega_{\alpha+1}$.*

Proof. Let β be such that $X \in L_\beta[A]$. Let S be such that:

$$
\text{(15.12)}\quad
\begin{array}{rl}
(\ ) & S \models \text{Extensionality};\\
\text{(i)} & S \models \Theta^*;\\
\text{(ii)} & L_{\omega_\alpha}[A] \subseteq S,\ X \in S,\ \beta \in S;\\
\text{(iii)} & |S| = \aleph_\alpha;\\
\text{(iv)} & S \models \beta \text{ is an ordinal and } X \in L_\beta[A].
\end{array}
$$

(Compare with (13.18).) As in Lemma 13.1, we collapse S onto a transitive set $M = \pi[S]$, and use absoluteness to show that $X \in L_\gamma[A]$ where $\gamma = \pi(\beta)$. ■

This completes the proof of Theorem 37.

A consequence of Theorem 37(iv) is that if $V = L[A]$ and $A \subseteq \omega$, then the generalized continuum hypothesis holds. For a slightly better result, see the following exercises.

Exercise 15.2. Prove (in ZFC) the following variant of the reflection principle: For a given formula φ, there exists a countable M that reflects φ and, moreover, $\alpha \in M$ implies $\alpha \subseteq M$ for every $\alpha < \omega_1$.

Exercise 15.3. If $V = L[A]$ where $A \subseteq \omega_1$, then $2^{\aleph_0} = \aleph_1$. (Consequently, the GCH holds.)

[Modify the proof of Lemma 15.7 using Exercise 15.2; show that if $X \subseteq \omega$, then $X \in L_\alpha[A \cap \xi]$ for some $\alpha < \omega_1$, $\xi < \omega_1$. It follows that $|P(\omega)| = \aleph_1$.]

Although in practical applications of relative constructibility one often uses $L[X]$ where X is not necessarily a set of ordinals, for theoretical purposes it is sufficient to consider only $L[X]$ where $X \subset Ord$:

Exercise 15.4. For every X there is a set of ordinals A such that $L[X] = L[A]$.

[Let $\bar{X} = X \cap L[X]$, and let (θ, E) be isomorphic to $\mathrm{TC}\{\bar{X}\}$ (in $L[X]$). Let $A = \Gamma[E]$ where Γ is the canonical mapping of Ord^2 onto Ord. Then $A \in L[X]$ and $X \in L[A]$, and hence $L[A] = L[X]$.]

Exercise 15.5. Let $\alpha \geq \omega$ be a countable ordinal. There exists $A \subseteq \omega$ such that α is countable in $L[A]$.

[Let $W \subseteq \omega \times \omega$ be a well-ordering of ω of order type α; let $A \subseteq \omega$ be such that $L[A] = L[W]$.]

Exercise 15.6. If ω_1 (in V) is not a limit cardinal in L, then there exists $A \subseteq \omega$ such that $\omega_1 = \omega_1^{L[A]}$.

[There exists $\alpha < \omega_1$ such that in L, ω_1 is the successor of α. Let A be such that α is countable in $L[A]$.]

Exercise 15.7. (In ZFC) There exists $A \subseteq \omega_1$ such that $\omega_1 = \omega_1^{L[A]}$.

[For each $\alpha < \omega_1$, choose $A_\alpha \subseteq \omega$ such that α is countable in $L[A_\alpha]$. Let $A \subseteq \omega_1 \times \omega_1$ be such that $A_\alpha = \{\xi : (\alpha, \xi) \in A\}$ for all α; then $\omega_1^{L[A]} = \omega_1$.]

Exercise 15.8. (In ZFC) If ω_2 is not inaccessible in L, then there exists $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$.

Some generalizations of $L[A]$: If $\mathbf{A}$ is a class, let us define $L[\mathbf{A}]$ as in (15.9), where $\operatorname{def}_{\mathbf{A}}(U)$ is defined as in (15.7).

Exercise 15.9. $L[\mathbf{A}] = L[\bar{\mathbf{A}}]$, where $\bar{\mathbf{A}} = \mathbf{A} \cap L[\mathbf{A}]$, and $L[\mathbf{A}]$ is a model of ZFC. Moreover, $L[\mathbf{A}]$ is the least model $\mathfrak{M} \supseteq Ord$ such that $V_\alpha^{\mathfrak{M}} \cap \mathbf{A} \in \mathfrak{M}$ for all α.

*Exercise 15.10.** Assume that there exists a choice function $\mathbf{F}$ on V. Then there is a class $\mathbf{A}$ (even a class $\mathbf{A} \subseteq Ord$) such that $V = L[\mathbf{A}]$.

[Combine the various coding tricks used in the preceding exercises.]

If T is a transitive set, let us define $L(T)$ as follows

$$
\begin{aligned}
(15.13)\qquad L_0(T) &= \emptyset \\
L_\alpha(T) &= \bigcup_{\beta<\alpha} L_\beta(T) \qquad \text{if } \alpha \text{ is a limit} \\
L_{\alpha+1}(T) &= \bigcup_{A \in T} \operatorname{def}_A(L_\alpha(T)) \\
L(T) &= \bigcup_{\alpha \in Ord} L_\alpha(T)
\end{aligned}
$$

Exercise 15.11. $L(T)$ is a model of ZF. Moreover, $L(T)$ is the least medel $\mathfrak{M} \supseteq Ord$ such that $\mathbf{T} \subseteq \mathfrak{M}$.

$L(T)$ is not necessarily a model of ZFC: If $T = P(\omega)$, then $P(\omega) \subset L(T)$ and is not necessarily well-ordered.

Exercise 15.12. If $\mathbf{T}$ is a transitive class, then (15.13) defines a model $L(\mathbf{T})$ of ZF, the least model $\mathfrak{M} \supseteq Ord$ such that $\mathbf{T} \subseteq \mathfrak{M}$.

Exercise 15.13. Let $\mathfrak{M}$ be a transitive model of ZF, $\mathfrak{M} \supseteq Ord$, and let X be a subset of $\mathfrak{M}$. Then there is a least model $\mathfrak{M}[X]$ of ZF such that $\mathfrak{M} \subseteq \mathfrak{M}[X]$ and $X \in \mathfrak{M}[X]$. If $\mathfrak{M} \models \text{AC}$, then $\mathfrak{M}[X] \models \text{AC}$.

[Let $\mathfrak{M}[X] = L(\mathfrak{M} \cup \{X\})$.]

Relativization of the Lévy–Shoenfield Absoluteness Lemma

Theorem 36′ *Let $\varphi(x, Z)$ be a Σ_0-formula. If $Z \subseteq \omega$, then*

$$(15.14)\qquad \exists x\, \varphi(x, Z) \;\rightarrow\; (\exists x \in L[Z])\, \varphi(x, Z)$$

Proof. (Sketch) Rather than giving the proof in full, we shall describe how to generalize each step of the proof of Theorem 36 so as to obtain the relativized version of the theorem.

To prove the theorem from Lemma 14.6, we considered a transitive model A of $\exists x\, \varphi(x)$ and obtained a constructible transitive model M of $\exists x\, \varphi(x)$. To generalize, we add to the language constant symbols for each $n \in \omega$, and for ω, and a unary predicate symbol Z. Instead of one sentence Θ we consider a set of sentences S; in addition to the sentence $\exists x\ \varphi(x, Z)$, S contains sentences describing the natural numbers, the set ω and the set Z (i.e., sentences like $Z(n)$ (or $\neg Z(n)$), 0 is the empty set, $1 = \{0\}$, $2 = \{0, 1\}, \ldots, \omega$ is the set of all natural numbers). Notice that $L[S] = L[Z]$.

Thus in the generalized Lemma 14.6 Θ is replaced by a set of sentences S, and the language of the model $\mathfrak{A}$ is assumed to contain infinitely many constant symbols; the lemma asserts that S has a well-founded model $\mathfrak{B} \in L[S]$. By the Skolem normal form theorem we may assume that all sentences in S are $\forall\exists$-sentences. (In fact, the sentences $Z(n)$ (or $\neg Z(n)$) and the sentences describing the natural numbers are already $\forall\exists$, so that we add only finitely many predicate symbols $R_1, \ldots, R_k$ to the language.)

It remains to construct a well-founded model $\mathfrak{B}$ of S such that $\mathfrak{B} \in L[S]$. This is done as in (14.14) and (14.15). We consider finite models whose universe is a subset of a given countable set and functions that demonstrate that the models are well-founded. We let P be the set of triples $(\mathfrak{B}, f, k)$, where k is a natural number, and define $(\mathfrak{B}, f, k) < (\mathfrak{B}', f', k')$ as in (14.15) with the additional condition

$$\text{(o)} \quad k' < k$$

and with the requirement that (iii) holds for the first k sentences in S.

With these modifications, the proof proceeds as in Lemma 14.6 and we obtain a well-founded model $\mathfrak{B}$ of S such that $\mathfrak{B} \in L[S]$. Upon collapsing $\mathfrak{B}$ onto a transitive model M, the predicate Z becomes the set Z and M satisfies $\exists x\ \varphi(x, Z)$. Since φ is Σ_0, we have $L[Z] \models \exists x\ \varphi(x, Z)$. ■

Exercise 15.14. If $\sigma(Z)$ is Σ_1 and $Z \subseteq \omega$, then $\sigma(Z)$ is absolute for all transitive $M \supseteq L_\theta[Z]$ where $\theta = \omega_1^{L[Z]}$.

Exercise 15.15. If $\sigma(Z) \in \Sigma_1$, $Z \subseteq \omega$, and $\sigma(Z)$ holds in $\langle \mathrm{HC}, \in\rangle$, then $\sigma(Z)$ holds in $\langle \mathrm{HC}^{L[Z]}, \in\rangle$.

[Relativized Exercise 14.19.]

A set of reals A is Σ_1 *over* HC if $A = \{x \in \boldsymbol{R} : \mathrm{HC} \models \sigma(x)\}$, for some $\sigma(x) \in \Sigma_1$. If $\mathfrak{M}$ is a transitive model of ZF, we denote $A^{\mathfrak{M}}$ the set $\{x \in \boldsymbol{R}^{\mathfrak{M}} : \mathrm{HC}^{\mathfrak{M}} \models \sigma(x)\}$.

Exercise 15.16. Let $\mathfrak{M}$ be a transitive model of ZF containing all countable ordinals. If A is a set of reals Σ_1 over HC, then $A^{\mathfrak{M}} = A \cap \mathfrak{M}$.

[If $x \in \mathbf{R}^{\mathfrak{M}}$, then $\mathfrak{M} \supseteq L_\theta[x]$, where $\theta = \omega_1^{L[x]}$, and $\mathrm{HC} \models \sigma(x)$ iff $\mathrm{HC}^{\mathfrak{M}} \models \sigma(x)$.]

Ordinal-Definable Sets

A set X is *ordinal-definable* if there is a formula φ such that

$$X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n)\} \tag{15.15}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$.

It is not immediately clear that the property "ordinal definable" is expressible in the language of set theory. Thus we give a different definition of ordinal definable sets and show that it is equivalent to (15.15).

We recall that $\mathrm{cl}(M)$ denotes the closure of a set M under Gödel operations. The class OD of all *ordinal-definable sets* is defined as follows:

$$\mathrm{OD} = \bigcup_{\alpha \in \mathit{Ord}} \mathrm{cl}\{V_\beta : \beta < \alpha\} \tag{15.16}$$

In other words, OD is the Gödel closure of $\{V_\alpha : \alpha \in \mathit{Ord}\}$, that is, ordinal definable sets are obtained from the V_α by application of Gödel operations. We shall show that the elements of the class OD are exactly the sets satisfying (15.15).

Lemma 15.8. *There exists a definable well ordering of the class* OD (*and a definable mapping* F *of Ord onto* OD).

Proof. In (13.7) we described how to construct from a given well-ordering of a set M, a well-ordering of the set $\mathrm{cl}(M)$. For every α, the set $\{V_\beta : \beta < \alpha\}$ has an obvious well-ordering, and we call the canonical well-ordering of $\mathrm{cl}\{V_\beta : \beta < \alpha\}$ the well-ordering obtained from it. Thus we get a well-ordering of the class OD, and denote F the corresponding (definable) one-to-one mapping of *Ord* onto OD. ■

Now it follows that every $X \in \mathrm{OD}$ has the form (15.15). There exists α such that X is the αth set in the canonical well-ordering of OD, or, in other words,

$$X = \{u : \varphi(u, \alpha)\}$$

where $\varphi(u, \alpha)$ is the formula $u \in \mathbf{F}(\alpha)$.

We shall show that on the other hand, if φ is a formula and X is the set in (15.15), then $X \in \mathrm{OD}$. By the reflection principle, let β be such that $X \subseteq V_\beta$, $\alpha_1, \ldots, \alpha_n < \beta$ and that V_β reflects φ. Then we have

$$X = \{u \in V_\beta : V_\beta \vDash \varphi(u, \alpha_1, \ldots, \alpha_n)\}$$

Since $U \vDash \varphi$ is a Σ_0-formula, we may apply the normal form theorem and find a Gödel operation $\mathfrak{F}$ such that

$$X = \mathfrak{F}(V_\beta, \alpha_1, \ldots, \alpha_n)$$

(see also Exercise 12.2). Using the normal form theorem once again, we find a Gödel operation $\mathfrak{G}$ such that for every α,

$$\alpha = \{x \in V_\alpha : x \text{ is an ordinal}\} = \mathfrak{G}(V_\alpha)$$

and so there is a Gödel operation $\mathfrak{H}$ such that

$$X = \mathfrak{H}(V_{\alpha_1}, \ldots, V_{\alpha_n}, V_\beta)$$

and therefore X belongs to OD.

Exercise 15.17. Assume that not every ordinal number is definable and show that the property "X is definable" is not expressible in the language of set theory.

[Consider the least α that is not definable.]

Exercise 15.18. If $X \in \mathrm{OD}$, then there exists γ such that X is a definable subset of $\langle V_\gamma, \in\rangle$ (without parameters). Hence OD is the class of all X definable in some V_γ.

[If $X = \{u \in V_\beta : V_\beta \vDash \varphi(u, \alpha)\}$, consider $\gamma = \Gamma(\alpha, \beta)$.]

Exercise 15.19. If **F** is a definable function on *Ord*, then ran $\mathbf{F} \subseteq \mathrm{OD}$. Thus: OD is the largest class for which there exists a definable one-to-one correspondence with the class of all ordinals.

For any class **A**, we have defined in (9.6) the class of all x *hereditarily* in **A**:

$$\mathscr{H}\mathbf{A} = \{x \in \mathbf{A} : x \subseteq \mathscr{H}\mathbf{A}\} = \{x \in \mathbf{A} : \mathrm{TC}(x) \subseteq A\}$$

Thus let HOD denote the class of *hereditarily ordinal-definable* sets

$$\mathrm{HOD} = \{x : \mathrm{TC}(\{x\}) \subset \mathrm{OD}\}$$

The class HOD is transitive and contains all ordinals.

Theorem 38. *The class* HOD *is a transitive model of* ZFC.

This theorem provides an alternative proof of consistency of the axiom of choice.

Proof. The class HOD is transitive, and it is easy to see that it is closed under Gödel operations. Thus to show that HOD is a model of ZF, it suffices to show that HOD is almost universal. For that, it is enough to verify that $V_\alpha \cap \mathrm{HOD} \in \mathrm{HOD}$, for all α. For any α, the set $V_\alpha \cap \mathrm{HOD}$ is a subset of HOD, and so it is sufficient to prove that $V_\alpha \cap \mathrm{HOD}$ is ordinal-definable. This is indeed true because $V_\alpha \cap \mathrm{HOD}$ is the set of all u satisfying the formula

$$u \in V_\alpha \wedge (\forall z \in \mathrm{TC}(\{u\}))\ \exists\beta[z \in \mathrm{cl}\{V_\gamma : \gamma < \beta\}]$$

and thus $V_\alpha \cap \mathrm{HOD} \in \mathrm{OD}$.

It remains to prove that HOD satisfies the axiom of choice. We shall show that for each α there exists a one-to-one function $g \in \mathrm{HOD}$ of $V_\alpha \cap \mathrm{HOD}$ into the ordinals. Since every such function is a subset of HOD, it suffices to find $g \in \mathrm{OD}$.

By Lemma 15.8, there is a definable one-to-one mapping **G** of the class OD into the ordinals. If we let g be the restriction of **G** to the ordinal-definable set $V_\alpha \cap \mathrm{HOD}$, then g is clearly ordinal-definable. ■

The model HOD contains all constructible sets; $\mathrm{HOD} \supseteq L$. It is consistent that $\mathrm{HOD} \neq L$. Also, unlike constructibility, ordinal definability is not an absolute property, and it is consistent that $\mathrm{HOD}^{\mathrm{HOD}} \neq \mathrm{HOD}$. I shall return to this in Section 25.

Exercise 15.20. HOD is the largest transitive model of ZF for which there exists a definable one-to-one correspondence with the class of all ordinals.

[Use Exercise 15.19.]

We conclude this section with several generalizations of ordinal definability.

A set X is *ordinal-definable from* A, $X \in \mathrm{OD}[A]$, if there is a formula φ such that

$$(15.17) \qquad X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, A)\}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$.

As above, this notion is expressible in the language of set theory:

$$(15.18) \qquad \mathrm{OD}[A] = \mathrm{cl}(\{V_\alpha : \alpha \in Ord\} \cup \{A\})$$

The class $\mathrm{OD}[A]$ has a well-ordering definable from A and thus every set in $\mathrm{OD}[A]$ is of the form (15.17). Conversely (using the reflection principle), every set X in (15.17) belongs to $\mathrm{OD}[A]$.

The proof of Theorem 38 generalizes easily to the case of $\mathrm{HOD}[A]$. Thus $\mathrm{HOD}[A]$, the class of all set hereditarily-ordinal-definable from A, is a transitive model of ZFC.

As a further generalization, we call X *ordinal-definable over* A, $X \in \mathrm{OD}(A)$, if it belongs to the Gödel closure of $\{V_\alpha : \alpha \in Ord\} \cup \{A\} \cup A$. If $X \in \mathrm{OD}(A)$, then $X \in \mathrm{cl}(\{V_\alpha : \alpha \in Ord\} \cup \{A\} \cup E)$, where $E = \{x_0, \ldots, x_k\}$, a finite subset of A. Hence there is a finite sequence $s = \langle x_0, \ldots, x_k \rangle$ in A such that X is ordinal-definable from A and s. On the other hand, if s is a finite sequence in A, then obviously $s \in \mathrm{OD}(A)$ and thus we have

$$(15.19) \quad \mathrm{OD}(A) = \{X : X \in \mathrm{OD}[A, s] \text{ for some finite sequence } s \text{ in } A\}$$

In other words, $X \in \mathrm{OD}(A)$ iff there is a formula φ such that

$$(15.20) \qquad X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, A, \langle x_0, \ldots, x_k \rangle)\}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$ and a finite sequence $\langle x_0, \ldots, x_k \rangle$ in A.

The class $\mathrm{HOD}(A)$ of all sets hereditarily ordinal-definable over A is a transitive model of ZF. To show that $\mathrm{HOD}(A)$ is almost universal, it suffices to verify that $V_\alpha \cap \mathrm{HOD}(A) \in \mathrm{OD}(A)$. In fact, $V_\alpha \cap \mathrm{HOD}(A)$ is ordinal-definable from A: It is the set

$$\{u \in V_\alpha : (\forall z \in \mathrm{TC}(\{u\}))[z \in \mathrm{cl}(\{V_\alpha : \alpha \in Ord\} \cup \{A\} \cup A]\}$$

Exercise 15.21. Let **C** be a definable class. Let OD(**C**), the class of all sets ordinal definable over **C**, be the Gödel closure of $\{V_\alpha : \alpha \in Ord\} \cup \mathbf{C}$. Show that HOD(**C**) is a model of ZF.

Let $\mathfrak{M}$ be a transitive model of ZF, $\mathfrak{M} \supseteq Ord$. A set X is *$\mathfrak{M}$-definable*, $X \in \mathfrak{M}\mathrm{D}$, if there is a formula φ such that

$$X = \{u : \varphi(u, x)\}$$

for some $x \in \mathfrak{M}$.

Exercise 15.22. Show that "X is $\mathfrak{M}$-definable" is expressible in the language of set theory, and prove that $\mathrm{H}\mathfrak{M}\mathrm{D}$ is a model of ZF.

Exercise 15.23. $\mathrm{LD} = \mathrm{OD}$.

PART II

More Sets

CHAPTER 3

FORCING AND GENERIC MODELS

16. GENERIC MODELS

One of the consequences of Gödel's proof of consistency of the axiom of constructibility is that one cannot prove in ZFC that there is more than one transitive model of set theory, namely the constructible universe L. And L satisfies the generalized continuum hypothesis. Thus if we wish to show, e.g., that the continuum hypothesis is unprovable in ZFC we cannot just look for a transitive model $\mathfrak{M}$ that would satisfy $2^{\aleph_0} > \aleph_1$. What is needed is a different way of constructing models of set theory. There are several possible approaches to this problem:

(a) to construct a model $\mathfrak{M} = (\mathrm{M}, \mathrm{E})$ where E is not well-founded (a nonstandard model);

(b) to make an assumption that there exists a countable transitive model $\mathfrak{M}$ of ZFC and extend $\mathfrak{M}$ by adjoining a set $G \notin \mathfrak{M}$ to obtain a model $\mathfrak{M}[G]$ of ZFC such that $Ord^{\mathfrak{M}} = Ord^{\mathfrak{M}[G]}$;

(c) to introduce "imaginary" sets and adjoin an imaginary set G to the universe to obtain an extension $V[G]$ that is a model of ZFC;

(d) to generalize the notions of model and satisfaction and construct a "generalized" model of ZFC.

There is an inherent disadvantage in approach (a): Unlike in the case of transitive models, one cannot use absoluteness of fundamental notions of set theory to establish truth or falsity of various statements in the model. In case (b) the main problem is to choose the set G properly, so that $\mathfrak{M}[G]$ is a model of ZFC with the same ordinals as $\mathfrak{M}$. For instance, one has to avoid the case when $G \subset \omega \times \omega$ is a well-ordering of order-type $Ord^{\mathfrak{M}}$.

It turns out that a combination of the ideas in (c) and (d) is most fruitful. The key to the problem is the method of forcing, invented by Cohen. This method not only establishes independence of the continuum hypothesis and of the axiom of choice, but proves to be a powerful method of construction of models of set theory, providing a number of consistency results.

The idea of extending a mathematical structure by adjoining new "ideal" objects is very frequent. Typical is the construction of the ring of polynominals over a given field. Of course, to extend the universe of set theory to a model of set theory is more difficult, first because there are no sets outside the universe, and secondly because the extension has to satisfy the axioms of ZFC.

Cohen's method of extending the universe of set theory is general enough to apply to any model of set theory. We are however interested primarily in transitive models. Thus let $\mathfrak{M}$ be an arbitrary but fixed transitive model of ZFC (possibly V, possibly a countable model). We shall call $\mathfrak{M}$ the *ground model.* We shall describe how to adjoin to $\mathfrak{M}$ a *generic set* G to obtain a *generic extension* $\mathfrak{M}[G]$. We shall do it in such a way that:

(i) $\mathfrak{M}[G]$ is a model of ZFC with the same ordinals as $\mathfrak{M}$; and
(ii) $\mathfrak{M}[G]$ can be described inside $\mathfrak{M}$.

Before defining the notion of generic set, let us consider a typical example. Suppose that we want to enlarge the ground model $\mathfrak{M}$ in such a way that the ordinal $\omega_1^{\mathfrak{M}}$ becomes countable (or, as we say, is *collapsed*). We want to describe a model $\mathfrak{M}[G]$ that contains a mapping f of ω onto $\omega_1^{\mathfrak{M}}$. Let f be such a—hypothetical—mapping. For every n, the initial segment $f \upharpoonright n = \langle f(0), \ldots, f(n-1)\rangle$ lies in $\mathfrak{M}$. Thus let P be the set of all finite sequences $\langle \alpha_0, \ldots, \alpha_{n-1}\rangle$ of countable ordinals in $\mathfrak{M}$. We shall call the elements of P *forcing conditions.*

We may look at forcing conditions as giving us information about f, namely telling us that $\langle \alpha_0, \ldots, \alpha_{n-1}\rangle$ is an initial segment of f. We say that $p = \langle \alpha_0, \ldots, \alpha_{n-1}\rangle$ *forces* that $f(0) = \alpha_0, \ldots, f(n-1) = \alpha_{n-1}$. Apparently, if $p \subset q$, then q gives us more information than p; we say that q is *stronger* than p.

The relation "q is stronger than p" is a partial ordering of P. The convention is to use

$$q < p$$

to denote that q is stronger than p. Thus in our example, $q < p$ iff $q \supset p$. Note that both the set P and its partial ordering are sets in $\mathfrak{M}$.

Now consider the set $G = \{f \upharpoonright n : n \in \omega\}$; G is a subset of P. Obviously, f can be recovered from G, viz. $f = \bigcup G$, and therefore G cannot be a set in $\mathfrak{M}$. It is easy to see that if we want $\bigcup G$ to be a function of ω onto $\omega_1^{\mathfrak{M}}$, then G has to have the following properties:

(16.1) (i) if $p, q \in G$, then p and q are *compatible*, i.e., $p(k) = q(k)$ for all $k \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$;
(ii) for every n there exists $p \in G$ such that $n \in \mathrm{dom}(p)$;
(iii) for every $\alpha < \omega_1^{\mathfrak{M}}$ there exists $p \in G$ such that $\alpha \in \mathrm{ran}(p)$.

We shall return to this example after giving the general definition of generic sets to verify that an $\mathfrak{M}$-generic set G has the properties (i)–(iii) in (16.1).

Forcing Conditions and Generic Sets

In the ground model $\mathfrak{M}$, let us consider a nonempty partially ordered set $(P, <)$. We call $(P, <)$ a *notion of forcing* and the elements of P *forcing conditions*. We say that p is *stronger* than q if $p < q$. If p and q are conditions and there exists r such that both $r \leq p$ and $r \leq q$, then p and q are *compatible*; otherwise they are *incompatible*. A set $W \subseteq P$ is *incompatible* if its elements are pairwise incompatible. A set $D \subseteq P$ is *dense* in P if for every $p \in P$ there is $q \in D$ such that $q \leq p$. A set $F \subseteq P$ is a *filter* on P if:

(16.2) (i) F is nonempty;
(ii) if $p \leq q$ and $p \in F$, then $q \in F$;
(iii) if $p, q \in F$, then there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

A set of conditions $G \subset P$ is *generic* (over $\mathfrak{M}$) if:

(16.3) (i) G is a filter on P;
(ii) if D is dense in P and $D \in \mathfrak{M}$, then $G \cap D \neq \emptyset$.

Note how (16.3)(ii) depends on the ground model $\mathfrak{M}$: What matters is which dense subsets of P are in $\mathfrak{M}$. Thus if $\mathscr{D}$ is any collection of sets, let us say that a set $G \subseteq P$ is a *$\mathscr{D}$-generic filter* on P if it is a filter and if $G \cap D \neq \emptyset$ for every dense subset of P that is in $\mathscr{D}$. Then G is generic over $\mathfrak{M}$ just in case it is $\mathscr{D}$-generic where $\mathscr{D}$ is the collection of all $D \in \mathfrak{M}$ dense in P.

Exercise 16.1. Show that in the definition of generic set one can replace (16.2)(iii) by the following weaker property: If $p, q \in G$, then p and q are compatible.

[To prove (16.2)(iii), show that $D = \{r \in P : \text{either } r \text{ is incompatible with } p \text{ or } r \text{ is incompatible with } q, \text{ or } r \leq p \text{ and } r \leq q\}$ is dense.]

A set $D \subseteq P$ is *dense below* p if for every $p' \leq p$ there is $q \in D$ such that $q \leq p'$. A set $U \subseteq P$ is *open* if $p \in U$ and $q \leq p$ imply $q \in U$.

Exercise 16.2. A filter G on P is generic over $\mathfrak{M}$ if and only if for every $p \in G$, if $D \in \mathfrak{M}$ is dense below p then $G \cap D \neq \emptyset$.

Exercise 16.3. A filter G on P is generic over $\mathfrak{M}$ if and only if $G \cap D \neq \emptyset$ whenever $D \in \mathfrak{M}$ is open and dense in P.

A set $C \subseteq P$ is a *cover* of P if every $p \in P$ is compatible with some $q \in C$. A set $W \subseteq P$ is a *partition* of P if it is a cover of P and if any two distinct $p, q \in W$ are incompatible.

Exercise 16.4. A filter G on P is generic over $\mathfrak{M}$ if and only if $G \cap C \neq \emptyset$ whenever $C \in \mathfrak{M}$ is a cover of P.

Exercise 16.5. A filter G on P is generic over $\mathfrak{M}$ if and only if $G \cap W \neq \emptyset$ whenever $W \in \mathfrak{M}$ is a partition of P.

We shall now return to our first example: we consider the partially ordered set $(P, \supset)$ of finite sequences of ordinals less than $\omega_1^{\mathfrak{M}}$. Let $G \subseteq P$ be generic over $\mathfrak{M}$. Since G is a filter, it is clear that $f = \bigcup G$ is a function. It is rather easy to see that for every $n \in \omega$, the set

$$D_n = \{p \in P : n \in \operatorname{dom}(p)\}$$

is dense in P. Similarly, for every $\alpha < \omega_1^{\mathfrak{M}}$, the set

$$E_\alpha = \{p \in P : \alpha \in \operatorname{ran}(p)\}$$

is dense in P. Therefore, $G \cap D_n \neq \emptyset$ for every n and $G \cap E_\alpha \neq \emptyset$ for every α, and so $\operatorname{dom}(f) = \omega$ and $\operatorname{ran}(f) = \omega_1^{\mathfrak{M}}$.

Thus we have proved that any transitive model $\mathfrak{N} \supset \mathfrak{M}$ that contains G satisfies that $\omega_1^{\mathfrak{M}}$ is a countable ordinal.

If the ground model $\mathfrak{M}$ is *countable*, then for every notion of forcing $(P, <)$ in $\mathfrak{M}$ there exists a generic set. This follows from the next lemma:

Lemma 16.1. *If $(P, <)$ is a partially ordered set and $\mathscr{D}$ is a countable collection of dense subsets of P, then there exists a $\mathscr{D}$-generic filter on P. In fact, for every $p \in P$ there exists a $\mathscr{D}$-generic filter G on P such that $p \in G$.*

Proof. Let $D_1, D_2, \ldots$ be the sets in $\mathscr{D}$. Let $p_0 = p$, and for each n, let p_n be such that $p_n \leq p_{n-1}$ and $p_n \in D_n$. The set

$$G = \{q \in P : q \geq p_n \text{ for some } n \in \mathcal{N}\}$$

is a $\mathscr{D}$-generic filter on P and $p \in G$. ■

We shall now state the first of two main theorems on generic models.

Theorem 39 (*Generic Model Theorem*). *Let $\mathfrak{M}$ be a transitive model of* ZFC *and let $(P, <)$ be a notion of forcing in $\mathfrak{M}$. If $G \subseteq P$ is generic over P, then there exists a transitive model $\mathfrak{M}[G]$ such that:*

(a) *$\mathfrak{M}[G]$ is a model of* ZFC;
(b) *$\mathfrak{M} \subseteq \mathfrak{M}[G]$ and $G \in \mathfrak{M}[G]$;*
(c) *$Ord^{\mathfrak{M}[G]} = Ord^{\mathfrak{M}}$;*
(d) *if $\mathfrak{N}$ is a transitive model of* ZF *such that $\mathfrak{M} \subseteq \mathfrak{N}$ and $G \in \mathfrak{N}$, then $\mathfrak{M}[G] \subseteq \mathfrak{N}$.*

The model $\mathfrak{M}[G]$ is called a *generic extension* of $\mathfrak{M}$. The proof of Theorem 39 (as well as of the other theorem of this section) will be given in Section 18.

We shall now give another example of a generic set; in fact, this is the simplest possible nontrivial example of a generic set.

Let P be the set of all finite 0–1 sequences $\langle p(0), \ldots, p(n-1)\rangle$, ordered by inverse inclusion: $(P, \supset)$. If G is a generic set of conditions (over the ground model $\mathfrak{M}$), then $f = \bigcup G$ is a 0–1 function on ω; as in the first example, f is a function because G is a filter on P, and $\operatorname{dom}(f) = \omega$ because for every n, the set

$$D = \{p \in P : n \in \operatorname{dom}(p)\}$$

is dense in P and therefore $D \cap G \neq \emptyset$.

The function $f: \omega \to \{0, 1\}$ is the characteristic function of a unique set $A \subseteq \omega$. We shall show that A is not in the ground model $\mathfrak{M}$. Equivalently, let us show that $f \notin \mathfrak{M}$. Let $g: \omega \to \{0, 1\}$ be a function in $\mathfrak{M}$, and let us consider the set

$$D_g = \{p \in P : p \not\subset g\}$$

It is easy to verify that D_g is dense in P and since $D_g \in \mathfrak{M}$, we have $G \cap D_g \neq \emptyset$ and hence $f \neq g$.

The above example describes the simplest way of adjoining a new set of natural numbers to the ground model. A set $A \subset \omega$ obtained in this way is usually called a *Cohen generic* set (or a Cohen generic real).

Exercise 16.6. Let $(P, <)$ be a notion of forcing in $\mathfrak{M}$ with the following property: For every $p \in P$ there exist q and r such that $q \leq p$, $r \leq p$ and such that q and r are incompatible. Show that if $G \subseteq P$ is generic over $\mathfrak{M}$, then $G \notin \mathfrak{M}$.

[If F is a filter on P, then $\{p \in P : p \notin F\}$ is dense in P.]

The last example (adding a generic real) suggests how to attack the problem of independence of the continuum hypothesis: One may attempt to adjoin to the ground model $\mathfrak{M}$ a large number of generic reals.

Let P be (in $\mathfrak{M}$) the set of all functions p such that

(16.4) (i) $\mathrm{dom}(p)$ is a finite subset of $\omega_2 \times \omega$,
(ii) $\mathrm{ran}(p) \subseteq \{0, 1\}$,

and let p be stronger than q just in case $p \supset q$.

If G is a generic set of conditions, then if we let $f = \bigcup G$, we can use the same argument as before to show that

(16.5) (i) f is a function;
(ii) $\mathrm{dom}(f) = \omega_2^{\mathfrak{M}} \times \omega$.

Now, for each $\alpha < \omega_2^{\mathfrak{M}}$, let $f_\alpha: \omega \to \{0, 1\}$ be the function defined as follows:

$$f_\alpha(n) = f(\alpha, n)$$

(That is, we may look at f as coding a collection of 0–1 functions on ω:

$$\langle f_\alpha : \alpha < \omega_2^{\mathfrak{M}} \rangle)$$

Again, it is easy to see that each f_α is outside $\mathfrak{M}$; for any $g \in \mathfrak{M}$, the set

$$\{p \in P : p(\alpha, n) \neq g(n) \text{ for some } n\}$$

is dense in P and thus $f_\alpha \neq g$.

Moreover, if $\alpha \neq \beta$, then $f_\alpha \neq f_\beta$; this is because the set

$$D = \{p \in P : p(\alpha, n) \neq p(\beta, n) \text{ for some } n\}$$

is dense in P and hence $G \cap D \neq \emptyset$. Thus in $\mathfrak{M}[G]$ we have a one-to-one mapping $\alpha \mapsto f_\alpha$ of $\omega_2^{\mathfrak{M}}$ into ${}^\omega\{0, 1\}$. However, in order to conclude that the continuum hypothesis is violated in $\mathfrak{M}[G]$, we have to know that $\omega_2^{\mathfrak{M}}$ is not collapsed in $\mathfrak{M}[G]$. To prove that, one has to use more subtle arguments.

This leads us to the other important feature of Cohen's method, namely that the generic model $\mathfrak{M}[G]$ can be described within the ground model.

Associated with the notion of forcing $(P, <)$ is a *forcing language*. This forcing language as well as the *forcing relation* $\Vdash$ are defined in the ground model $\mathfrak{M}$. The forcing language contains a *name* for every element of $\mathfrak{M}$, and a constant $\boldsymbol{G}$ (the *canonical name* for a generic set). In fact, the way the forcing language will be set up will provide for a name for every potential element of the generic extension $\mathfrak{M}[G]$. Once we select a generic set G, then every constant of the forcing language is interpreted as an element of the model $\mathfrak{M}[G]$.

The forcing relation is a relation between the forcing conditions and sentences of the forcing language:

$$p \Vdash \sigma$$

(*p forces* σ). The forcing relation, which is defined in $\mathfrak{M}$, is in a sense a generalization of the notion of satisfaction (in the generic model). For instance, if $p \Vdash \sigma$ and if σ' is a logical consequence of σ, then $p \Vdash \sigma'$.

In due course, we shall prove various rules for the forcing relation. For instance,

(16.6) if $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$

(16.7) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$

or

(16.8) $p \Vdash \neg\varphi$ iff no $q \leq p$ forces φ

In view of the forthcoming theorem, these rules will enable us to decide (within $\mathfrak{M}$) whether a given set theoretical statement holds in $\mathfrak{M}[G]$. Let $(P, <)$ be a fixed notion of forcing in the ground model $\mathfrak{M}$.

Theorem 40 (*Forcing Theorem*). *If* σ *is a sentence of the forcing language, then for every* $G \subseteq P$ *generic over* $\mathfrak{M}$,

(16.9) $\mathfrak{M}[G] \vDash \sigma$ *if and only if* $(\exists p \in G) p \Vdash \sigma$

[In the left-hand-side σ one interprets the constants of the forcing language according to G.]

Theorem 40 will also be proved in Section 18. Prior to proving Theorems 39 and 40 we shall develop the theory of forcing and prove the basic rules for the forcing relation.

In the following exercises, we give a definition of the forcing relation under an assumption that is true if $\mathfrak{M}$ is countable, thus giving a definition of $\Vdash$ for countable models. However, this definition is in terms of generic sets, and there is no straightforward way of showing that $\Vdash$ is definable inside $\mathfrak{M}$. On the other hand, using the forcing theorem one can readily derive the basic rules for $\Vdash$.

In the following exercises, we assume that for each $p \in P$ there is $G \subseteq P$ generic over $\mathfrak{M}$ containing p. Let us define

$$(16.10) \qquad p \Vdash \sigma \quad \text{iff} \quad \mathfrak{M}[G] \models \sigma \quad \text{for all } G \text{ such that } p \in G$$

Exercise 16.7.

(a) if $p \Vdash \sigma$ and $q \leq p$, then $q \Vdash \sigma$;
(b) no p forces both σ and $\neg\sigma$;
(c) for every p there is $q \leq p$ such that either $q \Vdash \sigma$ or $q \Vdash \neg\sigma$.

[Use (16.9) to prove (c).]

Exercise 16.8.

(a) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$;
(b) $p \Vdash \forall x \varphi$ iff $p \Vdash \varphi(\boldsymbol{a})$ for every constant $\boldsymbol{a}$ of the forcing language.

[For (b), use: $\mathfrak{M}[G] \models \forall x \varphi$ iff $\mathfrak{M}[G] \models \varphi(\boldsymbol{a})$ for every constant of the forcing language.]

Exercise 16.9. $p \Vdash \sigma$ iff no $q \leq p$ forces $\neg\sigma$.
[Use Exercise 16.7c.]

Exercise 16.10.

(a) $p \Vdash \varphi \to \psi$ iff $(\forall q \leq p)\ q \Vdash \varphi$ implies $q \Vdash \psi$;
(b) $p \Vdash \varphi \vee \psi$ iff $(\forall q \leq p)\ (\exists r \leq q)\ r \Vdash \varphi$ or $r \Vdash \psi$;
(c) $p \Vdash \exists x \varphi$ iff $(\forall q \leq p)\ (\exists r \leq q)\ \exists \boldsymbol{a}\ r \Vdash \varphi(\boldsymbol{a})$.

[Express $\to$, $\vee$, and $\exists$ in terms of $\wedge$, $\neg$, and $\forall$.]

To conclude this section, I shall briefly touch the question of existence of generic sets. Of course, if $\mathfrak{M}$ is countable, then every $P \in \mathfrak{M}$ has an $\mathfrak{M}$-generic filter. On the other hand, in general one can at least prove something like: Given $P \in \mathfrak{M}$, then it is consistent that P has an $\mathfrak{M}$-generic filter. (A theorem of this sort will be proved in Section 18.) Even when formulated correctly, one has to be careful to exclude the obvious false cases (like when $\mathfrak{M}$ = the universal class). In fact, it seems necessary to formulate the theorem only for $\mathfrak{M}$ having an absolute definition (like $\mathfrak{M}$ = the constructible universe).

However, the problem of consistency of existence of generic sets in general is not really a problem since in practice, when dealing with the consistency of a particular set theoretical statement, the correct existence theorem can be easily formulated. Moreover, the use of a generic extension can be converted directly into a consistency proof, avoiding the question of existence of a generic set altogether. I shall elaborate on this in Section 18.

17. COMPLETE BOOLEAN ALGEBRAS

Boolean Algebras

A *field of sets* (an algebra of sets) is a collection $\mathcal{F}$ of subsets of a given nonempty set S with the following properties:

(17.1) (i) $S \in \mathcal{F}$;
(ii) if $X, Y \in \mathcal{F}$, then $X \cup Y \in \mathcal{F}$, $X \cap Y \in \mathcal{F}$, and $X - Y \in \mathcal{F}$.

Obviously, $P(S)$ is a field of sets, and also, $\{\emptyset, S\}$ is a field of sets. Some less trivial examples:

(a) Let S be an infinite set and let $\mathcal{F}$ be the collection of all X such that either X is finite or $S - X$ is finite.
(b) Let S be a topological space and let $\mathcal{F}$ be the collection of all clopen (closed and open) sets in S.
(c) The collection of all Borel sets of reals.
(d) The collection of all Lebesgue measurable sets of reals.

The algebraic properties of fields of sets are abstracted in the notion of Boolean algebra. A *Boolean algebra* is a set B with at least two elements, 0 and 1 (*zero* and *unit*), endowed with binary operations $+$ and and a unary operation $-$.

The Boolean operations satisfy the following axioms:

$$\text{(17.2)}\qquad \begin{array}{ll} u+v=v+u & u\cdot v=v\cdot u \qquad \text{(commutativity)} \\ u+(v+w)=(u+v)+w & u\cdot(v\cdot w)=(u\cdot v)\cdot w \\ & \qquad\qquad\qquad \text{(associativity)} \\ u\cdot(v+w)=u\cdot v+u\cdot w & u+(v\cdot w)=(u+v)\cdot(u+w) \\ & \qquad\qquad\qquad \text{(distributivity)} \\ u+u=u & u\cdot u=u \\ u\cdot(u+v)=u & u+(u\cdot v)=u \\ x+0=x & x\cdot 0=0 \\ x+1=1 & x\cdot 1=x \\ x+(-x)=1 & x\cdot(-x)=0 \\ -(-x)=x & \\ -(x+y)=(-x)\cdot(-y) & -(x\cdot y)=(-x)+(-y) \\ & \qquad\qquad\qquad \text{(deMorgan laws)} \end{array}$$

Two elements u, v of B are *disjoint* if $u \cdot v = 0$. Let us write

$$u - v$$

instead of $u \cdot (-v)$, and let us define

$$u \leq v \quad \text{iff} \quad u - v = 0 \tag{17.3}$$

It is easy to see that $\leq$ is a partial ordering of B and that

$$u \leq v \quad \text{iff} \quad u + v = v \quad \text{iff} \quad u \cdot v = u \tag{17.4}$$

Moreover, 1 is the greatest element of B and 0 is the least element. Also, for any u, $v \in B$, $u + v$ is the least upper bound of u, v, and $u \cdot v$ is the greatest lower bound of $\{u, v\}$. Since $-u$ is the unique v such that $u + v = 1$ and $u \cdot v = 0$, it follows that all Boolean-algebraic operations can be defined in terms of the partial ordering of B.

For any field of sets $\mathscr{F}$, the set $\mathscr{F}$ with Boolean operations $X \cup Y$, $X \cap Y$, and $S - X$ is a Boolean algebra (the unit is S and the zero is $\emptyset$). The Boolean algebraic $\leq$ is the same as inclusion.

Exercise 17.1. Let B be a Boolean algebra and define

$$u \oplus v = (u - v) + (v - u)$$

Then B with operations $\oplus$ and $\cdot$ is a ring (with zero 0 and unit 1).

We shall now give an example showing the relation of Boolean algebras and logic. This relation will later be utilized in Boolean-valued models.

Let $\mathscr{L}$ be a first order language and let S be the set of all sentences of $\mathscr{L}$. We consider the equivalence relation $\vdash \varphi \leftrightarrow \psi$ over S. The set B of all equivalence classes $[\varphi]$ is a Boolean algebra under the following operations:

$$\begin{aligned} [\varphi] + [\psi] &= [\varphi \vee \psi] & 1 &= [\varphi \vee \neg\varphi] \\ [\varphi] \cdot [\psi] &= [\varphi \cdot \psi] & 0 &= [\varphi \wedge \neg\varphi] \\ -[\varphi] &= [\neg\varphi] \end{aligned}$$

This algebra is called the *Lindenbaum algebra.*

Another important example of a Boolean algebra is this: Let S be a topological space. An open set U in S is called *regular open* if U = interior (closure(U)). Let B be the collection of all regular open sets in the space S, and let

$$\begin{aligned} U \cdot V &= U \cap V \\ U + V &= \text{int cl}(U \cup V) \\ -U &= S - \text{cl}(U) \end{aligned} \tag{17.5}$$

Although tedious to verify, it is nevertheless true that B with the operations defined above is a Boolean algebra. Note that the complement of U in B is not the set-theoretical complement. However, $\leq$ is again the same as inclusion.

Exercise 17.2. If U is an open set, let $U^{\perp} = S - \text{cl}(U)$. Show that:

(i) $U^{\perp\perp\perp} = U^{\perp}$;
(ii) U is regular open iff $U^{\perp\perp} = U$;
(iii) $U^{\perp\perp}$ is the unique regular open set V such that $\text{cl}(U) = \text{cl}(V)$; it is maximal among all open V such that $\text{cl}(V) = \text{cl}(U)$.

A subset A of a Boolean algebra B is a *subalgebra* if it contains 0 and 1 and is closed under the Boolean operations:

(17.6) (i) $0 \in A$, $1 \in A$;
(ii) if $u, v \in A$, then $u + v \in A$, $u \cdot v \in A$, $-u \in A$.

If $X \subseteq B$, then there is a least subalgebra A of B that contains X; A can be described either as $\bigcap \{A : X \subseteq A \subseteq B$ and A is a subalgebra$\}$, or as the set of all Boolean combinations in B of elements of X. The subalgebra A is said to be *generated* by X. As for other abstract algebras, if X is infinite, then $|A| = |X|$. However, if X is finite, we can do better in case of Boolean algebras:

Exercise 17.3. A finitely generated Boolean algebra is finite. In fact, if A has k generators, then $|A| \leq 2^{2^k}$.

[Using distributivity and the deMorgan laws, show that the following finite sums constitute a subalgebra of B:

$$u_{s_1} + \cdots + u_{s_n}$$

where u_s, $s \in {}^k\{0, 1\}$ are pairwise disjoint elements of the form

$$u_s = \pm x_1 \cdot \pm x_2 \cdot \cdots \cdot \pm x_k$$

where $x_1, \ldots, x_k$ are the generators and $\pm x_i$ is either x_i or $-x_i$ according to whether $s(i) = 0$ or $s(i) = 1$.]

Let B and C be two Boolean algebras. A mapping h: $B \to C$ is a *homomorphism* if it preserves the operations:

(17.7) (i) $h(0) = 0, \quad h(1) = 1$;
(ii) $h(u + v) = h(u) + h(v)$,
$h(u \cdot v) = h(u) \cdot h(v)$,
$h(-u) = -h(u)$.

Note that the range of a homomorphism is a subalgebra of C and that $h(u) \leq h(v)$ whenever $u \leq v$. A one-to-one homomorphism of B onto C is called an *isomorphism*. An *embedding* of B in C is an isomorphism of B onto a subalgebra of C.

Exercise 17.4. If h: $B \to C$ is an order-isomorphism of B and C (i.e., a one-to-one mapping such that $u \leq v$ iff $hu \leq hv$), then h is an isomorphism.

An isomorphism of a Boolean algebra onto itself is called an *automorphism*. A nonzero element $a \in B$ is called an *atom* if it is a minimal element of $B - \{0\}$;

equivalently, if there is no x such that $0 < x < a$. A Boolean algebra is *atomic* if for every nonzero $u \in B$ there is an atom $a \le u$; B is *atomless* if it has no atoms.

Exercise 17.5. Every finite Boolean algebra B is atomic. If $A = \{a_1, \ldots, a_n\}$ are the atoms of B, then B is isomorphic to the field of sets $P(A)$. Hence B has 2^n elements.

Exercise 17.6. Any two countable atomless Boolean algebras are isomorphic.
[Use the method of Cantor's Theorem 11.]

Ideals and Filters

The definition of filter (and ideal) given in Section 7 generalizes to arbitrary Boolean algebras. Let B be a Boolean algebra. An *ideal* on B is a subset I of B such that:

(17.8) (i) $0 \in I$, $1 \notin I$;
(ii) if $u \in I$ and $v \in I$, then $u + v \in I$;
(iii) if $u, v \in B$, $u \in I$ and $v \le u$, then $v \in I$.

A *filter* on B is a subset F of B such that

(17.9) (i) $1 \in F$, $0 \notin F$;
(ii) if $u \in F$ and $v \in F$, then $u \cdot v \in F$;
(iii) if $u, v \in B$, $u \in F$, and $u \le v$, then $v \in F$.

In (16.2), we defined the notion of filter on a partially ordered set P. Note that if B is a Boolean algebra, then if we consider the partially ordered set $B - \{0\}$ for all nonzero elements of B then the present definition of filter on B coincides with the definition (16.2).

The *trivial* ideal is the ideal $\{0\}$; an ideal is *principal* if $I = \{u \in B : u \le u_0\}$ for some $u_0 \ne 1$. Similarly for filters.

Note that two nonzero elements u, v of B are disjoint just in case they are incompatible in the partially ordered set $B - \{0\}$. Thus we may call u, v *compatible* if $u \cdot v \ne 0$. A subset G of $B - \{0\}$ has the *finite intersection property* if for every finite $\{u_1, \ldots, u_n\} \subseteq G$,

$$u_1 \cdot \cdots \cdot u_n \ne 0$$

Every $G \subset B$ that has the finite intersection property generates a filter on B; this and the other two clauses of Lemma 7.1 hold also for Boolean algebras.

There are two important examples of ideals on the Boolean algebra of all Borel sets of reals:

1. The Borel sets of Lebesgue measure zero form an ideal.
2. A set X of reals is *nowhere dense* if the closure of X does not contain a nonempty open subset: $\operatorname{int}(\operatorname{cl} X) = \emptyset$. A union of countably many nowhere dense sets is called *meager*.

Exercise 17.7. The meager Borel sets form an ideal on the algebra of all Borel sets.
[The Baire category theorem states that $\boldsymbol{R}$ is not meager.]

There is a relation between ideals and homomorphisms. If $h: B \to C$ is a homomorphism, then

$$I = \{u \in B : h(u) = 0\} \tag{17.10}$$

is an ideal on B (the *kernel* of the homomorphism). On the other hand, let I be an ideal on B. Let us consider the following equivalence relation over B:

$$u \sim v \quad \text{iff} \quad u \mathbin{\triangle} v \in I \tag{17.11}$$

where

$$u \mathbin{\triangle} v = (u - v) + (v - u)$$

Let C be the set of all equivalence classes, $C = B/\sim$, and endow C with the following operations:

$$\begin{aligned} [u] + [v] &= [u + v], \qquad [u] \cdot [v] = [u \cdot v] \\ -[u] &= [-u] \\ 0 &= [0], \qquad\qquad\quad 1 = [1] \end{aligned} \tag{17.12}$$

(Of course, the conscientious reader will verify that (17.11) is an equivalence relation and that (17.12) does not depend on the choice of representatives.) Then C is a Boolean algebra, the *quotient* of B mod I, and is a homomorphic image of B under the homomorphism

$$h(u) = [u] \tag{17.13}$$

The quotient algebra is denoted B/I.

Exercise 17.8. Regarding a Boolean algebra as a ring (with operations $\oplus$ and $\cdot$, see Exercise 17.1), show that the definition of ideal coincides with the ring theoretical definition of ideal.

An ideal I on B is a *prime ideal* if

$$\text{for every } u \in B, \qquad \text{either } u \in I \text{ or } -u \in I \tag{17.14}$$

The dual of a prime ideal is an *ultrafilter.*

Lemma 7.2 holds in general: An ideal is a prime ideal (and a filter is an ultrafilter) iff it is maximal. Also, an ideal I on B is prime iff the quotient of B modulo I is the trivial algebra $\{0, 1\}$.

The ultrafilter theorem easily generalizes to Boolean algebras:

Theorem 21′ (*Prime Ideal Theorem*). *Every ideal on B can be extended to a prime ideal.*

The proof of the prime ideal theorem (PIT) uses the axiom of choice. It is known that PIT cannot be proved without using the axiom of choice (see Example V in Section 21). However, it is also known that PIT is weaker than full AC: It does not imply the axiom of choice. Still, in many proofs in algebra and topology, one can do with PIT in place of full AC (examples: compactification theorems, Hahn–Banach theorem).

Exercise 17.9. Show that PIT follows from its weaker version: Every Boolean algebra has a prime ideal.

[Given an ideal I on B, consider a prime ideal K on B/I and use K to construct a prime ideal $J \supseteq I$ on B.]

The prime ideal theorem is of particular importance in logic. We recall the basic two theorems of the first order predicate calculus:

Completeness Theorem. *Every consistent set of sentences has a model.*

Compactness Theorem. *A set Σ of sentences has a model if each finite subset of Σ has a model.*

The usual proofs of the completeness theorem use the axiom of choice; namely to show that a consistent set of sentences can be extended to a complete consistent set of sentences. Since this can be proved using an ultrafilter on the Lindenbaum algebra, it follows that the completeness theorem can be proved using PIT. The compactness theorem is an easy corollary of the completeness theorem. On the other hand, one can give a proof of PIT using the compactness theorem, thus showing that they are in fact equivalent:

Exercise 17.10. Using the compactness theorem, prove that every Boolean algebra has a prime ideal.

[Given B, consider the language containing $+, \cdot, -$, names for all $u \in B$, and a unary predicate I. Let Σ be the set of all quantifier-free sentences true in B, plus the sentences stating that I is a prime ideal: $I(0)$, $\neg I(1)$, $I(u) \vee I(-u)$, $I(u) \wedge I(v) \rightarrow I(u+v)$, $u, v \in B$.]

The prime ideal theorem also gives the following important representation theorem for Boolean algebras.

Theorem 41 (*Stone's Representation Theorem*). *Every Boolean algebra is isomorphic to a field of sets.*

Proof. Let B be a Boolean algebra. We let

$$S = \{p : p \text{ is an ultrafilter on } B\} \tag{17.15}$$

For every $u \in B$, let X_u be the set of all $p \in S$ such that $u \in p$. Let

$$\mathcal{F} = \{X_u : u \in B\} \tag{17.16}$$

Let us consider the mapping π: $u \mapsto X_u$. Clearly, $\pi(1) = S$ and $\pi(0) = \emptyset$. It follows from the definition of ultrafilter that

$$\pi(u \cdot v) = \pi(u) \cap \pi(v), \qquad \pi(u+v) = \pi(u) \cup \pi(v), \qquad \pi(-u) = S - \pi(u)$$

Thus π is a homomorphism of B onto $\mathcal{F}$ (and $\mathcal{F}$ is a field of sets). It remains to show that π is one-to-one.

If $u \neq v$, then using PIT, one can find an ultrafilter p on B containing one of these two elements but not the other. Thus π is an isomorphism. ■

Stone's representation theorem is usually considered in a topological form: Call the *Stone space* of B the space S of (17.15) with the topology given by the base $\mathscr{F}$ of (17.16).

Exercise 17.11.* The Stone space is a compact Hausdorff space with a base of clopen (closed and open) sets.

[To prove compactness: Every subset of B with the finite intersection property can be extended to an ultrafilter.]

Exercise 17.12.* B is isomorphic to the algebra of all clopen sets of its Stone space.

[Use compactness of S to show that every clopen set is in $\mathscr{F}$.]

The following two exercises show that if S is the Stone space of an infinite Boolean algebra B then $|S| \geq |B|$.

Exercise 17.13. Let A be a subalgebra of a Boolean algebra B and let $u \in B - A$. Then there exist ultrafilters F, G on B such that $u \in F$, $u \notin G$, and $F \cap A = G \cap A$.

Exercise 17.14. Let B be an infinite Boolean algebra, $|B| = \kappa$. There are at least κ ultrafilters on B.

[Assume otherwise. For each pair $(F, G) \in S \times S$ pick $u \in F - G$, and let these u's generate a subalgebra A. Since $|A| \leq |S| < \kappa$, let $u \in B - A$. Use Exercise 17.13 to get a contradiction.]

Complete Boolean Algebras

The partial ordering $\leq$ of a Boolean algebra can be used to define infinitary operations on B, generalizing $+$ and $\cdot$. Let us recall that

$$u + v = \sup\{u, v\}$$

$$u \cdot v = \inf\{u, v\}$$

in the partial ordering of B. Thus for any nonempty $X \subseteq B$, we define

$$\sum \{u : u \in X\} = \sup X \qquad (17.17)$$

$$\prod \{u : u \in X\} = \inf X$$

provided that the least upper bound (the greatest lower bound) exists. We also define $\sum \emptyset = 0$ and $\prod \emptyset = 1$.

If the infinitary sum and product is defined for all $X \subseteq B$, the Boolean algebra is called *complete*. Similarly, we call B *κ-complete* (where κ is a regular uncountable cardinal) if sums and products exist for all X of cardinality $< \kappa$. As usual, an $\aleph_1$-complete Boolean algebra is called *σ-complete* or *countably complete*.

An ideal I on a κ-complete Boolean algebra is *κ-complete* if

$$\sum \{u : u \in X\} \in I$$

whenever $X \subset I$ and $|X| < \kappa$. A κ-complete filter is the dual notion.

A field of sets $\mathscr{F}$ is *κ-complete* if it is closed under unions and intersections of collections of less that κ sets in $\mathscr{F}$. A κ-complete field of sets is a κ-complete Boolean algebra, and $\sum X = \bigcup X$ for every $X \subseteq \mathscr{F}$, $|X| < \kappa$.

A trivial example of a complete Boolean algebra is the complete field of sets $P(S)$. For a nontrivial example, let S be a topological space and let B be the collection of all regular open sets of the space S. B is a Boolean algebra, with the operations defined in (17.5). With a little additional effort the reader can verify that B is complete and that

$$\sum \{U : U \in X\} = \text{int cl} \bigcup X$$
$$\prod \{U : U \in X\} = \text{int cl} \bigcap X$$

Exercise 17.15. Let B be a complete Boolean algebra.

(a) Verify the distributive laws:

$$a \cdot \sum \{u : u \in X\} = \sum \{a \cdot u : u \in X\}$$
$$a + \prod \{u : u \in X\} = \prod \{a + u : u \in X\}$$

(b) Verify the deMorgan laws:

$$-\sum \{u : u \in X\} = \prod \{-u : u \in X\}$$
$$-\prod \{u : u \in X\} = \sum \{-u : u \in X\}$$

Exercise 17.16. If I is a κ-complete ideal on a κ-complete Boolean algebra B, then B/I is κ-complete, and

$$\sum \{[u] : u \in X\} = [\sum \{u : u \in X\}]$$

for every $X \subseteq B$, $|X| < \kappa$. Similarly for products.

Exercise 17.17. For B to be complete it is sufficient that all the sums $\sum X$ exist.
[$\prod X = \sum \{u : u \leq x \text{ for all } x \in X\}$.]

Let B be a complete Boolean algebra. A subalgebra A of B is a *complete subalgebra* if $\sum X \in A$ and $\prod X \in A$ for all $X \subseteq A$. (Caution: A subalgebra A of B that is itself complete is not necessarily a complete subalgebra of B.) Similarly, a *complete homomorphism* is a homomorphism h of B into C such that for all $X \subseteq B$,

$$h(\sum X) = \sum h[X], \qquad h(\prod X) = \prod h[X] \tag{17.18}$$

A *complete embedding* is an embedding that satisfies (17.18). Note that every isomorphism is complete.

Since the intersection of any collection of complete subalgebras of B is a complete subalgebra, every $X \subseteq B$ is included in a least complete subalgebra of B. This algebra is called the complete subalgebra of B (completely) *generated* by X.

Completion of Boolean Algebras

Let A be a subalgebra of a Boolean algebra B. A is said to be *dense* in B if for every nonzero $u \in B$ there is a nonzero $v \in A$ such that $v \leq u$. (Note that this is just in case that $A - \{0\}$ is dense in the partially ordered set $B - \{0\}$. We therefore call a set $D \subseteq B - \{0\}$ *dense in* B if D is dense in $B - \{0\}$.)

A *completion* of a Boolean algebra B is a complete Boolean algebra C such that B is a dense subalgebra of C.

Lemma 17.1. *The completion of a Boolean algebra B is unique up to isomorphism.*

Proof. Let C and D be completions of B. We define an isomorphism $\pi: C \to D$ as

$$\pi(c) = \textstyle\sum^D \{u \in B : u \leq c\} \tag{17.19}$$

To verify that π is an isomorphism, one uses the fact that B is a dense subalgebra of both C and D. For example, to show that $\pi c \neq 0$ whenever $c \neq 0$: There is $u \in B$ such that $0 < u \leq c$, and we have $0 < u \leq \pi c$. ■

Theorem 42 (*Completion of Boolean algebras*). *Every Boolean algebra has a completion.*

We shall prove a somewhat more general theorem (Lemma 17.2), using the method of Dedekind cuts. For a topological proof of Theorem 42, see the following exercise:

*Exercise 17.18.** Let S be the Stone space of a Boolean algebra B. The algebra of regular open sets of S is (isomorphic to) the completion of B.

If P is a dense subset of a Boolean algebra B (i.e., dense in $B - \{0\}$), then it has the following property:

(17.20) If $p \nleq q$, then there is $r \leq p$ that is incompatible with q.

We shall call a partially ordered set P satisfying (17.20) *separative.*

Lemma 17.2. *Let $(P, <)$ be a separative partially ordered set. Then there is a complete Boolean algebra B such that:*

(i) *$P \subseteq B - \{0\}$ and $<$ agrees with the partial ordering of B;*
(ii) *P is dense in B.*

The algebra B is unique up to isomorphism.

Proof. We use a construction similar to the method of Dedekind cuts (Theorem 12). Let us call a set $U \subseteq P$ a *cut* if

(17.21) $$p \leq q \quad \text{and} \quad q \in U \qquad \text{implies} \qquad p \in U$$

For every $p \in P$, let U_p denote the cut $\{x : x \leq p\}$.

A cut U is *regular* if

(17.22) whenever $p \notin U$, then $\exists q \leq p$ such that $U_q \cap U = \emptyset$.

Note that (17.20) defines $(P, <)$ to be separative just in case every U_p is regular.

We let B be the set of all regular cuts in P. We claim that B, under the partial ordering by inclusion, is a complete Boolean algebra. Note that the intersection of any collection of regular cuts is a regular cut, and hence each cut U is included in a least regular cut $\bar{U}$. In fact,

$$\bar{U} = \{p : (\forall q \leq p) U \cap U_q \neq \emptyset\}$$

Thus for $u, v \in B$ we have

$$u \cdot v = u \cap v, \qquad u + v = \overline{u \cup v}$$

The complement of $u \in B$ is the regular cut

$$-u = \{p : U_p \cap u = \emptyset\}$$

And, of course, $\emptyset$ and P are the zero and the unit of B. It is not difficult to verify that B is a complete Boolean algebra, and we leave the verification to the reader.

Since $(P, <)$ is separative, every U_p is a regular cut and so the function $e : p \mapsto U_p$ maps P into $B - \{0\}$. The function e is one-to-one and order-preserving. Moreover, the set of all U_p is dense in B and thus we have an embedding of P onto a dense subset of B.

Uniqueness of the complete Boolean algebra B is proved essentially by the same argument as in Lemma 17.1: If B and C are two complete Boolean algebras, both having P as a dense subset, then $\pi : B \to C$ defined is an isomorphism:

$$\pi(b) = \textstyle\sum^C \{p \in P : p \leq b\} \quad \blacksquare$$

The above proof of Lemma 17.2 can be regarded as a topological proof in disguise. When one defines a topology on P by taking the U_p as basic open sets then regular cuts are in fact regular open sets, and B is the algebra of regular open sets. Thus we shall use the following notation:

$$B = \text{r.o.}(P)$$

When $(P, <)$ is an arbitrary partially ordered set, we can still define the above topology on P and associate with P the algebra of regular open sets. However, if P is not separative, we do not get an embedding of P into B; the exact relation is described in the corollary below.

Lemma 17.3. *Let $(P, <)$ be a partially ordered set. There exists a separative partially ordered set $(Q, \prec)$ and a mapping h of P onto Q such that*

(i) *$x \leq y$ implies $h(x) \preccurlyeq h(y)$;*
(ii) *x and y are compatible in P iff $h(x)$ and $h(y)$ are compatible in Q.*

Proof. Let us define the following equivalence relation over P:

$$x \sim y \qquad \text{iff} \qquad \forall z (z \text{ is compatible with } x \leftrightarrow z \text{ is compatible with } y)$$

Let $Q = P/\sim$ and let us define

$$[x] \preccurlyeq [y] \qquad \text{iff} \qquad (\forall z \leq x)\,[z \text{ and } y \text{ are compatible}]$$

The relation $\preccurlyeq$ over Q is a partial ordering, and it does not take long to verify that $(Q, \prec)$ is separative. The mapping

$$h(x) = [x]$$

satisfies (i) and (ii). ■

Exercise 17.19. The partially ordered set Q in Lemma 17.3 is unique up to isomorphism.

[Use this: If $(Q, \prec)$ is separative, then $\preccurlyeq$ can be defined in terms of compatibility. $x \preccurlyeq y$ iff $\forall z(z$ and x are compatible $\rightarrow z$ and y are compatible.]

Corollary. *For every partially ordered set $(P, <)$ there is a complete Boolean algebra $B =$ r.o.(P) and a mapping e: $P \rightarrow B - \{0\}$ (the* canonical embedding[†]*) of P in B) such that:*

(17.23) (i) *if $p \leq q$, then $e(p) \leq e(q)$;*
(ii) *p and q are compatible iff $e(p) \cdot e(q) \neq 0$;*
(iii) *$\{e(p) : p \in P\}$ is dense in B.*

Generic Ultrafilters on Complete Boolean Algebras

An important consequence of Lemma 17.2 is that in theoretical considerations dealing with forcing and generic sets we can restrict ourselves to complete Boolean algebras (see Lemma 17.4).

In the ground model $\mathfrak{M}$, let B be a complete Boolean algebra. In the universe, B is still a Boolean algebra, though not necessarily complete. An ultrafilter G on B is called *generic* (over $\mathfrak{M}$) if

$$X \subseteq G \text{ and } X \in \mathfrak{M} \text{ implies } \prod X \in G \tag{17.24}$$

(G is also called *$\mathfrak{M}$-complete* since it is closed under products of collections in $\mathfrak{M}$.)

[†] e is not necessarily one-to-one.

Lemma 17.4. *Let P be a partially ordered set, let $B = \text{r.o.}(P)$ and let $e\colon P \to B$ be the canonical embedding of P in B (all this in the ground model $\mathfrak{M}$).*

If G is generic on B, then

$$G' = \{p \in P : e(p) \in G\}$$

is generic on P. Conversely, if G' is generic on P, then

$$G = \{u \in B : (\exists p \in G')\ e(p) \le u\}$$

is generic on B.

In either case, $\mathfrak{M}[G] = \mathfrak{M}[G']$.

Proof. (a) Let G be a generic ultrafilter on B. First we notice that G meets every dense set D in B (if $D \in \mathfrak{M}$). Let $X_D = \{-u : u \in D\}$. If $G \cap D = \emptyset$, then $X \subseteq G$ and hence $0 = \prod X_D \in G$, a contradiction.

Next we observe that if D is a dense set in P, then $e[P]$ is a dense set in B. Thus $G' = \{p \in P : e(p) \in G\}$ meets every dense set D in P which is in $\mathfrak{M}$, and it suffices to show that G' is a filter.

If p and q are arbitrary elements of P, then the set

$$E_{p,q} = \{r \in P : \text{either } r \le p \text{ and } r \le q, \text{ or } r \text{ is incompatible with } p, \text{ or } r \text{ is incompatible with } q\}$$

is dense in P: Given $s \in P$, if s is compatible with p, then there is $t \le s$ such that $t \le p$; if t is compatible with q, then there is $r \le t$ such that $r \le q$.

Let p and q be in G'. Then because $E_{p,q}$ is dense, there exists $r \in E_{p,q}$ such that $e(r) \in G$. It is obvious that r cannot be incompatible with either p or q and hence $r \le p$ and $r \le q$. Hence G' is a filter.

(b) Let G' be a generic filter on P and let $G = \{u \in B : (\exists p \in G')\ e(p) \le u\}$. Since G' is a filter on P, G is a filter on B.

If D is a dense set in B, then the set

$$E_D = \{p \in P : e(p) \le d \text{ for some } d \in D\}$$

is a dense set in P. Given $q \in P$, there is $d \in D$ such that $e(q) \cdot d \ne 0$; then there is $r \in P$ such that $e(r) \le d$ and that $e(q) \cdot e(r) \ne 0$. It follows that q and r are compatible and hence there is $p \in P$ such that $p \le q$ and $e(p) \le d$.

Now if $u \in B$, then the set $\{v \ne 0 : v \le u \text{ or } v \le -u\}$ is dense in B and hence

$$D_u = \{p \in P : e(p) \le u \text{ or } e(p) \le -u\}$$

is dense in P; thus $D_u \cap G' \ne \emptyset$, and therefore either u or $-u$ is in G. Hence G is an ultrafilter.

For every subset X of B (in $\mathfrak{M}$), let

$$D_X = \{p \in P : \text{either } e(p) \le \prod X \text{ or } e(p) \cdot x = 0 \text{ for some } x \in X\}$$

The set D_X is dense in P and hence meets G'. It follows that if $X \subseteq G$ and $X \in \mathfrak{M}$, then $\prod X \in G$. Thus G is generic over $\mathfrak{M}$.

(c) It is clear that in either case

$$p \in G' \quad \leftrightarrow \quad e(p) \in G$$

and that the filters G and G' can be obtained from one another in every model $\mathfrak{N} \supseteq \mathfrak{M}$ that contains either G or G'. Hence $\mathfrak{M}[G] = \mathfrak{M}[G']$.

Corollary. *$G \subseteq B$ is a generic ultrafilter on B if and only if G is a generic filter on the partially ordered set $B - \{0\}$.*

A *partition* of B is a pairwise disjoint subset W of B such that $\sum W = 1$. In general, a *partition of $u \in B$* is a pairwise disjoint W such that $\sum W = u$.

Exercise 17.20. An ultrafilter G is generic iff for every partition W of B such that $W \in \mathfrak{M}$, there exists a unique $u \in G \cap W$.

The following version of Lemma 16.1 will be useful later. If B is a Boolean algebra and $\mathscr{X}$ a collection of subsets of B, we call an ultrafilter F on B *$\mathscr{X}$-complete* if

$$\prod X \in F \qquad \text{whenever} \qquad X \subset F, \quad X \in \mathscr{X}, \quad \text{and} \quad \prod X \text{ exists}$$

Lemma 17.5. *If B is a Boolean algebra and $\mathscr{X}$ is a countable family of subsets of B, then there exists an $\mathscr{X}$-complete ultrafilter on B.*

Proof. Let $\mathscr{X} = \{X_0, X_1, \ldots, X_n, \ldots\}$ and assume that each $\prod X_n$ exists. We construct a sequence $a_0 \geq a_1 \geq \cdots$ of nonzero elements as follows: Let $a_0 = 1$. Having a_n, let $a_{n+1} = a_n \cdot \prod X_n$ if this is nonzero, or else let $a_{n+1} = a_n - x$ for some $x \in X_n$. Then let F be any ultrafilter that contains all the a_n. ■

A Few Facts on Complete Boolean Algebras

Given a Boolean algebra B and a nonzero $a \in B$, consider the set

$$B_a = \{u \in B : u \leq a\}$$

endowed with the following operations:

$$u +_a v = u + v, \qquad u \cdot_a v = u \cdot v, \qquad -_a u = a - u$$

B_a is a Boolean algebra. (Its Stone space is a subspace of the Stone space of B, namely the clopen set X_a.)

If W is a partition of a complete Boolean algebra B, then B can be viewed at as a *product* of the algebras B_u, $u \in W$; see Fig. 17.1. Each $b \in B$ corresponds to the function $\langle b \cdot u : u \in W \rangle$, and $b = \sum \{b \cdot u : u \in W\}$.

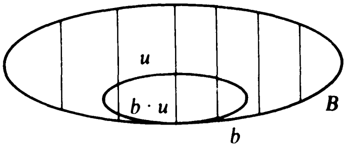

FIGURE 17.1

Exercise 17.21. **If B is an infinite complete Boolean algebra, then $|B|^{\aleph_0} = |B|$.**

[First consider the case when $|B_a| = |B|$ for all $a \neq 0$: There is a partition W such that $|W| = \aleph_0$, and $|B| = \prod\{|B_a| : a \in W\} = |B|^{\aleph_0}$. In general, call $a \neq 0$ *stable* if $|B_x| = |B_a|$ for all $x \leq a$, $x \neq 0$. The set of all stable $a \in B$ is dense, and $|B_a| = 2$ or $|B_a|^{\aleph_0} = |B_a|$ if a is stable. Let W be a partition of B such that each $a \in W$ is stable; we have $|B| = \prod\{|B_a| : a \in W\}$ and the theorem follows.]

The Chain Condition

Let κ be an infinite cardinal. A Boolean algebra B is *κ-saturated* if there is no partition W of B such that $|W| = \kappa$, and

$$\text{sat}(B) = \text{least } \kappa \text{ such that } B \text{ is } \kappa\text{-saturated} \tag{17.25}$$

B is also said to satisfy the *κ-chain condition*; this is because if B is complete, B is κ-saturated if and only if there exists no descending κ-sequence $u_0 > u_1 > \cdots > u_\alpha > \cdots$, $\alpha < \kappa$, of elements of B. The $\aleph_1$-chain condition is called the *countable chain condition* (c.c.c.).

Lemma 17.6. *If B is an infinite complete Boolean algebra, then* $\text{sat}(B)$ *is a regular uncountable cardinal.*

Proof. Let $\kappa = \text{sat}(B)$. It is clear that κ is uncountable. Let us assume that κ is singular; we shall obtain a contradiction by constructing a partition of size κ.

For $u \in B$, $u \neq 0$, let $\text{sat}(u)$ denote $\text{sat}(B_u)$. Let us call $u \in B$ *stable* if $\text{sat}(v) = \text{sat}(u)$ for every nonzero $v \leq u$. The set S of stable elements is dense in B; otherwise, there would be a descending sequence $u_0 > u_1 > u_2 > \cdots$ with decreasing cardinals $\text{sat}(u_0) > \text{sat}(u_1) > \cdots$. Let T be a maximal incompatible subset of S. Thus T is a partition of B, and $|T| < \kappa$.

First we show that $\sup\{\text{sat}(u) : u \in T\} = \kappa$. For every regular $\lambda < \kappa$ such that $\lambda > |T|$, consider a partition W of B of size λ. Then at least one $u \in T$ is partitioned by W into λ pieces.

Thus we consider two cases:

Case I. There is $u \in T$ such that $\text{sat}(u) = \kappa$. Since $\text{cf}\,\kappa < \kappa$, there is a partition W of u of size $\text{cf}\,\kappa$: $W = \{u_\alpha : \alpha < \text{cf}\,\kappa\}$. Let κ_α, $\alpha < \text{cf}\,\kappa$, be an increasing sequence with limit κ. For each α, $\text{sat}(u_\alpha) = \text{sat}(u) = \kappa$ and so let W_α be a partition of u_α of size κ_α. Then $\bigcup_{\alpha < \text{cf}\,\kappa} W_\alpha$ is a partition of u of size κ.

Case II. For all $u \in T$, $\text{sat}(u) < \kappa$, but $\sup\{\text{sat}(u) : u \in T\} = \kappa$. Again, let $\kappa_\alpha \to \kappa$, $\alpha < \text{cf}\,\kappa$. For each $\alpha < \text{cf}\,\kappa$ (by induction), we find $u_\alpha \in T$, distinct from all u_β, $\beta < \alpha$, which admits a partition W_α of size κ_α. Then $\bigcup_{\alpha < \text{cf}\,\kappa} W_\alpha$ is a pairwise disjoint subset of B of size κ. ■

A partially ordered set $(P, <)$ is *κ-saturated* (satisfies the *κ-chain condition*) if there is no set $W \subseteq P$ of pairwise incompatible elements such that $|W| = \kappa$. It can be easily observed that if P and Q are partially ordered sets and P is dense in Q, then $\text{sat}(P) = \text{sat}(Q)$. In particular, if B is a Boolean algebra and C is its completion, then $\text{sat}(B) = \text{sat}(C)$. Moreover, if a partially ordered set P is mapped into the complete Boolean algebra $B = \text{r.o.}(P)$, then the embedding $e\colon P \to B$ preserves compatibility (see (17.23)) and thus we have

$$\text{sat}(P) = \text{sat}(\text{r.o. } P) \tag{17.26}$$

Therefore, by Lemma 17.6, $\text{sat}(P)$ is always a regular uncountable cardinal (provided P admits infinite incompatible subsets).

Exercise 17.22. If $B = \text{r.o.}(P)$ and $\kappa = \text{sat}(P)$, then $|B| \leq |P|^{<\kappa}$.

[For each nonzero $u \in B$, there is an incompatible $W \subseteq P$ such that $u = \sum W$.]

Exercise 17.23. If B is a κ-complete, κ-saturated Boolean algebra, then B is complete.

[It suffices to show that $\sum X$ exists for every *open* X (i.e., $u \leq v \in X$ implies $u \in X$). If $X \subseteq B$ is open, show that $\sum X = \sum W$ where W is a maximal incompatible subset of X.]

Distributivity of Complete Boolean Algebras

The following distributive law holds for every complete Boolean algebra:

$$\sum_{i \in I} u_{0i} \cdot \sum_{j \in J} u_{1j} = \sum_{(i,j) \in I \times J} u_{0i} \cdot u_{1j}$$

(and dually, a similar law for a sum of two infinite products).

To formulate a general distributive law, let κ be a cardinal, and let us call B *κ-distributive* if

$$\prod_{\alpha<\kappa} \sum_{i \in I_\alpha} u_{\alpha i} = \sum_{f \in \prod_{\alpha<\kappa} I_\alpha} \prod_{\alpha<\kappa} u_{\alpha, f(\alpha)} \tag{17.27}$$

(Every complete field of sets satisfies (17.27).) We shall see later that distributivity plays an important role in generic models. For now, let us give two equivalent formulations of κ-distributivity.

If W and Z are partitions of B, then W is a *refinement* of Z if for every $w \in W$ there is $z \in Z$ such that $w \leq z$. A set $D \subseteq B$ is *open dense* if it is dense in B and $0 \neq u \leq v \in D$ implies $u \in D$.

Lemma 17.7. *The following are equivalent, for any complete Boolean algebra B:*

- (i) *B is κ-distributive.*
- (ii) *The intersection of κ open dense subsets of B is open dense.*
- (iii) *Every collection of κ partitions of B has a common refinement.*

Proof. (i) → (ii). Let D_α, $\alpha < \kappa$, be open dense, $D = \bigcap_{\alpha<\kappa} D_\alpha$. D is certainly open; thus let $u \neq 0$. If we let $\{u_{\alpha i} : i \in I_\alpha\} = \{u \cdot v : v \in D_\alpha\}$, then $\sum_i u_{\alpha i} = u$ for every α and the left-hand side of (17.27) is u. For each $f \in \prod_\alpha I_\alpha$, let $u_f = \prod_\alpha u_{\alpha, f(\alpha)}$; clearly, each nonzero u_f is in D. However, $\sum_f u_f = u$, by (17.27), and so some u_f is nonzero.

(ii) → (iii). Let W_α, $\alpha < \kappa$ be partitions of B. For each α, let $D_\alpha = \{u : u \leq v$ for some $v \in W_\alpha\}$; each D_α is open dense. Let $D = \bigcap_{\alpha<\kappa} D_\alpha$, and let W be a maximal incompatible subset of D. Since D is dense, W is a partition of B, and clearly, W is a refinement of each W_α.

(iii) → (i). Let $\{u_{\alpha i} : \alpha < \kappa, i \in I_\alpha\}$ be a collection of elements of B. First we show that the right-hand side of (17.27) is always $\leq$ the left-hand side. For each $f \in \prod_{\alpha<\kappa} I_\alpha$, let $u_f = \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}$; we have $u_f \leq u_{\alpha,f(\alpha)}$ and so $u_f \leq \sum_{i\in I_\alpha} u_{\alpha i}$ for each α. Thus, for each α,

$$\sum_f u_f \leq \sum_i u_{\alpha i}$$

and so

$$\sum_f \prod_\alpha u_{\alpha,f(\alpha)} = \sum_f u_f \leq \prod_\alpha \sum_i u_{\alpha i}$$

To prove (17.27), assume that (iii) holds, and let $u = \prod_\alpha \sum_i u_{\alpha i}$; we want to show that $\sum_f \prod_\alpha u_{\alpha,f(\alpha)} = u$. Without loss of generality, we can assume that $u = 1$ (otherwise we argue in the algebra B_u). For each α, let us replace $\{u_{\alpha i} : i \in I_\alpha\}$ by pairwise disjoint $\{v_{\alpha i} : i \in I_\alpha\} = W_\alpha$ such that $v_{\alpha i} \leq u_{\alpha i}$ and $\sum_i v_{\alpha i} = \sum u_{\alpha i}$ (some of the $v_{\alpha i}$ may be 0). Clearly $\sum_f \prod_\alpha v_{\alpha,f(\alpha)} \leq \sum_f \prod_\alpha u_{\alpha,f(\alpha)}$. Each W_α is a partition of B and so there is a partition W that is a refinement of each W_α. Now for each $w \in W$ there exists f such that $w \leq \prod_\alpha v_{\alpha,f(\alpha)}$, and so $\sum_f \prod_\alpha v_{\alpha,f(\alpha)} = 1$. ■

18. FORCING AND BOOLEAN-VALUED MODELS

Boolean-Valued Models

Let B be a complete Boolean algebra. A *Boolean-valued model* (of the language of set theory) $\mathfrak{U} = (\mathbf{U}, \mathbf{I}, \mathbf{E})$ consists of a *Boolean universe* $\mathbf{U}$ and binary functions $\mathbf{I}$ and $\mathbf{E}$ on $\mathbf{U}$ with values in B. Instead of $\mathbf{I}(x, y)$ and $\mathbf{E}(x, y)$, we write

$$\|x = y\|, \qquad \|x \in y\|$$

(the *Boolean values* of $=$ and $\in$), and assume that $\mathbf{I}$ and $\mathbf{E}$ satisfy the following:

$$\text{(18.1)}\quad \begin{aligned} &\text{(i)} && \|x = x\| = 1 \\ &\text{(ii)} && \|x = y\| = \|y = x\| \\ &\text{(iii)} && \|x = y\| \cdot \|y = z\| \leq \|x = z\| \\ &\text{(iv)} && \|x \in y\| \cdot \|v = x\| \cdot \|w = y\| \leq \|v \in w\| \end{aligned}$$

For every formula $\varphi(x_1, \ldots, x_n)$, we define the *Boolean value* of φ

$$\text{(18.2)}\qquad \|\varphi(a_1, \ldots, a_n)\| \qquad (a_1, \ldots, a_n \in \mathbf{U})$$

as follows:

(a) If φ is an atomic formula,

$$\|a = b\| = \mathbf{I}(a, b), \qquad \|a \in b\| = \mathbf{E}(a, b)$$

(b) If φ is a negation, conjunction, etc.,

$$\begin{aligned}
\|\neg\psi(a_1, \ldots, a_n)\| &= -\|\psi(a_1, \ldots, a_n)\| \\
\|\psi \wedge \chi(a_1, \ldots, a_n)\| &= \|\psi(a_1, \ldots, a_n)\| \cdot \|\chi(a_1, \ldots, a_n)\| \\
\|\psi \vee \chi(a_1, \ldots, a_n)\| &= \|\psi(a_1, \ldots, a_n)\| + \|\chi(a_1, \ldots, a_n)\| \\
\|\psi \rightarrow \chi(a_1, \ldots, a_n)\| &= \|\neg\psi \vee \chi(a_1, \ldots, a_n)\| \\
\|\psi \leftrightarrow \chi(a_1, \ldots, a_n)\| &= \|(\psi \rightarrow \chi) \wedge (\chi \rightarrow \psi)(a_1, \ldots, a_n)\|
\end{aligned}$$

(c) If φ is $\exists x\psi$ or $\forall x\psi$,

$$\begin{aligned}
\|\exists x\psi(x, a_1, \ldots, a_n)\| &= \sum_{a \in \mathbf{U}} \|\psi(a, a_1, \ldots, a_n)\| \\
\|\forall x\psi(x, a_1, \ldots, a_n)\| &= \prod_{a \in \mathbf{U}} \|\psi(a, a_1, \ldots, a_n)\|
\end{aligned}$$

Note how the notion of a Boolean-valued model generalizes the notion of a model (Section 10); the Boolean value of φ is a generalization of the satisfaction predicate $\models$. If B is the trivial algebra $\{0, 1\}$, then a Boolean-valued model is just a (two-valued) model; i.e., consider $\mathbf{U}/\equiv$ where $x \equiv y$ iff $\|x = y\| = 1$.

We say that $\varphi(a_1, \ldots, a_n)$ is *valid in* $\mathfrak{U}$, if $\|\varphi(a_1, \ldots, a_n)\| = 1$. Note that an implication $\varphi \rightarrow \psi$ is valid just in case $\|\varphi\| \leq \|\psi\|$. Hence it is postulated in (18.1) that the axioms for the equality predicate = are valid in a Boolean-valued model. It can be easily verified that all the other axioms of predicate calculus are valid, and that the rules of inference applied to valid sentences result in valid sentences. Thus every sentence provable in predicate calculus has Boolean value 1, and if two formulas φ, ψ are provably equivalent, we have $\|\varphi\| = \|\psi\|$. For example, we have

$$\|x = y\| \cdot \|\varphi(x)\| \leq \|\varphi(y)\|$$

The Boolean-valued models can therefore be used in consistency proofs in much the same way as two-valued models. Let $\mathfrak{U}$ be a Boolean-valued model such that all the axioms of ZFC are valid in $\mathfrak{U}$. (We say that $\mathfrak{U}$ is Boolean-valued model of ZFC.) Let A be set theoretical statement and assume that $\|\mathrm{A}\| \neq 0$. Then we can conclude that A is consistent relative to ZFC; otherwise, $\neg$A would be provable in ZFC and therefore valid in $\mathfrak{U}$: $\|\neg \mathrm{A}\| = -\|\mathrm{A}\| = 1$.

There is an important special case of Boolean-valued models, and in this special case, the Boolean valued model can be transformed into a two valued model.

We say that a Boolean valued model $\mathfrak{U}$ is *full* if for any formula $\varphi(x, x_1, \ldots, x_n)$ the following holds:

(18.3) for all $a_1, \ldots, a_n \in \mathbf{U}$, there exists $a \in \mathbf{U}$ such that

$$\|\varphi(a, a_1, \ldots, a_n)\| = \|\exists x\, \varphi(x, a_1, \ldots, a_n)\|$$

Let F be an ultrafilter on B. We define an equivalence relation over $\mathbf{U}$ by

$$(18.4) \qquad x \equiv y \quad \text{iff} \quad \|x = y\| \in F$$

and a binary relation $\mathbf{E}$ over $\mathbf{U}/\equiv$ by

$$(18.5) \qquad [x]\,\mathbf{E}\,[y] \quad \text{iff} \quad \|x \in y\| \in F$$

That $\equiv$ is an equivalence, and that (18.5) does not depend on the choice of representatives are easy consequences of (18.1) and the fact that F is a filter (cf. the definition of reduced power in Section 7). Thus $\mathfrak{U}/F = (\mathbf{U}/\equiv, \mathbf{E})$ is a model. Moreover, we have the following relationship between the Boolean valued model $\mathfrak{U}$ and the model $\mathfrak{U}/F$:

Lemma 18.1. *Let* $\mathfrak{U}$ *be full. For any formula* $\varphi(x_1, \ldots, x_n)$,

$$(18.6) \qquad \mathfrak{U}/F \vDash \varphi([a_1], \ldots, [a_n]) \quad \textit{iff} \quad \|\varphi(a_1, \ldots, a_n)\| \in F,$$

for all $a_1, \ldots, a_n \in \mathbf{U}$.

Proof. (a) If φ is atomic, then (18.6) is true by definition.

(b) If φ is a negation, conjunction, etc., we use the basic properties of an ultrafilter, and the definition of $\| \;\; \|$; e.g., we use

$$\|\neg\psi\| \in F \quad \text{iff} \quad \|\psi\| \notin F$$

$$\|\psi \wedge \chi\| \in F \quad \text{iff} \quad \|\psi\| \in F \text{ and } \|\chi\| \in F$$

(c) If φ is $\exists x \psi(x, \ldots)$, we use fullness of $\mathfrak{U}$ to prove (18.6), assuming it holds for ψ. By (18.3), we pick $a \in \mathbf{U}$ such that $\|\varphi(a, \ldots)\| = \|\exists x\, \varphi(x, \ldots)\|$ and then we have

$$\|\exists x\, \varphi(x, \ldots)\| \in F \quad \text{iff} \quad (\exists a \in \mathbf{U})\, \|\varphi(a, \ldots)\| \in F$$

which enables us to do the induction step in this case. ■

The Boolean-Valued Model V^B

We shall now define the Boolean-valued model V^B that we shall be using in subsequent independence proofs. Let B be a complete Boolean algebra.

Our intention is to define a Boolean-valued model in which all the axioms of ZFC are valid. In particular, we want V^B to be *extensional*, i.e., the Axiom of Extensionality to be valid in V^B:

$$(18.7) \qquad \|\forall u(u \in X \leftrightarrow u \in Y)\| \leq \|X = Y\|$$

We shall define V^B as a generalization of V: Instead of (two-valued) sets, we consider " Boolean-valued " sets, i.e., functions that assign Boolean values to its "elements." Thus we define V^B as follows:

(18.8)

$$\begin{aligned}
V_0^B &= \emptyset \\
V_\alpha^B &= \bigcup_{\beta<\alpha} V_\beta^B \qquad \text{if } \alpha \text{ is a limit ordinal} \\
V_{\alpha+1}^B &= \text{the set of all functions } x \text{ such that } \operatorname{dom}(x) \subseteq V_\alpha^B \text{ and } \operatorname{ran}(x) \subseteq B \\
V^B &= \bigcup_{\alpha \in Ord} V_\alpha^B
\end{aligned}$$

The definition of $\|x \in y\|$ and $\|x = y\|$ is motivated by (18.7), and the requirement

$$x(t) \leq \|t \in x\| \tag{18.9}$$

We define the Boolean values by induction. Each $x \in V^B$ is assigned the rank in V^B,

$$\rho(x) = \text{least } \alpha \text{ such that } x \in V_{\alpha+1}^B$$

The forthcoming definition is by induction on pairs $(\rho(x), \rho(y))$, under the canonical well ordering.

To make the notation more suggestive, we introduce the following Boolean operation (corresponding to the implication):

$$u \Rightarrow v = -u + v \tag{18.10}$$

(Compare with $\varphi \rightarrow \psi$ iff $\neg\varphi \vee \psi$.)

$$\|x \in y\| = \sum_{t \in \operatorname{dom} y} (\|x = t\| \cdot y(t)) \tag{18.11}$$

$$\|x \subseteq y\| = \prod_{t \in \operatorname{dom} x} (x(t) \Rightarrow \|t \in y\|) \tag{18.12}$$

$$= \prod_{t \in \operatorname{dom} x} \sum \{p \in B : \text{either } p \cdot x(t) = 0 \text{ or } p \leq \|t \in y\|\} \tag{18.13}$$

$$\|x = y\| = \|x \subseteq y\| \cdot \|y \subseteq x\| \tag{18.14}$$

We are going to show that V^B is a Boolean-valued model. To do that, we have to verify (18.1). To start, the clause (ii) in (18.1) is trivially satisfied since the definition of $\|x = y\|$ is symmetric in x and y:

$$\|x = y\| = \|y = x\|$$

Lemma 18.2. $\|x = x\| = 1$ *for all* $x \in V^B$.

Proof. By induction on $\rho(x)$. Clearly, it suffices to show that $\|x \subseteq x\| = 1$, i.e., by (18.12) we wish to show that

$$x(t) \Rightarrow \|t \in x\| = 1 \tag{18.15}$$

for all $t \in \operatorname{dom}(x)$. If $t \in \operatorname{dom}(x)$, then by induction hypothesis we have $\|t = t\| = 1$ and hence, by definition of $\|t \in x\|$,

$$x(t) = \|t = t\| \cdot x(t) \leq \|t \in x\| \tag{18.16}$$

Now (18.15) follows immediately from (18.16). ■

Incidentally, the proof of Lemma 18.2 also proves (18.9). To prove (18.1)(iii) and (iv), we prove them simultaneously by induction:

Lemma 18.3. *For all* $x, y, z \in V^B$,

(a) $\|x = y\| \cdot \|y = z\| \leq \|x = z\|$,
(b) $\|x \in y\| \cdot \|x = z\| \leq \|z \in y\|$,
(c) $\|y \in x\| \cdot \|x = z\| \leq \|y \in z\|$.

Proof. By induction on triples $(\rho(x), \rho(y), \rho(z))$, under the canonical well-ordering.

(a) It suffices to prove

$$\|x \subseteq y\| \cdot \|y = z\| \leq \|x \subseteq z\| \tag{18.17}$$

Let $t \in \operatorname{dom}(x)$ be arbitrary; we wish to show that

$$\|y = z\| \cdot (x(t) \Rightarrow \|t \in y\|) \leq x(t) \Rightarrow \|t \in z\| \tag{18.18}$$

(Then we get (18.17) from (18.18) using the definition (18.12) of $\|x \subseteq z\|$.) Note that

$$\|y = z\| \cdot (x(t) \Rightarrow \|t \in y\|) = \sum \{p \leq \|y = z\| : \text{either } p \cdot x(t) = 0 \text{ or } p \leq \|t \in y\|\} \tag{18.19}$$

By the induction hypothesis, we have

$$\|t \in y\| \cdot \|y = z\| \leq \|t \in z\|$$

Thus if $p \leq \|y = z\|$ satisfies the condition on the right-hand side of (18.19), it also satisfies the condition in (18.13) and we obtain (18.18).

(b) Let $t \in \operatorname{dom}(y)$ be arbitrary. By the induction hypothesis we have

$$\|x = z\| \cdot \|x = t\| \leq \|z = t\|$$

and so

$$\|x = z\| \cdot \|x = t\| \cdot y(t) \leq \|z = t\| \cdot y(t) \tag{18.20}$$

Taking the sum of (18.20) over all $t \in \operatorname{dom}(y)$, we get

$$\|x = z\| \cdot \sum_{t \in \operatorname{dom} y} (\|x = t\| \cdot y(t)) \leq \sum_{t \in \operatorname{dom} y} (\|z = t\| \cdot y(t))$$

that is,

$$\|x = z\| \cdot \|x \in y\| \leq \|z \in y\|$$

(c) Let $t \in \operatorname{dom}(x)$. By the definition of $\|t \in z\|$ we have

$$x(t) \cdot \|x = z\| \leq \|t \in z\|$$

and so

$$\|y = t\| \cdot x(t) \cdot \|x = z\| \leq \|y = t\| \cdot \|t \in z\| \tag{18.21}$$

By the induction hypothesis,

$$\|y = t\| \cdot \|t \in z\| \leq \|y \in z\|, \tag{18.22}$$

and therefore

$$\|y = t\| \cdot x(t) \cdot \|x = z\| \leq \|y \in z\| \tag{18.23}$$

Taking the sum of the left-hand side of (18.23) over all $t \in \operatorname{dom}(y)$, we get

$$\sum_{t \in \operatorname{dom} y} (\|y = t\| \cdot x(t)) \cdot \|x = z\| \leq \|y \in z\|$$

that is,

$$\|y \in x\| \cdot \|x = z\| \leq \|y \in z\| \quad \blacksquare$$

Thus V^B is a Boolean-valued model. We shall later show that all axioms of ZFC are valid in V^B. For the moment we shall just show that V^B is extensional, and also that V^B is full.

To show that V^B is extensional, let X, $Y \in V^B$. By the definition of $a \Rightarrow b$ (18.10), we observe that if $a \leq a'$, then $(a' \Rightarrow b) \leq (a \Rightarrow b)$. Thus for any $u \in V^B$ we have, using (18.9),

$$\|u \in X\| \Rightarrow \|u \in Y\| \leq X(u) \Rightarrow \|u \in Y\|$$

and therefore

$$\prod_{u \in V^B} (\|u \in X\| \Rightarrow \|u \in Y\|) \leq \prod_{u \in V^B} (X(u) \Rightarrow \|u \in Y\|) \tag{18.24}$$

While the left-hand side of (18.24) is equal to $\|\forall u(u \in X \rightarrow u \in Y)\|$, the right-hand side is easily seen to equal $\|Y \subseteq X\|$ (cf. (18.12)). Consequently,

$$\|\forall u(u \in X \leftrightarrow u \in Y)\| \leq \|X = Y\|$$

and we have:

Lemma 18.4. *V^B is extensional.* $\blacksquare$

We shall now show that V^B is full.

Lemma 18.5. *If W is a set of pairwise disjoint elements of B and if a_u, $u \in W$, are elements of V^B, then there exists $a \in V^B$ such that*

$$u \leq \|a = a_u\| \tag{18.25}$$

for all $u \in W$.

Proof. Let $D = \bigcup_{u \in W} \operatorname{dom}(a_u)$, and for every $t \in D$, let

$$a(t) = \sum \{u \cdot a_u(t) : u \in W\}$$

Since the u's are pairwise disjoint, we have for each $u \in W$ and each $t \in D$,

$$u \cdot a(t) = u \cdot a_u(t)$$

In other words,

$$u \le (a(t) \Rightarrow a_u(t)), \qquad u \le (a_u(t) \Rightarrow a(t))$$

and by (18.12)

$$u \le \|a = a_u\| \quad \blacksquare$$

Lemma 18.6. *V^B is full. Given a formula $\varphi(x, \ldots)$, there exists $a \in V^B$ such that*

$$\|\varphi(a, \ldots)\| = \|\exists x\, \varphi(x, \ldots)\| \tag{18.3}$$

Proof. In (18.3), $\le$ holds for every a. We wish to find $a \in V^B$ such that $\ge$ holds. Let $u_0 = \|\exists x \varphi(x, \ldots)\|$. Let

$$D = \{u \in B : \text{there is } a_u \text{ such that } u \le \|\varphi(a_u, \ldots)\|\}$$

It is clear that D is open and dense below u_0. Let W be a maximal set of pairwise disjoint elements of D; clearly, $\sum \{u : u \in W\} \ge u_0$. By Lemma 18.5 there exists $a \in V^B$ such that $u \le \|a = a_u\|$ for all $u \in W$. Thus for each $u \in W$ we have $u \le \|\varphi(a, \ldots)\|$, and hence $u_0 \le \|\varphi(a, \ldots)\|$. $\blacksquare$

Exercise 18.1.

(a) $\|(\exists y \in x)\varphi(y)\| = \sum_{y \in \operatorname{dom}(x)} (x(y) \cdot \|\varphi(y)\|)$

(b) $\|(\forall y \in x)\varphi(y)\| = \prod_{y \in \operatorname{dom}(x)} (x(y) \Rightarrow \|\varphi(y)\|)$

Exercise 18.2. Lemma 18.6 was the only place in the present section where we used the axiom of choice.

Forcing

Let $(P, <)$ be a notion of forcing, i.e., a partially ordered set. As I promised in Section 16, I shall now define the forcing language associated with $(P, <)$, and the forcing relation

$$p \Vdash \varphi$$

(p *forces* φ), where $p \in P$ and φ is a sentence of the forcing language.

As proved in Section 17, there exists a complete Boolean algebra $B = \text{r.o.}(P)$, and a canonical embedding $e\colon P \to B - \{0\}$ such that

(17.23) (i) if $p \le q$, then $e(p) \le e(q)$;
(ii) p and q are compatible iff $e(p) \cdot e(q) \ne 0$;
(iii) $\{e(p) : p \in P\}$ is dense in B.

Exercise 18.3. For each $S \subseteq P$, let $S^\perp = \{q \in P : q$ is incompatible with every $p \in S\}$. Let $B = \{u \subseteq P : u^{\perp\perp} = u\}$, and partially ordered by inclusion. Then $B = \text{r.o.}(P)$ and:

(a) $e(p) = \{p\}^{\perp\perp}$;
(b) $-u = u^\perp$, $u \cdot v = u \cap v$, $u + v = (u \cup v)^{\perp\perp}$.

[Compare with Exercise 17.2 and Lemma 17.2.]

The *forcing language* associated with the notion of forcing $(P, <)$ is the language of set theory with additional constants (*names*), namely the elements of the Boolean-valued model V^B (where $B = \text{r.o.}(P)$). The forcing relation is defined as follows

$$(18.26) \qquad p \Vdash \varphi(a_1, \ldots, a_n) \qquad \text{iff} \qquad e(p) \leq \|\varphi(a_1, \ldots, a_n)\|$$

where φ is a formula and $a_1, \ldots, a_n \in V^B$ (i.e., $\varphi(a_1, \ldots, a_n)$ is a sentence of the forcing language).

The following two lemmas are easy consequences of the definition of $\| \ \|$ and (18.26):

Lemma 18.7.

(a) *If p forces* φ *and* $q \leq p$, *then* $q \Vdash \varphi$.
(b) *No p forces both* φ *and* $\neg\varphi$.
(c) *For every p there is* $q \leq p$ *such that q* decides φ, i.e., *either* $q \Vdash \varphi$ *or* $q \Vdash \neg\varphi$. ■

Lemma 18.8.

(a) $p \Vdash \neg\varphi$ *iff no* $q \leq p$ *forces* φ.
(b) $p \Vdash \varphi \wedge \psi$ *iff* $p \Vdash \varphi$ *and* $p \Vdash \psi$.
$p \Vdash \forall x \varphi$ *iff* $p \Vdash \varphi(a)$ *for every* $a \in V^B$.
(c) $p \Vdash \varphi \vee \psi$ *iff* $\forall q \leq p\ \exists r \leq q (r \Vdash \varphi$ *or* $r \Vdash \psi)$.
$p \Vdash \exists x \varphi$ *iff* $\forall q \leq p\ \exists r \leq q\ \exists a\ r \Vdash \varphi(a)$. ■

Exercise 18.4. When does p force $\varphi \to \psi$? When $\varphi \leftrightarrow \psi$?

Exercise 18.5. If $\{q : q \Vdash \varphi\}$ is dense below p, then p forces φ.

Proof of the Forcing Theorem

We are now ready to prove Theorem 40 stated in Section 16. Let us recall the background: We are considering a fixed transitive model $\mathfrak{M}$ of ZFC (the ground model), and in $\mathfrak{M}$, a notion of forcing $(P, <)$.

To define the forcing relation associated with P, we let (in $\mathfrak{M}$) $B = \text{r.o.}(P)$ and construct the Boolean-valued model V^B. Since all this is done inside $\mathfrak{M}$, we denote this Boolean-valued model by $\mathfrak{M}^B$. $\mathfrak{M}^B$ is a class in $\mathfrak{M}$, and

$\mathfrak{M}^B = \bigcup_{\alpha \in Ord \cap \mathfrak{M}} \mathfrak{M}^B_\alpha$, where

$$\mathfrak{M}^B_{\alpha+1} = \{x \in \mathfrak{M} : x \text{ is a function, } \operatorname{dom}(x) \subseteq \mathfrak{M}^B_\alpha, \operatorname{ran}(x) \subseteq B\}$$

etc. Still in $\mathfrak{M}$, we define the Boolean values $\|\varphi\|$ and the forcing relation $p \Vdash \varphi$. Just for completeness, let us state explicitly that for any formula φ, the relation

$$p \Vdash \varphi(a_1, \ldots, a_n), \qquad p \in P, \quad a_1, \ldots, a_n \in \mathfrak{M}^B$$

is in the model $\mathfrak{M}$.

Now let G be a filter on P generic over $\mathfrak{M}$. We shall define a transitive model $\mathfrak{M}[G]$, give the elements of $\mathfrak{M}[G]$ names in $\mathfrak{M}^B$, and prove:

Forcing Theorem. *For any formula* φ,

$$(18.27) \qquad \mathfrak{M}[G] \vDash \varphi(x_1, \ldots, x_n) \quad \textit{iff} \quad (\exists p \in G) p \Vdash \varphi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathfrak{M}^B$ *are names for* $x_1, \ldots, x_n$.

To define $\mathfrak{M}[G]$, we construct, by induction on $\rho(x)$, a function i_G on $\mathfrak{M}^B$ called the *G-interpretation* of $\mathfrak{M}^B$:

$$(18.28) \quad \text{(i)} \quad i_G(\emptyset) = \emptyset;$$
$$\text{(ii)} \quad i_G(x) = \{i_G(y) : (\exists p \in G)\, e(p) \le x(y)\}.$$

Here e is the canonical embedding of P in B; in (ii), $i_G(y)$ is already defined for all $y \in \operatorname{dom}(x)$. Now we define

$$\mathfrak{M}[G] = \{i_G(x) : x \in \mathfrak{M}^B\}$$

It is clear from (18.28) that $\mathfrak{M}[G]$ is a transitive class. For each $x \in \mathfrak{M}[G]$, there exists $\boldsymbol{x} \in \mathfrak{M}^B$ such that $x = i_G(\boldsymbol{x})$. We call (any such) $\boldsymbol{x}$ a *name* for x.

We showed in Lemma 17.4 that if we let

$$G' = \{u \in B : (\exists p \in G)\, e(p) \le u\}$$

then G' is an $\mathfrak{M}$-generic ultrafilter on B. Note that for every $x \in \mathfrak{M}^B$,

$$(18.29) \qquad i_{G'}(x) = \{i_{G'}(y) : x(y) \in G'\}$$

Moreover, it follows easily by induction that $i_{G'}(x) = i_G(x)$ for all $x \in \mathfrak{M}^B$, and consequently, $\mathfrak{M}[G'] = \mathfrak{M}[G]$. Similarly, it is easy to verify that the right-hand side of (18.27) is equivalent to $\|\varphi\| \in G'$. Thus we can restate the Forcing Theorem as follows:

If G is an $\mathfrak{M}$-generic ultrafilter on B, then

$$(18.30) \qquad \mathfrak{M}[G] \vDash \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad \|\varphi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\| \in G$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathfrak{M}^B$ are names for $x_1, \ldots, x_n$.

We prove the forcing theorem (18.30) in two steps; first for atomic formulas, and then for arbitrary φ.

Lemma 18.9. *Let G be a generic ultrafilter on B and let $i = i_G$ be the G-interpretation of $\mathfrak{M}^B$. Then*

(a) $i(x) \in i(y)$ *iff* $\|x \in y\| \in G$,
(b) $i(x) = i(y)$ *iff* $\|x = y\| \in G$,

for all $x, y \in \mathfrak{M}^B$.

Proof. We prove (a) and (b) simultaneously, by induction on pairs $(\rho(x), \rho(y))$. ① holds by the genericity of G, ② by the induction hypothesis.

$$
\begin{array}{lll}
\text{(a)}\ \|x \in y\| \in G & \text{iff} & \sum_{t \in \operatorname{dom} y} (y(t) \cdot \|x = t\|) \in G. \\
① & \text{iff} & (\exists t \in \operatorname{dom}(y))[y(t) \in G \text{ and } \|x = t\| \in G] \\
② & \text{iff} & \exists t[y(t) \in G \text{ and } i(x) = i(t)] \\
 & \text{iff} & i(x) \in \{i(t) : y(t) \in G\} \\
 & \text{iff} & i(x) \in i(y)
\end{array}
$$

$$
\begin{array}{lll}
\text{(b)}\ \|x \subseteq y\| \in G & \text{iff} & \prod_{t \in \operatorname{dom}(x)} (x(t) \Rightarrow \|t \in y\|) \in G. \\
① & \text{iff} & (\forall t \in \operatorname{dom}(x))[x(t) \in G \text{ implies } \|t \in y\| \in G] \\
② & \text{iff} & \forall t[x(t) \in G \text{ implies } i(t) \in i(y)] \\
 & \text{iff} & \{i(t) : x(t) \in G\} \subseteq i(y) \\
 & \text{iff} & i(x) \subseteq i(y) \quad \blacksquare
\end{array}
$$

Exercise 18.6. Let G be a generic ultrafilter on B and let $\mathfrak{M}^B/G$ be the model defined as in (18.4) and (18.5). Then $\mathfrak{M}^B/G$ is isomorphic to $\mathfrak{M}[G]$.
[Let $h([x]) = i_G(x)$.]

Once we have proved (18.30) for atomic formulas, we finish the proof for arbitrary φ by induction on the complexity of φ.

(a) φ is $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, etc. Assuming (18.30) for ψ and χ, the induction step works because G is an ultrafilter. For instance,

$$
\begin{array}{lll}
\mathfrak{M}[G] \vDash \psi \wedge \chi & \text{iff} & \mathfrak{M}[G] \vDash \psi \text{ and } \mathfrak{M}[G] \vDash \chi \\
 & \text{iff} & \|\psi\| \in G \text{ and } \|\chi\| \in G \\
 & \text{iff} & \|\psi\| \cdot \|\chi\| \in G \\
 & \text{iff} & \|\psi \wedge \chi\| \in G
\end{array}
$$

Similarly for $\neg$, $\vee$, etc.

(b) φ is $\exists x\, \psi(x, \ldots)$ or $\forall x\, \psi(x, \ldots)$. We assume (18.30) for ψ and use the genericity of G:

$$\begin{array}{rll}
\mathfrak{M}[G] \vDash \exists x\; \psi(x, \ldots) & \text{iff} & (\exists x \in \mathfrak{M}[G])\mathfrak{M}[G] \vDash \psi(x, \ldots) \\
 & \text{iff} & (\exists \mathbf{x} \in \mathfrak{M}^B)\mathfrak{M}[G] \vDash \psi(i(\mathbf{x}), \ldots) \\
 & \text{iff} & (\exists \mathbf{x} \in \mathfrak{M}^B)\|\psi(\mathbf{x}, \ldots)\| \in G \\
\text{①} & \text{iff} & \sum_{x \in \mathfrak{M}^B} \|\psi(x, \ldots)\| \in G \\
 & \text{iff} & \|\exists x \psi(x, \ldots)\| \in G
\end{array}$$

The equivalence ① holds because if we let $A = \{\|\psi(\mathbf{x}, \ldots)\| : \mathbf{x} \in \mathfrak{M}^B\}$, then $A \subseteq B$ and $A \in \mathfrak{M}$, and since G is generic we have

$$(\exists a \in A)\, a \in G \qquad \text{iff} \qquad \sum A \in G$$

Similarly for $\forall x\; \psi(x, \ldots)$. ■

Proof of the Generic Model Theorem

Having defined $\mathfrak{M}[G]$, we shall now use the forcing theorem to prove that:

(a) $\mathfrak{M}[G]$ is a model of ZFC.
(b) $\mathfrak{M} \subseteq \mathfrak{M}[G]$ and $G \in \mathfrak{M}[G]$.
(c) $Ord^{\mathfrak{M}[G]} = Ord^{\mathfrak{M}}$.
(d) If $\mathfrak{N}$ is a transitive model of ZF such that $\mathfrak{M} \subseteq \mathfrak{N}$ and $G \in \mathfrak{N}$, then $\mathfrak{M}[G] \subseteq \mathfrak{N}$.

Throughout, G is a fixed $\mathfrak{M}$-generic ultrafilter on B, a complete Boolean algebra in $\mathfrak{M}$. We first dispose of (d). If $\mathfrak{M}$ is a class in $\mathfrak{N}$, then the assertion is trivial: The function i_G is a class in $\mathfrak{N}$ and thus $\mathfrak{M}[G]$ is a class in $\mathfrak{N}$. However, $\mathfrak{M}$ need not be a class in $\mathfrak{N}$; but the assertion is still true: For each $\alpha \in Ord^{\mathfrak{M}}$, $\mathfrak{M}_\alpha^B \in \mathfrak{M} \subseteq \mathfrak{N}$, and thus, by induction, $i_G | \mathfrak{M}_\alpha^B \in \mathfrak{N}$. Therefore $i_G(x) \in \mathfrak{N}$ for all $x \in \mathfrak{M}^B$ and we have $\mathfrak{M}[G] \subseteq \mathfrak{N}$.

Next we prove (b). To show that $\mathfrak{M} \subseteq \mathfrak{M}[G]$, we define *canonical names* for sets in the ground model. We define (in $\mathfrak{M}$) an embedding of $\mathfrak{M}$ in $\mathfrak{M}^B$, by $\in$-induction:

(18.31) (i) $\check{\emptyset} = \emptyset$;
(ii) for every $x \in \mathfrak{M}$, let $\check{x} \in \mathfrak{M}^B$ be the function whose domain is $\{\check{y} : y \in x\}$, and for all $y \in x$,

$$\check{x}(\check{y}) = 1$$

It is easy to see that for each $x \in \mathfrak{M}$,

$$i_G(\check{x}) = x$$

and so $\check{x}$ is a name for x (for any generic G). Thus $\mathfrak{M} \subseteq \mathfrak{M}[G]$. To show that $G \in \mathfrak{M}[G]$, we define a canonical name for G, the *canonical generic ultrafilter*: Let $\boldsymbol{G} \in \mathfrak{M}^B$ be as follows:

$$(18.32) \qquad \operatorname{dom}(\boldsymbol{G}) = \{\check{u} : u \in B\}, \qquad \boldsymbol{G}(\check{u}) = u \text{ for every } u \in B$$

If G is any generic ultrafilter, then

$$i_G(\boldsymbol{G}) = \{i_G(x) : \boldsymbol{G}(x) \in G\} = \{i_G(\check{u}) : u \in G\} = G$$

Thus $G \in \mathfrak{M}[G]$ and we have proved (b).

Exercise 18.7. If $x, y \in \mathfrak{M}$, then $\|\check{x} = \check{y}\| = 1$ if $x = y$ and 0 if $x \neq y$. Similarly for $\|\check{x} \in \check{y}\|$.

Exercise 18.8. For every $u \in B$, $\|\check{u} \in \boldsymbol{G}\| = u$.

As for (c), this is easy. Since $\mathfrak{M} \subseteq \mathfrak{M}[G]$, we have $Ord^{\mathfrak{M}} \subseteq Ord^{\mathfrak{M}[G]}$. On the other hand, note that for every $x \in \mathfrak{M}^B$,

$$\operatorname{rank} i_G(x) \leq \rho(x)$$

(by induction) and therefore all sets in $\mathfrak{M}[G]$ have rank equal to an ordinal in $\mathfrak{M}$.

It remains to show that $\mathfrak{M}[G]$ is a model of ZFC:

A. *Extensionality, Regularity, Infinity.* These axioms hold in $\mathfrak{M}[G]$ since $\mathfrak{M}[G]$ is transitive and $\mathfrak{M}[G] \supseteq \mathfrak{M}$.

B. *Separation.* If φ is a formula (we suppress the parameter),

$$\mathfrak{M}[G] \vDash \forall X\ \exists Y\ \forall x(x \in Y \leftrightarrow x \in X \wedge \varphi(x))$$

Let $X \in \mathfrak{M}[G]$ and let $\boldsymbol{X} \in \mathfrak{M}^B$ be a name for X. We let $\boldsymbol{Y} \in \mathfrak{M}^B$ be as follows:

$$(18.33) \qquad \operatorname{dom}(\boldsymbol{Y}) = \operatorname{dom}(\boldsymbol{X}), \qquad \boldsymbol{Y}(t) = \boldsymbol{X}(t) \cdot \|\varphi(t)\|$$

Now, if $Y = i_G(\boldsymbol{Y})$, it is clear that $Y \subseteq X$. Moreover, for every $\boldsymbol{x} \in \mathfrak{M}^B$ we have

$$\|\boldsymbol{x} \in \boldsymbol{Y}\| = \|\boldsymbol{x} \in \boldsymbol{X}\| \cdot \|\varphi(\boldsymbol{x})\|$$

and so for every $x \in X$, $\mathfrak{M}[G]$ satisfies

$$x \in Y \leftrightarrow x \in X \wedge \varphi(x)$$

C. *Pairing, Union, Power set.* Before we verify these axioms in $\mathfrak{M}[G]$, let us make the following observation: Let $S \in \mathfrak{M}$ be a set of names (i.e., $S \subset \mathfrak{M}^B$). Then the set $i_G[S] = \{i_G(t) : t \in S\}$ is in $\mathfrak{M}[G]$. For, consider this $\boldsymbol{X} \in \mathfrak{M}^B$: $\operatorname{dom}(\boldsymbol{X}) = S$, $\boldsymbol{X}(t) = 1$ for all $t \in S$. Clearly, $i_G(\boldsymbol{X}) = i_G[S]$.

In particular, if $x, y \in \mathfrak{M}^B$, we denote $\{x, y\}^B$ (the Boolean-valued pair), the $z \in \mathfrak{M}^B$ such that $\operatorname{dom}(z) = \{x, y\}$, and $z(x) = z(y) = 1$. Also, $(x, y)^B = \{\{x\}^B, \{x, y\}^B\}^B$. Clearly, $i_G(\{x, y\}^B) = \{i_G(x), i_G(y)\}$.

Exercise 18.9. If $\mathfrak{M}$ contains all ordinals, then $\mathfrak{M}[G]$ is almost universal. Consequently, $\mathfrak{M}[G]$ is a model of ZF.

Since we are proving Theorem 39 even in the case when $\mathfrak{M}$ is only a set, let us continue with the proof. We have just shown that $\mathfrak{M}[G]$ satisfies the Pairing Axiom. To verify Union and Power set, it suffices to show that for every $X \in \mathfrak{M}[G]$:

$$\bigcup X \subseteq Y \text{ for some } Y \in \mathfrak{M}[G]; \tag{18.34}$$

$$P(X) \cap \mathfrak{M}[G] \subseteq Y \text{ for some } Y \in \mathfrak{M}[G]. \tag{18.35}$$

Let $X \in \mathfrak{M}[G]$ and let $\mathbf{X}$ be a name for X. As for the union, let

$$S = \bigcup \{\operatorname{dom}(\mathbf{y}) : \mathbf{y} \in \operatorname{dom} \mathbf{X}\}$$

and let $Y = i_G[S]$. If $z \in \bigcup X$, then $z \in i_G(\mathbf{y})$ for some $\mathbf{y} \in \operatorname{dom}(\mathbf{X})$ and so $z \in Y$; hence (18.34) holds.

As for the power set, let us first show that every subset of X has a name of a certain form. If $Z \subseteq X$ and if $\mathbf{Z}$ is a name for Z, let $\mathbf{Z}'$ be as follows:

$$\operatorname{dom}(\mathbf{Z}') = \operatorname{dom}(\mathbf{X}), \qquad \mathbf{Z}'(t) = \mathbf{X}(t) \cdot \|t \in \mathbf{Z}\| \qquad \text{for all } t \tag{18.36}$$

Clearly, $i_G(\mathbf{Z}') = X \cap i_G(\mathbf{Z}) = Z$ and so $\mathbf{Z}'$ is a name for Z. Therefore, if we let

$$S = \{\mathbf{Z} : \operatorname{dom}(\mathbf{Z}) = \operatorname{dom}(\mathbf{X}) \text{ and } \mathbf{Z}(t) \leq \mathbf{X}(t) \text{ for all } t\}$$

then $Y = i_G[S]$ satisfies (18.35).

D. *Replacement.* As shown in Section 9, the replacement schema follows from the collection principle. Thus we show that $\mathfrak{M}[G]$ satisfies

$$\forall X \ \exists Y (\forall u \in X)[\exists v \ \varphi(u, v) \rightarrow (\exists v \in Y) \ \varphi(u, v)] \tag{9.4}$$

Let $X \in \mathfrak{M}[G]$ and let $\mathbf{X}$ be a name for X. In $\mathfrak{M}$, there is for each $\mathbf{u} \in \operatorname{dom}(\mathbf{X})$ a set of names $S_{\mathbf{u}}$ such that

$$\sum_{\mathbf{v} \in \mathfrak{M}^B} \|\varphi(\mathbf{u}, \mathbf{v})\| = \sum_{\mathbf{v} \in S_{\mathbf{u}}} \|\varphi(\mathbf{u}, \mathbf{v})\| \tag{18.37}$$

We let $S = \bigcup \{S_{\mathbf{u}} : \mathbf{u} \in \operatorname{dom} \mathbf{X}\}$, and $Y = i_G[S]$. Now, if $u \in X$ is such that $\mathfrak{M}[G] \Vdash \exists v \varphi(u, v)$, then $u = i_G(\mathbf{u})$ where $\mathbf{X}(\mathbf{u}) \in G$, and by (18.37), $\mathfrak{M}[G]$ satisfies

$$\exists v \ \varphi(u, v) \quad \rightarrow \quad (\exists v \in Y) \ \varphi(u, v)$$

Therefore $\mathfrak{M}[G]$ is a model of ZF. Note that so far we have not used the axiom of choice:

Exercise 18.10. If $\mathfrak{M}$ is a model of ZF, then $\mathfrak{M}[G]$ is a model of ZF.

To show that $\mathfrak{M}[G]$ satisfies the axiom of choice, we prove the following lemma:

Lemma 18.10. *Let i_G be the interpretation function. If $S \in \mathfrak{M}$ is a set of names, then $i_G | S \in \mathfrak{M}[G]$.*

Proof. Let $T = \{(\check{s}, s)^B : s \in S\}$. Clearly, $i_G | S = i_G[T]$. ∎

In fact, in the case that $\mathfrak{M}$ is a class in $\mathfrak{M}[G]$, then i_G is a class in $\mathfrak{M}[G]$. In any case, if $X \in \mathfrak{M}[G]$, then let $\dot{X}$ be a name for X and let $S = \operatorname{dom}(\dot{X})$. By Lemma 18.10, $i_G | S$ is in $\mathfrak{M}[G]$, and since S can be well-ordered, it follows that X, an image of a well-ordered set, can be well-ordered (in $\mathfrak{M}[G]$).

Generic Models and Independence Proofs

We shall conclude this section by providing answers to the following questions about generic models: Why can we assume that a generic filter exists? And how are generic models used to obtain an independence proof?

We shall first prove a theorem about the Boolean-valued model V^B.

Theorem 43

(a) *V^B is a Boolean-valued model of ZFC. Every axiom of ZFC has Boolean value* 1.

(b) *Let $\mathscr{D} = P(B)$, and let $\dot{G}$ be the canonical name for a generic ultrafilter on B. Then*

$$\|\dot{G} \text{ is a } \check{\mathscr{D}}\text{-generic ultrafilter on } \check{B}\| = 1 \tag{18.38}$$

Proof. (a) I am very reluctant to give a detailed proof that for every axiom φ of ZFC, $\|\varphi\| = 1$. The main reason is that the arguments are exactly the same as in the proof of the generic model theorem, only less lucid since we are dealing not with a transitive class but with a Boolean-valued model. Thus what follows is a sketch of the proof, and the incredulous reader is requested to work out the details.

When calculating the Boolean value of a formula, one may find the following observation helpful (cf. Exercise 18.1):

$$\|(\exists y \in x)\varphi(x)\| = \sum_{y \in \operatorname{dom}(x)} (x(y) \cdot \|\varphi(y)\|) \tag{18.39}$$

$$\|(\forall y \in x)\varphi(y)\| = \prod_{y \in \operatorname{dom}(x)} (x(y) \Rightarrow \|\varphi(y)\|)$$

Now, let us consider the axioms of ZFC one by one:

Extensionality: Proved in Lemma 18.4.

Pairing. We prove that for every $x, y \in V^B$ there is $z \in V^B$ such that

$$\|x \in z \wedge y \in z\| = 1$$

Let $z = \{x, y\}^B$.

Separation. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|Y \subseteq X\| = 1 \qquad \text{and} \qquad \|(\forall z \in X)\,(\varphi(z) \leftrightarrow z \in Y)\| = 1$$

Let $Y \in V^B$ be as follows (see (18.33)):

$$\operatorname{dom}(Y) = \operatorname{dom}(X), \qquad Y(t) = X(t) \cdot \|\varphi(t)\|$$

Union. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|(\forall u \in X)(\forall v \in u)(v \in Y)\| = 1$$

We let

$$\operatorname{dom} Y = \bigcup \{\operatorname{dom}(u) : u \in \operatorname{dom}(X)\}, \qquad Y(t) = 1 \text{ for all } t \in \operatorname{dom}(Y)$$

Power set. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|\forall u(u \subseteq X \to u \in Y)\| = 1 \tag{18.40}$$

Here we let

$$\operatorname{dom}(Y) = \{u \in V^B : \operatorname{dom}(u) = \operatorname{dom}(X) \text{ and } u(t) \leq X(t) \text{ for all } t\}$$
$$Y(u) = 1 \text{ for all } u \in \operatorname{dom}(Y)$$

The argument given in (18.36) shows that (18.40) holds.

Replacement. As in the proof of Theorem 39, we consider the axioms of collection; we prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|(\forall u \in X)(\exists v \varphi(u, v) \to (\exists v \in Y)\varphi(u, v)\| = 1$$

Here we let

$$\operatorname{dom}(Y) = \bigcup \{S_u : u \in \operatorname{dom}(X)\}, \qquad Y(t) = 1 \text{ for all } t \in \operatorname{dom}(Y)$$

where S_u is defined by the formula (18.37), for each $u \in \operatorname{dom}(X)$.

Infinity. Here we recommend the reader to verify that

$$\|\check{\omega} \text{ is an inductive set}\| = 1$$

(See also Lemma 18.11.)

Regularity. We prove that for every $X \in V^B$,

$$\|X \text{ is nonempty} \to (\exists y \in X)(\forall z \in y) z \notin X\| = 1 \tag{18.41}$$

If (18.41) is false, then

$$\|\exists u(u \in X) \wedge (\forall y \in X)(\exists z \in y) z \in X\| = b$$

is nonzero. Let $y \in V^B$ be of least $\rho(y)$ such that $\|y \in X\| \cdot b \neq 0$. Then $\|y \in X\| \cdot b \leq \|(\exists z \in y) z \in X\|$, so there exists $z \in \operatorname{dom}(y)$ such that $\|z \in X\| \cdot \|y \in X\| \cdot b \neq 0$. Since $\rho(z) < \rho(y)$, this is a contradiction.

Choice. As in the proof of Theorem 39. For every S, we have

$$\|\check{S} \text{ can be well ordered}\| = 1$$

(Either verify or wait for Lemma (18.11): "can be well-ordered" is Σ_1.) Now, as in Lemma 18.10, we prove that for every $X \in V^B$ there is S and there is $f \in V^B$ such that

$$\|f \text{ is a function on } \check{S} \text{ and } \operatorname{ran}(f) \supseteq X\| = 1 \tag{18.42}$$

Then it follows that the axiom of choice is valid in V^B.

We let $S = \mathrm{dom}(X)$ and $f \in V^B$ as follows:

$$\mathrm{dom}(f) = \{(\check{x}, x)^B : x \in S\}, \qquad f(t) = 1 \text{ for all } t \in \mathrm{dom}(f)$$

These f and S satisfy (18.42).

(b) Recall that $\|G \subseteq \check{B}\| = 1$ and $\|\check{u} \in G\| = u$ for every $u \in B$. Thus $\|\check{0} \in G\| = 0$ and $\|\check{u} \in G \vee (-u)^\vee \in G\| = u + -u = 1$, and we shall prove

$$(18.43) \qquad \|(\forall X \in \check{\mathscr{D}}) \text{ if } X \subseteq G, \text{ then } \textstyle\prod X \in G\| = 1$$

If $X \in \mathscr{D}$, then

$$\|\check{X} \subseteq G\| = \|(\forall u \in \check{X}) u \in G\| = \prod_{u \in X} \|\check{u} \in G\| = \prod X = \|(\textstyle\prod X)^\vee \in G\|$$

and so

$$\|\check{X} \subseteq G \rightarrow (\textstyle\prod X)^\vee \in G\| = 1$$

Now (18.43) follows. ■

The Boolean-valued version of absoluteness of Σ_0-formulas:

Lemma 18.11. *If $\varphi(x_1, \ldots, x_n)$ is a Σ_0-formula, then*

$$\varphi(x_1, \ldots, x_n) \quad \textit{iff} \quad \|\varphi(\check{x}_1, \ldots, \check{x}_n)\| = 1$$

Proof. By induction on the complexity of φ. ■

Corollary. *If φ is Σ_1, then $\varphi(x, \ldots)$ implies $\|\varphi(\check{x}, \ldots)\| = 1$.*

The next lemma states that V and V^B "have the same ordinals":

Lemma 18.12. *For every $x \in V^B$,*

$$\|x \text{ is an ordinal}\| = \sum_{\alpha \in Ord} \|x = \check{\alpha}\|$$

Proof. Since "x is an ordinal" is Σ_0, we have, by Lemma 18.11,

$$\sum_{\alpha \in Ord} \|x = \check{\alpha}\| \leq \|x \text{ is an ordinal}\|$$

On the other hand, let $\|x \text{ is an ordinal}\| = u$. We first observe that if γ is an ordinal, then

$$\|x \text{ is an ordinal and } x \in \check{\gamma}\| \leq \sum_{\alpha \in \gamma} \|x = \check{\alpha}\|$$

Also, for every α, we have

$$u \leq \|x \in \check{\alpha}\| + \|x = \check{\alpha}\| + \|\check{\alpha} \in x\|$$

However, there is only a set of α's such that $\|\check{\alpha} \in x\| \neq 0$ (because $\|\check{\alpha} \in x\| = \sum_{t \in \mathrm{dom}(x)} (\|\check{\alpha} = t\| \cdot x(t))$). Hence there is γ such that

$$u \leq \|x \subseteq \check{\gamma}\|$$

and we have $u \leq \sum_{\alpha \leq \gamma} \|x = \check{\alpha}\|$. ■

As a consequence, since $\alpha \mapsto L_\alpha$ is a Σ_1-function, we have

Corollary 18.13. *For every $x \in V^B$,*

$$\|x \text{ is constructible}\| = \sum_{a \in L} \|x = \check{a}\|$$

Converting Forcing Arguments into Independence Proofs

I shall use the example from Section 16 to indicate how one uses the method of forcing to prove independence of set-theoretical statements. Let us show that the statement

(18.44) ω_1^L is countable

is consistent relative to ZFC. We can use three different ways to do it.

I. *Consistency of a generic ultrafilter*

We let $\mathfrak{M} = L$ be the ground model, and we consider the notion of forcing $(P, <)$ defined as follows (see Section 16):

(18.45) $p \in P$ iff

p is a finite sequence $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ of ordinals less than ω_1^L

$p < q$ iff $p \supset q$

Now, let us assume that there is $G \subseteq P$ generic over L. Then, as shown in Section 16, if we let $f = \bigcup G$, then f is a function of ω onto ω_1^L and therefore

$$\mathfrak{M}[G] \vDash \omega_1^L \text{ is countable}$$

Thus all we have to do is to show that it is consistent that P has a filter generic over L.

We let $B = (\text{r.o. } P)^L$ and consider the Boolean-valued model L^B. If $\boldsymbol{G}$ is the canonical generic ultrafilter and if $\mathscr{D} = P^L(B)$, then we have, by Theorem 43b,

$$\|\boldsymbol{G} \text{ is a } \check{\mathscr{D}}\text{-generic ultrafilter on } \check{B}\| = 1$$

Now, since constructibility is absolute for Boolean-valued models (cf. Corollary 18.13), and so is the definition of (this particular) P and B, and since also

$$\|\check{\mathscr{D}} \text{ is the set of all constructible subsets of } \check{B}\| = 1$$

we conclude that the following sentence has Boolean value 1 in L^B:

(18.46) There exists a filter on the set P defined in (18.45), generic over the constructible universe.

Therefore, the sentence (18.46) is consistent and hence we may assume that an L-generic $G \subseteq P$ exists, and construct $\mathfrak{M}[G]$.

II. *The method of Boolean-valued models*

We can prove the consistency of (18.44) directly, by showing that

$$\|\text{the constructible } \omega_1 \text{ is countable}\| = 1 \tag{18.47}$$

in L^B, where B is defined as in I. To do that, we have to carry out the arguments not in $L[G]$ but in L^B. By manipulating the Boolean values, we can show that

$$\|\bigcup G \text{ is a function from } \check{\omega} \text{ onto } (\omega_1^L)^\vee\| = 1$$

Now, since constructibility is absolute for Boolean-valued models, we have

$$\|(\omega_1^L)^\vee \text{ is the constructible } \omega_1\| = 1$$

and therefore we conclude that (18.47) holds. Hence the statement (18.44) is consistent because it holds in L^B.

III. *The method of countable transitive models*

We can also use the following argument to show that a statement A is consistent with ZFC (in our Example A is "ω_1^L is countable"). We have defined a notion of forcing P and shown

(18.48)

$$(\forall \text{ model } \mathfrak{M} \text{ of ZFC})(\forall G \in P^{\mathfrak{M}})[\text{if } G \text{ is generic over } \mathfrak{M}, \text{ then } \mathfrak{M}[G] \vDash \text{A}]$$

If we analyze our proof of [if G is generic, then $\mathfrak{M}[G] \vDash$ A], we can see that although we use the fact that $\mathfrak{M}$ and $\mathfrak{M}[G]$ are models of ZFC, we in fact use only finitely many axioms of ZFC, say $\varphi_1, \ldots, \varphi_n$, in either $\mathfrak{M}$ or $\mathfrak{M}[G]$.

Assume that $\neg$A is provable in ZFC. Then $\neg$A holds in every model of ZFC, in particular in every generic model $\mathfrak{M}[G]$. Now, if we look at the proof of the generic model theorem, we see that it takes only finitely many axioms of ZFC in $\mathfrak{M}$, say $\psi_1, \ldots, \psi_k$, to show that $\mathfrak{M}[G]$ is a model of $\neg$A, $\varphi_1, \ldots, \varphi_n$ (and that $P^{\mathfrak{M}}$ exists). By the reflection principle, there is a *countable* transitive model M such that $M \vDash \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \psi_1 \wedge \cdots \wedge \psi_k$, and by Lemma 16.1, there exists $G \subseteq P^M$ generic over M.

Then $M[G]$ satisfies $\neg$A, and also $\varphi_1 \wedge \cdots \wedge \varphi_n$. The latter is then used, as in (18.48), to show that $\mathfrak{M}[G] \vDash$ A, a contradiction. Thus $\neg$A is not provable in ZFC.

19. INDEPENDENCE OF THE CONTINUUM HYPOTHESIS AND THE AXIOM OF CHOICE

In this section we apply the method of forcing to give independence proofs of the continuum hypothesis and of the axiom of choice.

Theorem 44 (Cohen). *There is a model of* ZFC *in which* $2^{\aleph_0} > \aleph_1$. *Thus the continuum hypothesis is independent of the axioms of* ZFC.

We shall prove Theorem 44 by constructing a generic model of ZFC in which the continuum hypothesis is violated. The independence of the continuum hypothesis then follows by one of the arguments at the end of Section 18.

Given a ground model $\mathfrak{M}$, it is possible to extend it to a model $\mathfrak{M}[G]$ in which the continuum is as large as prescribed. The precise statement follows.

Lemma 19.1. *For any cardinal κ in $\mathfrak{M}$, that satisfies (in $\mathfrak{M}$) $\kappa^{\aleph_0} = \kappa$, there exists a notion of forcing $(P, <)$ such that the generic extension $\mathfrak{M}[G]$ has these properties:*

(i) *$\mathfrak{M}$ and $\mathfrak{M}[G]$ have the same cardinals and same cofinalities;*
(ii) $\mathfrak{M}[G] \vDash 2^{\aleph_0} = \kappa$.

["Same cofinalities" means: $\mathrm{cf}^{\mathfrak{M}[G]}(\alpha) = \mathrm{cf}^{\mathfrak{M}}(\alpha)$ for all limit $\alpha \in \mathfrak{M}$.]

I have more or less started the construction in Section 16; cf. (16.4)–(16.6).

Adding κ Generic Reals

Let P be the set of all functions p such that

(19.1) (i) $\mathrm{dom}(p)$ is a finite subset of $\kappa \times \omega$,
(ii) $\mathrm{ran}(p) \subseteq \{0, 1\}$,

and let p be stronger than q iff $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. By a genericity argument, f is a function from $\kappa \times \omega$ into $\{0, 1\}$. For each $\alpha < \kappa$, we let f_α be the function on ω defined by

$$f_\alpha(n) = f(\alpha, n) \tag{19.2}$$

and

$$a_\alpha = \{n \in \omega : f_\alpha(n) = 1\} \tag{19.3}$$

Each a_α is a real (a subset of ω); and as shown in Section 16, $a_\alpha \notin \mathfrak{M}$ and if $\alpha \neq \beta$, then $a_\alpha \neq a_\beta$. For instance, the latter holds because the set $D = \{p \in P : p(\alpha, n) \neq p(\beta, n) \text{ for some } n\}$ is dense in P (and is in $\mathfrak{M}$) and hence $D \cap G \neq \emptyset$.

Thus if we show that $\mathfrak{M}[G]$ has the same cardinals as $\mathfrak{M}$ (that $(P, <)$ *preserves cardinals*), then it follows that $2^{\aleph_0}$ in $\mathfrak{M}[G]$ is at least κ. This brings us to the general problem of preservation of cardinals and the role of the chain condition.

We recall that P is λ-saturated (or satisfies the λ-chain condition) if there is no incompatible $W \subseteq P$ of size λ, and that P is λ-saturated if and only if $B = \mathrm{r.o.}(P)$ is.

Lemma 19.2. *Let λ be a regular uncountable cardinal (in $\mathfrak{M}$) and assume that $(P, <)$ is λ-saturated (in $\mathfrak{M}$). Then λ is a regular cardinal in $\mathfrak{M}[G]$.*

Corollary

1. *Every cardinal in* $\mathfrak{M}$ *greater than or equal to* sat(P) *remains a cardinal in* $\mathfrak{M}[G]$.
2. *If* P *satisfies the countable chain condition, then* $\mathfrak{M}[G]$ *and* $\mathfrak{M}$ *have the same cardinals and same cofinalities.*

Proof. Let $B =$ r.o. P. Assume that $\mathrm{cf}^{\mathfrak{M}[G]}(\lambda) < \lambda$. Then there exist $\nu < \lambda$, a name $\underset{\sim}{f} \in \mathfrak{M}^B$ and a condition $w \in B$ such that w forces:

(19.4) (i) $\underset{\sim}{f}$ is a function;
(ii) $\underset{\sim}{f}\colon \check{\nu} \to \check{\lambda}$ and the range of $\underset{\sim}{f}$ is cofinal in $\check{\lambda}$.

For each $\alpha < \nu$ and each $\beta < \lambda$, let

$$w(\alpha, \beta) = w \cdot \|\underset{\sim}{f}(\check{\alpha}) = \check{\beta}\|$$

It follows from (i) that $w(\alpha, \beta) \cdot w(\alpha, \gamma) = 0$ whenever $\beta \neq \gamma$; from (ii) it follows that for every $\beta < \lambda$ there exist $\alpha < \nu$ and $\gamma > \beta$ such that $w(\alpha, \gamma) \neq 0$. Thus

$$|\{\beta < \lambda : \exists\alpha(w(\alpha, \beta) \neq 0)\}| = \lambda \tag{19.5}$$

and since $\nu < \lambda$, there is $\alpha_0 < \nu$ such that

$$|\{\beta < \lambda : w(\alpha_0, \beta) \neq 0\}| = \lambda \tag{19.6}$$

However, the collection in (19.6) is incompatible, contrary to the assumption that B is λ-saturated.

Now if $\nu \geq$ sat(P), then if ν is regular in $\mathfrak{M}$, then ν is regular in $\mathfrak{M}[G]$; and if ν is a singular cardinal in $\mathfrak{M}$, then $\nu >$ sat(P) and is a limit of regular cardinals, thus a cardinal in $\mathfrak{M}[G]$.

In particular, if sat$(P) = \aleph_1$, then all cardinals and confinalities are preserved. ■

We shall now show that the notion of forcing $(P, <)$ defined in (19.1) satisfies the countable chain condition. The following lemma is a slightly more general statement:

Lemma 19.3. *Let* S *be an arbitrary set and let* C *be at most countable. Let* P *be the set of all functions* p *whose domain is a finite subset of* S, *with values in* C. *Let* P *be partially ordered by inverse inclusion. Then every incompatible subset* W *of* P *is at most countable.*

Proof. Let W be an incompatible subset of P; that is, if p and q are distinct elements of W, then $p(x) \neq q(x)$ for some $x \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. We shall show that there is a countable subset $A \subseteq S$ such that $\mathrm{dom}(p) \subseteq A$ for all $p \in W$. Then it follows that $|W| \leq \aleph_0$.

We construct a sequence $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of subsets of S, and a sequence $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n \subseteq \cdots$ of subsets of W as follows: First we pick

$q \in W$ and let $W_0 = \{q\}$ and $A_0 = \operatorname{dom}(q)$. Given W_n and A_n, we choose for each $p \in P$ with $\operatorname{dom}(p) \subseteq A_n$, some $q \in W$ whose restriction to A_n is p (if such q exists). Then we let W_{n+1} be W_n along with these chosen q's, and $A_{n+1} = \bigcup \{\operatorname{dom}(q) : q \in W_{n+1}\}$. Finally, we let $A = \bigcup_{n=0}^{\infty} A_n$.

It is obvious (by induction) that each W_n and each A_n is at most countable. We complete the proof by showing that $W = \bigcup_{n=0}^{\infty} W_n$. If $q \in W$, then there is n such that $\operatorname{dom}(q) \cap A = \operatorname{dom}(q) \cap A_n$. Thus if $p = q | A_n$, there exists $q' \in W_{n+1}$ such that $q' | A_n = p$. Since $\operatorname{dom}(q') \subseteq A_{n+1} \subseteq A$, it follows that q and q' are compatible; however, both are elements of W and thus $q = q'$. Hence $q \in W_{n+1}$. ■

Now the proof of Lemma 19.1 is almost complete. Since P is $\aleph_1$-saturated, it preserves cardinals, and $\mathfrak{M}[G]$ has at least κ subsets of ω (namely the generic reals a_α, $\alpha < \kappa$). It remains to show that $\mathfrak{M}[G] \vDash 2^{\aleph_0} \leq \kappa$. This is a consequence of the following general estimate of the number of subsets of a cardinal λ in a generic extension.

Lemma 19.4. *Let λ be a cardinal in $\mathfrak{M}$. If G is an $\mathfrak{M}$-generic ultrafilter on B, then*

$$(2^\lambda)^{\mathfrak{M}[G]} \leq (|B|^\lambda)^{\mathfrak{M}}$$

Proof. Every subset $A \subseteq \lambda$ in $\mathfrak{M}[G]$ has a name $A \in \mathfrak{M}^B$; every such A determines a function $\alpha \mapsto \|\check{\alpha} \in A\|$ from λ into B. Different subsets correspond to different functions, and thus the number of all subsets of λ in $\mathfrak{M}[G]$ is not greater than the number of all functions from λ into B in $\mathfrak{M}$. ■

In our case, $|P| = \kappa$; and since P satisfies the countable chain condition, it follows that $|\text{r.o. } P| \leq \kappa^{\aleph_0}$ (see also Exercise 17.22); this is because every $u \in \text{r.o. } P$ is the sum of countably many elements of P. Considering that $\kappa^{\aleph_0} = \kappa$, we have $|B|^{\aleph_0} = \kappa$, and so $(2^{\aleph_0})^{\mathfrak{M}[G]} = \kappa$.

Exercise 19.1. If P is defined as in (19.1) and if κ is arbitrary, then $(2^{\aleph_0})^{\mathfrak{M}[G]} = (\kappa^{\aleph_0})^{\mathfrak{M}}$.
[$(2^{\aleph_0})^{\mathfrak{M}[G]} = ((2^{\aleph_0})^{\aleph_0})^{\mathfrak{M}[G]} \geq (\kappa^{\aleph_0})^{\mathfrak{M}[G]} \geq (\kappa^{\aleph_0})^{\mathfrak{M}}$.]

Adding Subsets of Regular Cardinals

The proof of independence of the continuum hypothesis (Lemma 19.1) consisted of adding a large number of generic subsets of ω. This construction is easily generalized to arbitrary regular cardinals. (For what happens at singular cardinals, see Exercises 19.3 and 19.4.)

Let κ be a regular cardinal (in the ground model $\mathfrak{M}$) and assume that $2^{<\kappa} = \kappa$. (We shall discuss the case $2^{<\kappa} > \kappa$ in the exercises.) Let $(P, <)$ be defined as follows: P is the set of all functions p such that:

(19.7) (i) $\operatorname{dom}(p) \subseteq \kappa$ and $|\operatorname{dom}(p)| < \kappa$;
(ii) $\operatorname{ran}(p) \subseteq \{0, 1\}$.

A condition p is stronger than q iff $p \supset q$.

Let G be a set of conditions generic over $\mathfrak{M}$ and let $f = \bigcup G$. As before, f is a function from κ into $\{0, 1\}$, and

$$X = \{\alpha < \kappa : f(\alpha) = 1\} \tag{19.8}$$

is a subset of κ and $X \notin \mathfrak{M}$.

We shall show that $\mathfrak{M}$ and $\mathfrak{M}[G]$ have the same cardinals (and cofinalities). To start, since $|P| = \kappa^{<\kappa} = \kappa$, we have $\operatorname{sat}(P) \leq |P|^+ = \kappa^+$, and by Lemma 19.2 all regular cardinals $\lambda > \kappa$ remain regular cardinals in the extension. We shall now show that all the regular cardinals $\lambda \leq \kappa$ are preserved. In fact, we shall show that for every $\alpha < \kappa$, every α-sequence of ordinal numbers in $\mathfrak{M}[G]$ is a set in $\mathfrak{M}$. (In particular, $P^{\mathfrak{M}[G]}(\alpha) = P^{\mathfrak{M}}(\alpha)$ for all $\alpha < \kappa$.) Then it clearly follows that if $\lambda \leq \kappa$ is a regular cardinal in $\mathfrak{M}$, then λ is a regular cardinal in $\mathfrak{M}[G]$.

Let λ be an infinite cardinal. We say that a notion of forcing P is *λ-closed* if for every $\alpha \leq \lambda$, if

$$p_0 \geq p_1 \geq \cdots \geq p_\xi \geq \cdots \qquad (\xi < \alpha)$$

is a descending sequence of conditions, then there is $p \in P$ stronger than all p_ξ, $\xi < \alpha$. Let us call P *λ-distributive* if the intersection of λ open dense subsets of P is open dense. (If $P = B - \{0\}$, B a complete Boolean algebra, then by Lemma 17.7 this definition is equivalent to λ-distributivity of B.) Note that if $B = \text{r.o.}(P)$, then B is λ-distributive if and only if P is λ-distributive.

Lemma 19.5. *If P is λ-closed, then it is λ-distributive.*

Proof. Let $\{D_\alpha : \alpha < \lambda\}$ be a collection of open dense sets. The intersection $D = \bigcap_{\alpha<\lambda} D_\alpha$ is clearly open; to show that D is dense, let $p \in P$ be arbitrary. By induction on $\alpha < \lambda$, we construct a descending λ-sequence of conditions $p \geq p_0 \geq p_1 \geq \cdots$. We let p_α be a condition stronger than all p_ξ, $\xi < \alpha$, and such that $p_\alpha \in D_\alpha$. Finally, we let q be a condition stronger than all p_α, $\alpha < \lambda$. Clearly, $q \in D$. ∎

Lemma 19.6. *Let λ be an infinite cardinal and assume that $(P, <)$ is λ-distributive (in $\mathfrak{M}$). Then if $f \in \mathfrak{M}[G]$ is a function from λ into $\mathfrak{M}$, then $f \in \mathfrak{M}$. In particular, $P^{\mathfrak{M}[G]}(\lambda) = P^{\mathfrak{M}}(\lambda)$.*

Proof. Let $f\colon \lambda \to \mathfrak{M}$ and $f \in \mathfrak{M}[G]$, let $\underset{\sim}{f}$ be a name for f. There is $A \in \mathfrak{M}$ and a condition $p_0 \in G$ such that p_0 forces

$$\underset{\sim}{f} \text{ is a function from } \check{\lambda} \text{ into } \check{A} \tag{19.9}$$

For each $\alpha < \lambda$, the set

$$D_\alpha = \{p \leq p_0 : (\exists x \in A)\, p \Vdash \underset{\sim}{f}(\check{\alpha}) = \check{x}\}$$

is dense below p_0. (It goes without saying that $\{D_\alpha : \alpha < \lambda\} \in \mathfrak{M}$.) Thus $D = \bigcap_{\alpha<\lambda} D_\alpha$ is dense below p_0 and therefore there is some $p \in D \cap G$. Now we argue in $\mathfrak{M}$: for each $\alpha < \lambda$ there is x_α such that $p \Vdash \underset{\sim}{f}(\check{\alpha}) = \check{x}_\alpha$; let $g\colon \lambda \to A$

be the function defined by $g(\alpha) = x_\alpha$. However, it is easy to see that $f(\alpha) = x_\alpha = g(\alpha)$, for every $\alpha < \lambda$, and thus $f \in \mathfrak{M}$. ■

The notion of forcing defined in (19.7) is clearly λ-closed for all $\lambda < \kappa$ and therefore all regular cardinals $\leq \kappa$ remain regular cardinals. $\mathfrak{M}[G]$ has no new subsets of any $\lambda < \kappa$, while X is a new subset of κ.

Exercise 19.2. Let P be as in (19.7) and let $Q = \{p \in P : \text{dom}(p) \text{ is an initial segment of } \kappa\}$. Then Q is dense in P and hence r.o.(Q) = r.o.(P).

Exercise 19.3. Let κ be a singular cardinal and let $(P, <)$ be defined as in (19.7). Then P collapses κ to cf(κ): in the generic extension, there is a 1–1 function g from κ into cf κ.

[Let $\kappa = \aleph_\omega$, and let X be the added subset of $\aleph_\omega$; cf. (19.8). For each $\alpha < \aleph_\omega$, let $g(\alpha)$ = the least n such that the order type of $X \cap (\omega_{n+1} - \omega_n)$ is $\omega_n + \alpha$. Show that for every α and every $p \in P$ there is $q \supseteq p$ and some n such that $\text{dom}(q) \supseteq \omega_{n+1} - \omega_n$ and that the set $\{\xi \in \omega_{n+1} - \omega_n : q(\xi) = 1\}$ has order-type $\omega_n + \alpha$. By genericity of G, the function g is defined for every $\alpha < \aleph_\omega$; it is clearly one-to-one.]

Exercise 19.4. Again let κ be singular, and let P be the set of all 0, 1-functions whose domains are a bounded subset of κ; as usual, P is ordered by inverse inclusion. Show that P collapses κ to cf κ.

If we want to add a large number of subsets of a regular cardinal κ, we employ a technique similar to that in Lemma 19.1:

Lemma 19.7. *Let κ, λ be cardinals (in $\mathfrak{M}$) such that κ is regular and $2^{<\kappa} = \kappa$, and such that $\lambda^\kappa = \lambda$. There exists a notion of forcing $(P, <)$ such that the generic extension $\mathfrak{M}[G]$ has these properties:*

(i) *$\mathfrak{M}$ and $\mathfrak{M}[G]$ have the same cardinals and same cofinalities;*
(ii) $\mathfrak{M}[G] \vDash 2^\kappa = \lambda$.

Proof. Let P be the set of all functions p such that:

(19.10) (i) $\text{dom}(p) \subseteq \lambda \times \kappa$ and $|\text{dom}(p)| < \kappa$,
(ii) $\text{ran}(p) \subseteq \{0, 1\}$,

and let p be stronger than q iff $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. For each $\alpha < \lambda$, we let

$$a_\alpha = \{\xi < \kappa : f(\alpha, \xi) = 1\} \tag{19.11}$$

Each a_α is a subset of κ and by what is by now already a standard argument, each $a_\alpha \notin \mathfrak{M}$ and $a_\alpha \neq a_\beta$ whenever $\alpha \neq \beta$.

Since P is α-closed for each $\alpha < \kappa$, it follows that P preserves all cardinals and cofinalities below κ (including κ). We shall show in the lemma below that P is κ^+-saturated, and thus P also preserves cardinals and cofinalities above κ.

Clearly, κ has at least λ subsets in $\mathfrak{M}[G]$. On the other hand, we have (in $\mathfrak{M}$) $|P| = \lambda^{<\kappa} = \lambda$, and since P is κ^+-saturated, we have $|B| \leq |P|^\kappa = \lambda$ (where B = r.o. P), and by Lemma 19.4, $(2^\kappa)^{\mathfrak{M}[G]} \leq (|B|^\kappa)^{\mathfrak{M}} = \lambda$.

It remains to show that P is κ^+-saturated. The following lemma is a straightforward generalization of Lemma 19.3.

Lemma 19.8. *Let κ be a regular cardinal such that $2^{<\kappa} = \kappa$. Let S be an arbitrary set and let $|C| \le \kappa$. Let P be the set of all functions p whose domains are subsets of S of size $< \kappa$, with values in C. Then* $\mathrm{sat}(P, \supset) \le \kappa^+$.

Proof. The proof is as in Lemma 19.3. Let $W \subseteq P$ be incompatible. We construct a sequence $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots$ $(\alpha < \kappa)$ of subsets of S, and $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_\alpha \subseteq \cdots$ of subsets of W. If α is a limit ordinal, we let $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$ and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Given A_α and W_α, we choose for each $p \in P$ with $\mathrm{dom}(p) \subseteq A_\alpha$ some $q \in W$ (if there is one) such that $p = q \restriction A_\alpha$. Then we let $W_{\alpha+1} = W_\alpha \cup \{$the chosen q's$\}$, and $A_{\alpha+1} = \bigcup \{\mathrm{dom}(q) : q \in W_{\alpha+1}\}$; finally, $A = \bigcup_{\alpha<\kappa} A_\alpha$.

As in Lemma 19.3 one shows that $W = \bigcup_{\alpha<\kappa} W_\alpha$: If $q \in W$, then there is $\alpha < \kappa$ such that $\mathrm{dom}(q) \cap A = \mathrm{dom}(q) \cap A_\alpha$, etc. The proof is completed by showing that $|A_\alpha| \le \kappa$ and $|W_\alpha| \le \kappa$ for each $\alpha < \kappa$. This is proved by induction on α. If $|W_\alpha| \le \kappa$, then $|A_\alpha| \le \kappa$ because $A_\alpha = \bigcup \{\mathrm{dom}(q) : q \in W_\alpha\}$. If α is a limit ordinal and $|W_\beta| \le \kappa$ for all $\beta < \alpha$, then $|W_\alpha| = |\bigcup_{\beta<\alpha} W_\beta| \le \kappa$. Thus let us assume that $|W_\alpha| \le \kappa$ and let us show that $|W_{\alpha+1}| \le \kappa$. The set $W_{\alpha+1}$ is obtained by adding to W_α at most one $q \in W$ for each $p \in P$ with $\mathrm{dom}(p) \subseteq A_\alpha$. There are at most $\kappa^{<\kappa}$ subsets X of A_α of size $< \kappa$, and since κ is regular and $2^{<\kappa} = \kappa$, we have $\kappa^{<\kappa} = \kappa$. On each X there are $|C|^{|X|}$ functions with values in C, and therefore there are at most κ elements of p with $\mathrm{dom}(p) \subseteq A_\alpha$. Hence $|W_{\alpha+1}| \le \kappa$. Then it follows that $|W| \le \kappa$. ■

Collapsing Cardinals

In Section 16 we introduced a notion of forcing which collapses ω_1: The cardinal $\omega_1^{\mathfrak{M}}$ becomes a countable ordinal in $\mathfrak{M}[G]$. This simple method of collapsing a cardinal is described in the following lemma. We shall present a slightly more subtle way of collapsing in the next section (namely, how to make an inaccessible cardinal accessible).

Lemma 19.9. *Let κ be a regular cardinal and let $\lambda > \kappa$ be a cardinal. There is a notion of forcing $(P, <)$ that collapses λ onto κ, i.e., λ has cardinality κ in the generic extension. Moreover,*

(a) *every cardinal $\alpha \le \kappa$ in $\mathfrak{M}$ remains a cardinal in $\mathfrak{M}[G]$; and*
(b) *if $\lambda^{<\kappa} = \lambda$, then every cardinal $\alpha > \lambda$ remains a cardinal in the extension.*

[The condition in (b) is satisfied, e.g., if GCH holds and $\mathrm{cf}\,\lambda \ge \kappa$.]

Proof. Let P be the set of all functions p such that:

(19.12) (i) $\mathrm{dom}(p) \subseteq \kappa$ and $|\mathrm{dom}(p)| < \kappa$,
(ii) $\mathrm{ran}(p) \subseteq \lambda$,

and let $p < q$ iff $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. Clearly, f is a function, and it maps κ onto λ; this is because for all $\alpha < \kappa$ and all $\beta < \lambda$, the sets

$$D_\alpha = \{p : \alpha \in \text{dom}(p)\}, \qquad E_\beta = \{p : \beta \in \text{ran}(p)\}$$

are dense.

$(P, <)$ is α-closed for every $\alpha < \kappa$ and therefore all cardinals $\leq \kappa$ are preserved. If $\lambda^{<\kappa} = \lambda$, then $|P| = \lambda$ and hence $\text{sat}(P) \leq \lambda^+$; it follows that all cardinals $\geq \lambda^+$ are preserved. ■

If $\lambda^{<\kappa} > \lambda$ in Lemma 19.9, then not only λ is collapsed but $\lambda^{<\kappa}$ as well. For instance:

Exercise 19.5. In (19.12), let $\kappa = \aleph_1$ and $\lambda = \aleph_\omega$. Then in $\mathfrak{M}[G]$ there is a one-to-one function $g: \aleph_\omega^{\aleph_0} \to \aleph_1$.

[If X is a countable subset of $\aleph_\omega$, let $g(X) =$ the least α such that $f[(\alpha + \omega) - \alpha] = X$ (where $f = \bigcup G$ is the collapsing function). Use the fact that $X \in \mathfrak{M}$.]

If κ is singular in (19.12), then (as was the case in adding a subset of κ) κ is collapsed to cf κ. One of many ways to verify it:

Exercise 19.6. In (19.12), let $\kappa = \aleph_\omega$. Then in $\mathfrak{M}[G]$, there is a one-to-one function g from λ into ω.

[Let $f = \bigcup G$, and let $g(\alpha) =$ the least n such that the function $f \restriction (\omega_{n+1} - \omega_n)$ is eventually constantly equal to α.]

In the above examples of forcing, either when adding a subset of κ or collapsing a cardinal to κ, we assumed that $2^{<\kappa} = \kappa$. This is because if $2^{<\kappa} > \kappa$, then either notion of forcing collapses $2^{<\kappa}$ to κ. For instance:

Exercise 19.7. Let $(P, <)$ be the notion of forcing adding a subset of ω_1 via (19.7), and let $(Q, <)$ be the notion of forcing that adds a mapping of $\aleph_1$ onto $2^{\aleph_0}$ via (19.12). Then r.o.$(P) =$ r.o.(Q).

[Let $Q' = \{q \in Q : \text{dom}(q)$ is an initial segment of $\omega_1\}$; Q' is dense in Q. Show that P has a dense set P' isomorphic to Q': use the fact that every $p \in P$ has $2^{\aleph_0}$ mutually incompatible extensions.] [Another way to show that $(P, <)$ from (19.7) adjoins a one-to-one mapping of $2^{\aleph_0}$ into $\aleph_1$: Let $f = \bigcup G$, and for every $g \in {}^\omega\{0, 1\}$, let $F(g) =$ least α such that $f(\alpha + n) = g(n)$ for all n.]

Thus adding a subset of ω_1 via (19.7) makes the continuum hypothesis true, and we have a forcing proof of the consistency of $2^{\aleph_0} = \aleph_1$.

Independence of the Axiom of Choice

If the ground model satisfies the axiom of choice, then so does the generic extension. However, we can still use the method of forcing to construct a model in which AC fails; namely, we find a suitable submodel of the generic model, a model $\mathfrak{N}$ such that $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{M}[G]$.

I shall describe a general method of construction of models without choice in Section 21. In the present section I shall only prove the basic result.

Theorem 45 (Cohen). *There is a model of* ZF *in which the real numbers cannot be well-ordered. Thus the axiom of choice is independent of the axioms of* ZF.

Before we construct a model without choice, we shall prove an easy but useful lemma on automorphisms of Boolean-valued models. Let B be a complete Boolean algebra and let π be an automorphism of B. We define, by induction on $\rho(x)$, an automorphism of the Boolean-valued universe V^B, and denote it also by π:

(19.13) (i) $\pi(\emptyset) = \emptyset$;
(ii) if $\pi(y)$ has been defined for all $y \in \operatorname{dom}(x)$, let
$\operatorname{dom}(\pi x) = \pi[\operatorname{dom}(x)]$, and
$(\pi x)(\pi y) = \pi(x(y))$ for all $\pi(y) \in \operatorname{dom}(\pi x)$.

Clearly, π is a one-to-one function of V^B onto itself, and $\pi(\check{x}) = \check{x}$ for every x.

Lemma 19.10. *Let $\varphi(x_1, \ldots, x_n)$ be a formula. If π is an automorphism of B, then for all $x_1, \ldots, x_n \in V^B$,*

$$\|\varphi(\pi x_1, \ldots, \pi x_n)\| = \pi(\|\varphi(x_1, \ldots, x_n)\|) \tag{19.14}$$

Proof. (a) If φ is an atomic formula, (19.14) is proved by induction (as in the definition of $\|x \in y\|$, $\|x = y\|$). For instance,

$$\begin{aligned}\|\pi x \in \pi y\| &= \sum_{t \in \operatorname{dom}(\pi y)} (\|\pi x = t\| \cdot (\pi y)(t))\\ &= \sum_{z \in \operatorname{dom}(y)} (\|\pi x = \pi z\| \cdot (\pi y)(\pi z))\\ &= \pi\left(\sum_{z \in \operatorname{dom}(y)} (\|x = z\| \cdot y(z))\right)\\ &= \pi(\|x \in y\|)\end{aligned}$$

(b) For other formulas, the proof is by induction on the complexity of φ. ■

In practice, (19.14) is used as follows: Let $(P, <)$ be a separative partially ordered set. If π is an automorphism of P, then π extends to a unique automorphism of the complete Boolean algebra $B = \text{r.o. } P$:

$$\pi(u) = \sum \{\pi(p) : p \le u\} \tag{19.15}$$

Then (19.14) takes this form:

$$p \Vdash \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad \pi p \Vdash \varphi(\pi x_1, \ldots, \pi x_n) \tag{19.16}$$

where $x_1, \ldots, x_n \in V^B$.

Exercise 19.8. **If G is an $\mathfrak{M}$-generic ultrafilter on B and π an automorphism of B (in $\mathfrak{M}$), then $H = \pi[G]$ is $\mathfrak{M}$-generic and $\mathfrak{M}[H] = \mathfrak{M}[G]$.**

Now, let the ground model $\mathfrak{M}$ be the constructible universe: $\mathfrak{M} = L$. We first extend $\mathfrak{M}$ by adding countably many Cohen generic reals: Let P be the set of all functions p such that:

(19.17) (i) $\operatorname{dom}(p)$ is a finite subset of $\omega \times \omega$,
(ii) $\operatorname{ran}(p) \subseteq \{0, 1\}$,

and let $p < q$ iff $p \supset q$.

[Note that any one-to-one mapping between $\omega \times \omega$ and ω induces an isomorphism between this notion of forcing and the one that adds one Cohen generic real (when the conditions are functions with $\operatorname{dom}(p) \subseteq \omega$).]

Let G be a generic set of conditions. For each $i \in \omega$, let

$$a_i = \{n \in \omega : (\exists p \in G)\ p(i, n) = 1\} \tag{19.18}$$

and let

$$A = \{a_i : i \in \omega\} \tag{19.19}$$

Let $\boldsymbol{a}_i$, $i \in \omega$, and $\boldsymbol{A}$ be the canonical names for a_i and A:

$$\begin{aligned} &\operatorname{dom}(\boldsymbol{a}_i) = \{\check{n} : n \in \omega\} \\ &\boldsymbol{a}_i(\check{n}) = \textstyle\sum \{p \in P : p(i, n) = 1\} \qquad (n \in \omega) \end{aligned} \tag{19.20}$$

$$\begin{aligned} &\operatorname{dom}(\boldsymbol{A}) = \{\boldsymbol{a}_i : i \in \omega\} \\ &\boldsymbol{A}(\boldsymbol{a}_i) = 1 \qquad (i \in \omega) \end{aligned} \tag{19.21}$$

Lemma 19.11. *If $i \neq j$, then every p forces $\boldsymbol{a}_i \neq \boldsymbol{a}_j$:*

$$\|\boldsymbol{a}_i = \boldsymbol{a}_j\| = 0$$

Proof. For every p there exists $q \supset p$ and $n \in \omega$ such that $q(i, n) = 1$ and $q(j, n) = 0$. ■

From now on, let us work in $\mathfrak{M}[G]$. Let $\mathfrak{N}$ be the class of all sets hereditarily ordinal-definable over A:

$$\mathfrak{N} = \mathrm{HOD}(A)$$

As we have shown in Section 15, $\mathfrak{N}$ is a transitive model of ZF. Since the elements of A are sets of integers, it is clear that $A \in \mathfrak{N}$. We shall show that the set of all real numbers cannot be well-ordered in the model $\mathfrak{N}$ (in fact that A cannot be well-ordered).

If $\boldsymbol{R}^{\mathfrak{N}}$ is well-orderable in $\mathfrak{N}$, then there is a one-to-one mapping $f \in \mathfrak{N}$ of $P^{\mathfrak{N}}(\omega)$ into the ordinals. The following lemma shows that this is not true:

Lemma 19.12. (In $\mathfrak{M}[G]$) *There is no one-to-one function f: $A \to Ord$, ordinal-definable over A.*

Proof. Assume that f: $A \to Ord$ is one-to-one and ordinal-definable over A. Then there is a finite sequence $s = \langle x_0, \ldots, x_k\rangle$ in A such that f is ordinal-definable from s and A. Since f is one-to-one, it is easy to see that every $a \in A$ is ordinal-definable from s and A. In particular, pick some $a \in A$ that is not among the x_i, $i \leq k$.

Since $a \in \mathrm{OD}[s, A]$ (in $\mathfrak{M}[G]$), there is a formula φ such that

$$\text{(19.22)} \qquad \mathfrak{M}[G] \vDash a \text{ is the unique set such that } \varphi(a, \alpha_1, \ldots, \alpha_n, s, A)$$

for some ordinals $\alpha_1, \ldots, \alpha_n$. We shall show that (19.22) is impossible.

Let $\boldsymbol{a}$ be the canonical name for a, and let $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k$ be the canonical names for $x_0, \ldots, x_k$ (see 19.20); let $\boldsymbol{s}$ be the obvious canonical name for the sequence $\langle x_0, \ldots, x_k \rangle$. We shall show the following:

(19.23) For every p_0 that forces $\varphi(\boldsymbol{a}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \boldsymbol{s}, \boldsymbol{A})$ there exists $\boldsymbol{b}$ and $q \leq p_0$ that forces $\boldsymbol{a} \neq \boldsymbol{b}$ and $\varphi(\boldsymbol{b}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \boldsymbol{s}, \boldsymbol{A})$.

Let $p_0 \Vdash \varphi(\boldsymbol{a}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \boldsymbol{s}, \boldsymbol{A})$. Let $i, i_0, \ldots, i_k$ be such that $\boldsymbol{a} = \boldsymbol{a}_i$, $\boldsymbol{x}_0 = \boldsymbol{a}_{i_0}, \ldots, \boldsymbol{x}_k = \boldsymbol{a}_{i_k}$. Let us pick $j \in \omega$ such that $j \neq i$, and that for all $m \in \omega$, $(j, m) \notin \mathrm{dom}(p_0)$. See Fig. 19.1.

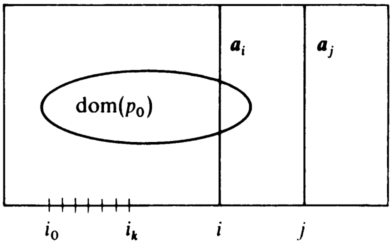

FIGURE 19.1

Now let π be the permutation of ω that interchanges i and j, and $\pi x = x$ otherwise. This permutation induces an automorphism of P: for every $p \in P$,

$$\text{(19.24)} \qquad \mathrm{dom}(\pi p) = \{(\pi x, m) : (x, m) \in \mathrm{dom}(p)\}$$
$$(\pi p)(\pi x, m) = p(x, m) \qquad (x \in \mathrm{dom}(p))$$

In turn, π induces an automorphism of B, and of $\mathfrak{M}^B$. It is easy to see (cf. 19.20, 19.21) that

$$\text{(19.25)} \qquad \pi(\boldsymbol{a}_i) = \boldsymbol{a}_j, \qquad \pi(\boldsymbol{a}_j) = \boldsymbol{a}_i \qquad \pi(\boldsymbol{a}_x) = \boldsymbol{a}_x \qquad (\text{all } x \neq i, j)$$
$$\pi(\boldsymbol{A}) = \boldsymbol{A} \qquad \text{and} \qquad \pi(\boldsymbol{s}) = \boldsymbol{s}$$

Since $(j, m) \notin \mathrm{dom}(p_0)$, for all m, it follows that $(i, m) \notin \mathrm{dom}(\pi p_0)$, for all m, and thus p_0 and πp_0 are compatible. Let $q = p_0 \cup \pi p_0$.

Now, on the one hand we have

$$p_0 \Vdash \varphi(\boldsymbol{a}_i, \check{\alpha}_1, \ldots, \check{\alpha}_n, \boldsymbol{s}, \boldsymbol{A})$$

and on the other hand, since $\pi\check{\alpha} = \check{\alpha}$, $\pi s = s$ and $\pi A = A$, we have

$$\pi p_0 \Vdash \varphi(\boldsymbol{a}_j, \check{\alpha}_1, \ldots, \check{\alpha}_n, s, A)$$

Hence

$$q \Vdash \varphi(\boldsymbol{a}_i, \ldots) \text{ and } \varphi(\boldsymbol{a}_j, \ldots)$$

and by Lemma 19.11, $q \Vdash \boldsymbol{a}_i \neq \boldsymbol{a}_j$. Thus we have proved (19.23), which contradicts (19.22). ■

20. MORE GENERIC MODELS

Products of Partially Ordered Sets

Let $(P_1, <)$ and $(P_2, <)$ be two partially ordered sets. The *product* $(P, <)$ of $(P_1, <)$ and $(P_2, <)$ is defined as follows:

(20.1)
$$P = P_1 \times P_2$$
$$(p_1, p_2) \leq (q_1, q_2) \quad \text{iff} \quad p_1 \leq q_1 \quad \text{and} \quad p_2 \leq q_2$$

Note that (p_1, p_2) and (q_1, q_2) are compatible iff p_1 is compatible with q_1, and p_2 is compatible with q_2; they are incompatible if either p_1 is incompatible with q_1, or p_2 is incompatible with q_2.

We shall investigate generic models obtained via products of notions of forcing. The first thing worth noticing is the following:

Exercise 20.1. If r.o. $P_1 =$ r.o. Q_1 and r.o. $P_2 =$ r.o. Q_2, then r.o.$(P_1 \times P_2) =$ r.o.$(Q_1 \times Q_2)$ (up to isomorphism).

Thus one can define a corresponding operation $\oplus$ on complete Boolean algebras: If B_1 and B_2 are complete Boolean algebras, let

$$B_1 \oplus B_2 = \text{r.o.}(P_1 \times P_2)$$

where P_1, P_2 are partially ordered sets, dense in B_1 and B_2. We call $B_1 \oplus B_2$ the *direct sum* of B_1 and B_2. (In fact $B_1 \oplus B_2$ *is* the direct sum of Boolean algebras B_1, B_2, defined algebraically.)

Exercise 20.2. The Stone space of $B_1 \oplus B_2$ is the topological product of the Stone spaces of B_1 and B_2.

While direct sum of complete Boolean algebras is important in the theory of forcing, its dual notion, the *direct product* of algebras is uninteresting. The direct product of B_1 and B_2 corresponds to the disjoint union of their Stone spaces (see also Figure 17.1). If B is the direct product of B_1, B_2, then there is $a \in B$ such that B_1 is isomorphic to B_a and B_2 is isomorphic to B_{-a}. Clearly, if G is a generic ultrafilter on B, then either $a \in G$ and $\mathfrak{M}[G] = \mathfrak{M}[G \cap B_a]$ or the same thing for $-a$.

If G is a generic filter on $P_1 \times P_2$, let G_1 and G_2 be the projections of G:

$$(20.2) \qquad G_1 = \{p_1 \in P_1 : (p_1, p_2) \in G \text{ for some } p_2 \in P_2\}$$
$$G_2 = \{p_2 \in P_2 : (p_1, p_2) \in G \text{ for some } p_1 \in P_1\}$$

It is fairly easy to show that G_1 is generic on P_1 and G_2 is generic on P_2, and $G = G_1 \times G_2$ (see the forthcoming lemma). On the other hand, to obtain a generic filter on the product, we cannot just take any two generic filters G_1 and G_2 on P_1 and P_2:

Exercise 20.3. Show that except in trivial cases, $G \times G$ is never generic on $P \times P$.

The characterization of generic filters on products is given in the following lemma.

Lemma 20.1. (Product Lemma). *Let P_1 and P_2 be two notions of forcing in $\mathfrak{M}$. In order that $G \subseteq P_1 \times P_2$ be generic over $\mathfrak{M}$, it is necessary and sufficient that $G = G_1 \times G_2$ where $G_1 \subseteq P_1$ is generic over $\mathfrak{M}$ and $G_2 \subseteq P_2$ is generic over $\mathfrak{M}[G_1]$. Moreover, $\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2]$.*

As a corollary, if G_1 is generic over $\mathfrak{M}$ and G_2 is generic over $\mathfrak{M}[G_1]$, then due to symmetry G_1 is generic over $\mathfrak{M}[G_2]$, and we have $\mathfrak{M}[G_1][G_2] = \mathfrak{M}[G_2][G_1]$.

Proof. First let G be an $\mathfrak{M}$-generic filter on $P_1 \times P_2$. We let G_1 and G_2 be the projections of G (20.2). Clearly G_1 and G_2 are filters, and $G \subseteq G_1 \times G_2$. If $(p_1, p_2) \in G_1 \times G_2$, then there are $p'_1 \in G_1$ and $p'_2 \in G_2$ such that $(p'_1, p_2) \in G$ and $(p_1, p'_2) \in G$. Since G is a filter, there exist $q_1 \leq p_1, p'_1$ and $q_2 \leq p_2, p'_2$ such that $(q_1, q_2) \in G$. Hence $(p_1, p_2) \in G$ and we have $G = G_1 \times G_2$.

It is easy to see that G_1 is generic over $\mathfrak{M}$: If $D_1 \in \mathfrak{M}$ is dense in P_1, then $D_1 \times P_2$ is dense in $P_1 \times P_2$; and since $(D_1 \times P_2) \cap G \neq \emptyset$, we have $D_1 \cap G_1 \neq \emptyset$. To show that G_2 is generic over $\mathfrak{M}[G_1]$, let $D_2 \in \mathfrak{M}[G_1]$ be dense in P_2. Let $\Vdash$ be the forcing relation corresponding to P_1. Let $\boldsymbol{D}_2$ be a name for D_2 and let $p_1 \in G_1$ be such that p_1 forces "$\boldsymbol{D}_2$ is dense in $\check{P}_2$." Let $p_2 \in G_2$ be arbitrary. For every $q_1 \leq p_1$ and every $q_2 \leq p_2$ there exist $r_1 \leq q_1$ and $r_2 \leq q_2$ such that $r_1 \Vdash \check{r}_2 \in \boldsymbol{D}_2$; thus

$$D = \{(r_1, r_2) : r_1 \leq p_1 \text{ and } r_1 \Vdash \check{r}_2 \in \boldsymbol{D}_2\}$$

is dense in $P_1 \times P_2$ below (p_1, p_2) and so there are r_1, r_2 such that $r_1 \in G_1$ and $r_1 \Vdash \check{r}_2 \in \boldsymbol{D}_2$. Hence $r_2 \in D_2 \cap G_2$.

On the other hand, let $G_1 \subseteq P_1$ be $\mathfrak{M}$-generic and let $G_2 \subseteq P_2$ be $\mathfrak{M}[G_1]$-generic. We let $G = G_1 \times G_2$. Clearly G is a filter on $P_1 \times P_2$. To show that G is $\mathfrak{M}$-generic, let $D \in \mathfrak{M}$ be dense in $P_1 \times P_2$. We let

$$D_2 = \{p_2 : (p_1, p_2) \in D \text{ for some } p_1 \in G_1\}$$

The set D_2 is in $\mathfrak{M}[G_1]$; we shall show that D_2 is dense in P_2 and thus $D \cap (G_1 \times G_2) \neq \emptyset$.

Let $q_2 \in P_2$ be arbitrary. Since D is dense in $P_1 \times P_2$, it follows that the set

$$D_1 = \{p_1 : (\exists p_2 \le q_2)(p_1, p_2) \in D\}$$

is dense in P_1. Hence there is $p_1 \in G_1 \cap D_1$ and so D_2 is dense in P_2. Since $G_1 \times G_2 \in \mathfrak{M}[G_1][G_2]$, it is obvious that $\mathfrak{M}[G_1 \times G_2] = \mathfrak{M}[G_1][G_2]$. ■

Products and κ-Products of a Collection of Partial Orders

Let P_i, $i \in I$, be a collection of partially ordered sets. We define the *product* of P_i, $i \in I$, as a generalization of the product of two partially ordered sets. The product of partially ordered sets is closely related to the product of topological spaces.

To simplify the definition, we shall assume that all the P_i have a greatest element 1. For every $p = \langle p_i : i \in I\rangle \in \prod_{i \in I} P_i$, we let

$$\text{(20.3)} \qquad s(p) = \{i \in I : p_i \neq 1\}$$

(the *support* of p) and we let

$$\text{(20.4)} \qquad P = \{p \in \prod_{i \in I} P_i : s(p) \text{ is finite}\}$$

$$p \le q \quad \text{iff} \quad p_i \le q_i \text{ for all } i$$

In practice, we consider finite functions p with $\text{dom}(p) \subseteq I$, and assume that $p(i) = 1$ whenever $i \notin \text{dom}(p)$.

Exercise 20.4. Compare the product of P_i, $i \in I$, with the topological product of the spaces P_i whose topology is given by the basic open sets $U_{p_i} = \{q_i \in P_i : q_i \le p_i\}$.

Exercise 20.5. If $\text{r.o.}(P_i) = \text{r.o.}(Q_i)$ for each $i \in I$, then $\text{r.o.}(P) = \text{r.o.}(Q)$, where P and Q are the respective products.

Exercise 20.6. If P is the product of P_i, $i \in I$, and if S is the topological product of the Stone spaces of $\text{r.o.}(P_i)$, $i \in I$, then $\text{r.o.}(P)$ is (isomorphic to) the algebra of regular open sets of S.

If G is a generic filter on the product of P_i, $i \in I$, then the projections G_i, $i \in I$, are generic filters on P_i.

An example of a product of notions of forcing is the notion of forcing (19.1), adding κ subsets of ω. There is an obvious isomorphism between this and the product of κ copies of the notion of forcing that produces a single Cohen-generic real.

The notion of forcing adding subsets of a regular cardinal κ can also be considered a product; however, since the conditions are not finite, we have to generalize the notion of product: We define the *κ-product* of P_i, $i \in I$, as follows:

$$\text{(20.5)} \qquad P = \{p \in \prod_{i \in I} P_i : |s(p)| < \kappa\}$$

$$p \le q \quad \text{iff} \quad p_i \le q_i \text{ for all } i$$

Chain Condition in Products and κ-Products

Since we want to use products and κ-products of notions of forcing, let us investigate the properties of products that are of significance in the theory of forcing. To start, let us make the following trivial observation:

Lemma 20.2. *Let κ be a regular cardinal, and let P be the κ-product of $P_i, i \in I$. Let $\alpha < \kappa$ be an infinite cardinal. If each P_i is α-closed, then so is P.*

Proof. Let $\alpha < \kappa$ and let $p^\xi = \langle p_i^\xi : i \in I \rangle$, $\xi < \alpha$, be a descending α-sequence of conditions in P. If we let $s = \bigcup_{\xi < \alpha} \mathrm{s}(p^\xi)$, then $|s| < \kappa$, and since each P_i is α-closed, it is easy to find $p = \langle p_i : i \in I \rangle$ such that $\mathrm{s}(p) = s$ and that $p_i \leq p_i^\xi$ for each $i \in I$ and each $\xi < \alpha$. ■

It is more interesting to determine the chain condition of a product (or a κ-product). It turns out that $\mathrm{sat}(P)$ does not depend on the number of factors in the product; it depends (a) on κ, and (b) on sizes of the factors.

The chain condition of a product (or a κ-product) can be evaluated by using the method of Lemmas 19.3 and 19.8. The following lemma gives an estimate of $\mathrm{sat}(P)$ in the most important cases of the κ-product.

Lemma 20.3. *Let κ be a regular cardinal, and let P be the κ-product of $P_i, i \in I$. Let W be an incompatible subset of P.*

(a) *If each $|P_i| \leq \kappa$, then $|W| \leq 2^{<\kappa}$. (In particular, if $\kappa = \aleph_0$ then W is at most countable.)*
(b) *If $\lambda \geq \kappa$ and if each $|P_i| \leq \lambda$, then $|W| \leq \lambda^{<\kappa}$.*
(c) *If $\lambda > \kappa$ is inaccessible and each $|P_i| < \lambda$, then $|W| < \lambda$.*

Proof. If $p = \langle p_i : i \in I \rangle$ and $q = \langle q_i : i \in I \rangle$ are incompatible in P, then for some $i \in \mathrm{s}(p) \cap \mathrm{s}(q)$, p_i and q_i are incompatible in P_i, and in particular $p_i \neq q_i$. Thus we can regard elements of W as functions whose domain is a subset of I of size $< \kappa$, with values in the P_i, and show that if W consists of pairwise incompatible functions then $|W|$ has the required bound.

This we do precisely as in Lemma 19.8 (or 19.3): We construct

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots \quad \text{and} \quad W_0 \subseteq W_1 \subseteq \cdots \subseteq W_\alpha \subseteq \cdots \qquad (\alpha < \kappa)$$

where each W_α is a subset of W and $A_\alpha = \bigcup \{\mathrm{dom}(p) : p \in W_\alpha\}$. Construction of the W_α is described in Lemma 19.8, and it is shown there that $W = \bigcup_{\alpha < \kappa} W_\alpha$.

The bound on $|W|$ is verified by induction on $\alpha < \kappa$. For example, let us prove (b); then (a) holds because $\kappa^{<\kappa} = 2^{<\kappa}$ if κ is regular, and (c) is proved similarly.

If $|W_\alpha| \leq \lambda^{<\kappa}$, then obviously $|A_\alpha| \leq \kappa \cdot \lambda^{<\kappa} = \lambda^{<\kappa}$. If $\alpha < \kappa$ is a limit and if $|W_\beta| \leq \lambda^{<\kappa}$ for each $\beta < \alpha$, then $|W_\alpha| \leq |\alpha| \cdot \lambda^{<\kappa} = \lambda^{<\kappa}$. Thus let us assume that $|W_\alpha| \leq \lambda^{<\kappa}$ and let us show that $|W_{\alpha+1}| \leq \lambda^{<\kappa}$. The set $W_{\alpha+1}$ is obtained by adding to W_α at most one q for each $p \in P$ with $\mathrm{dom}(p) \subseteq A_\alpha$. However, since

$|A_\alpha| \le \lambda^{<\kappa}$, there are at most $\lambda^{<\kappa}$ functions p with $\mathrm{dom}(p) \subseteq A_\alpha$, $|\mathrm{dom}(p)| < \kappa$, and λ possible values for each $i \in \mathrm{dom}(p)$. Thus $|W_{\alpha+1}| \le \lambda^{<\kappa}$. It follows that $|W| \le \lambda^{<\kappa}$. ■

The relationship between the chain condition of a product and the chain conditions of the factors, even in the case of a product of two factors $P \times Q$, is more complicated. The statement that the product of two c.c.c. partial orderings satisfies the c.c.c. is independent of ZFC; see Exercise 22.4 and Theorem 53 (Section 23). One can prove only that if P and Q satisfy the c.c.c., then $\mathrm{sat}(P \times Q) \le (2^{\aleph_0})^+$ (see Exercise 29.2), and the bound is best possible.

The Lévy Collapse

In Lemma 19.9 we have described a notion of forcing that collapses a cardinal λ onto a smaller regular cardinal κ. We shall now present a technique (due to Lévy) of collapsing all cardinals below an inaccessible cardinal λ while preserving λ, thus making λ a successor cardinal in the generic extension.

Lemma 20.4. *Let κ be a regular cardinal and let $\lambda > \kappa$ be an inaccessible cardinal. There is a notion of forcing $(P, <)$ such that:*

(a) *every α such that $\kappa \le \alpha < \lambda$ has cardinality κ in $\mathfrak{M}[G]$; and*
(b) *every cardinal $\le \kappa$ and every cardinal $\ge \lambda$ remains a cardinal in $\mathfrak{M}[G]$.*

In particular, $\mathfrak{M}[G] \vDash \lambda = \kappa^+$.

Proof. For each $\alpha < \lambda$, let P_α be the set of all functions p_α such that

$$\mathrm{dom}(p_\alpha) \subseteq \kappa \quad \text{and} \quad |\mathrm{dom}(p_\alpha)| < \kappa, \qquad \mathrm{ran}(p_\alpha) \subseteq \alpha \tag{20.6}$$

and let $p_\alpha < q_\alpha$ iff $p_\alpha \supset q_\alpha$.

Let $(P, <)$ be the κ-product of P_α, $\alpha < \lambda$. Note that we can regard the conditions $p \in P$ as functions on subsets of $\lambda \times \kappa$ such that

(20.7) (i) $|\mathrm{dom}(p)| < \kappa$;
(ii) $p(\alpha, \xi) < \alpha$ for each $(\alpha, \xi) \in \mathrm{dom}(p)$.

Let G be a generic set of conditions; and for each $\alpha < \lambda$, let G_α be the projection of G on P_α. Then G_α is a generic filter on P_α; and as in Lemma 19.9, the set $f_\alpha = \bigcup G_\alpha$ is a function that maps κ onto α. Thus $\mathfrak{M}[G] \vDash |\alpha| \le |\kappa|$, for every $\alpha < \lambda$.

The notion of forcing $(P, <)$ is γ-closed for every $\gamma < \kappa$, and hence it preserves all cardinals and cofinalities $\le \kappa$. In particular, κ is a cardinal in $\mathfrak{M}[G]$.

By Lemma 20.3, $(P, <)$ satisfies the λ-chain condition. Hence λ remains a cardinal in $\mathfrak{M}[G]$, and so do all cardinals greater than λ. It follows that in $\mathfrak{M}[G]$, λ is the cardinal successor of κ. ■

The Continuum Function 2^κ for Regular Cardinals κ

In Section 19 we have presented a method of adjoining a large number of subsets of a regular cardinal κ, thus making 2^κ as large as prescribed. We shall now describe a construction that does this simultaneously for a collection of regular cardinals. This method completely solves the problem of the behavior of the continuum function at *regular* cardinals. We recall that for each κ, $\mathrm{cf}(2^\kappa) > \kappa$.

Theorem 46 (Easton). *Let $\mathfrak{M}$ be a transitive model of* ZFC *and assume that the generalized continuum hypothesis holds in $\mathfrak{M}$. Let* $\mathbf{F}$ *be a function (in $\mathfrak{M}$) whose arguments are regular cardinals and whose values are cardinals, such that for all regular κ and λ:*

(20.8) (i) $\mathbf{F}(\kappa) > \kappa$;
(ii) $\mathbf{F}(\kappa) \leq \mathbf{F}(\lambda)$ *whenever* $\kappa \leq \lambda$;
(iii) $\mathrm{cf}\,\mathbf{F}(\kappa) > \kappa$.

Then there is a generic extension $\mathfrak{M}[G]$ of $\mathfrak{M}$ such that $\mathfrak{M}$ and $\mathfrak{M}[G]$ have the same cardinals and cofinalities, and for every regular κ,

$$\mathfrak{M}[G] \models 2^\kappa = \mathbf{F}(\kappa)$$

Remark 1. We have to point out that the generic extension is obtained by forcing with a *class* of conditions. By Lemma 19.4, a notion of forcing can only increase the size of 2^κ for $\kappa < |\mathrm{r.o.}(P)|$; thus we *have* to use a class of conditions. We shall describe the appropriate generalization of the forcing method.

Remark 2. Theorem 46 leaves open the problem of the behavior of 2^κ at *singular* cardinals. As I have remarked in Section 8, unless a certain large cardinal hypothesis holds, cardinal exponentiation is completely determined by the continuum function on regular cardinals (see Lemma 8.1). Besides, although some conditions on the continuum function are known (see Sections 6 and 8), no "necessary and sufficient" description has been formulated. The problem of adding subsets of singular cardinals is considerably more difficult (and of necessity involves large cardinals), and although substantial progress has been made recently, there are still many open questions and no analog of Easton's theorem for singular cardinals seems to be within reach.

Since the proof of Easton's theorem involves forcing with a class of conditions, we shall first give a proof of the special case, when the "continuum function" $\mathbf{F}$ is prescribed for only a *set* of regular cardinals. Thus let us work in a ground model $\mathfrak{M}$ that satisfies the GCH and let F be a function defined on a set A of regular cardinals and having the properties (20.8)(i)–(iii).

For each $\kappa \in \mathrm{dom}(F)$, let $(P_\kappa, \supset)$ be the notion of forcing that adjoins $F(\kappa)$ subsets of κ (cf. (19.10)):

(20.9) $\mathrm{dom}(p_\kappa) \subseteq \kappa \times F(\kappa)$ and $|\mathrm{dom}(p_\kappa)| < \kappa$, $\mathrm{ran}(p_\kappa) \subseteq \{0, 1\}$

We let $(P, <)$ be the *Easton product* of P_κ, $\kappa \in A$: A condition p is a function $p = \langle p_\kappa : \kappa \in A \rangle \in \prod_{\kappa \in A} P_\kappa$ such that if we denote $\mathrm{s}(p) = \{\kappa \in A : p_\kappa \neq \emptyset\}$ (cf. 20.3), then

$$\text{(20.10)} \qquad \text{for every regular cardinal } \gamma, \quad |\mathrm{s}(p) \cap \gamma| < \gamma$$

We can regard the conditions as functions with values 0 and 1, whose domain consists of triples (κ, α, β) where $\kappa \in A$, $\alpha < \kappa$, and $\beta < F(\kappa)$, and such that for every regular cardinal γ,

$$\text{(20.11)} \qquad |\{(\kappa, \alpha, \beta) \in \mathrm{dom}(p) : \kappa \leq \gamma\}| < \gamma$$

(and p is stronger than q iff $p \supset q$). Note that (20.11) implies that for each $\kappa \in A$, $|\mathrm{dom}(p_\kappa)| < \kappa$, where p_κ is defined by

$$p_\kappa(\alpha, \beta) = p(\kappa, \alpha, \beta)$$

Let G be a generic set of conditions, and let for each $\kappa \in A$, G_κ be the projection of G on P_κ. Each G_κ is a generic filter on P_κ and thus produces $F(\kappa)$ new subsets of κ:

$$a_\beta^\kappa = \{\alpha < \kappa : (\exists p \in G)p(\kappa, \alpha, \beta) = 1\} \qquad (\beta < F(\kappa))$$

We shall show that $(P, <)$ preserves cardinals, and that each $\kappa \in A$ has exactly $F(\kappa)$ subsets in $\mathfrak{M}[G]$. The condition (20.11) is instrumental in the proof.

Given a regular cardinal λ, we can break each condition $p \in P$ into two parts:

$$\text{(20.12)} \qquad p^{\leq\lambda} = p \restriction \{(\kappa, \alpha, \beta) : \kappa \leq \lambda\}, \qquad p^{>\lambda} = p \restriction \{(\kappa, \alpha, \beta) : \kappa > \lambda\}$$

Clearly $p = p^{\leq\lambda} \cup p^{>\lambda}$. We let

$$\text{(20.13)} \qquad P^{\leq\lambda} = \{p^{\leq\lambda} : p \in P\}, \qquad P^{>\lambda} = \{p^{>\lambda} : p \in P\}$$

Obviously, $P^{\leq\lambda}$ is the Easton product of P_κ, $\kappa \in A$ and $\kappa \leq \lambda$, and $P^{>\lambda}$ is the Easton product of P_κ, $\kappa \in A$ and $\kappa > \lambda$. Moreover, P is (isomorphic to) the product $P^{\leq\lambda} \times P^{>\lambda}$.

First we notice that $P^{>\lambda}$ is λ-closed: If $C \leq P^{>\lambda}$ consists of pairwise compatible conditions and $|C| \leq \lambda$, then $p = \bigcup C$ is a condition in $P^{>\lambda}$; (20.11) holds for all regular $\gamma > \lambda$, and holds trivially for $\gamma \leq \lambda$ because if $(\kappa, \alpha, \beta) \in \mathrm{dom}(p)$, then $\kappa > \lambda$.

Furthermore, $P^{\leq\lambda}$ satisfies the λ^+-chain condition: If $W \subseteq P^{\leq\lambda}$ is incompatible, then $|W| \leq \lambda$. The proof given in Lemma 19.8 works in this case as well because $|\mathrm{dom}(p)| < \lambda$ for each $p \in P^{\leq\lambda}$ (and because the GCH holds). Thus $P = P^{>\lambda} \times P^{\leq\lambda}$ where $P^{>\lambda}$ is λ-closed and $\mathrm{sat}(P^{\leq\lambda}) \leq \lambda^+$.

Lemma 20.5. *Let $G \times H$ be an $\mathfrak{M}$-generic filter on $P \times Q$, where P is λ-closed and* $\mathrm{sat}(Q) \leq \lambda^+$. *Then every function $f\colon \lambda \to \mathfrak{M}$ in $\mathfrak{M}[G \times H]$ is in $\mathfrak{M}[H]$. In particular,*

$$P^{\mathfrak{M}[G \times H]}(\lambda) = P^{\mathfrak{M}[H]}(\lambda)$$

Proof. Let $\dot{f}$ be a name for f; let us assume, without loss of generality, that for some A, $\|\dot{f}$ is a function from $\check{\lambda}$ into $\check{A}\| = 1$. For each $\alpha < \lambda$, let $D_\alpha \subseteq P$ be defined (in $\mathfrak{M}$) as follows:

$p \in D_\alpha$ iff there is a maximal incompatible $W \subseteq Q$ and a family $\{a^{(\alpha)}_{p,q} : q \in W\}$ such that for each $q \in W$,

$$(p, q) \Vdash \dot{f}(\check{\alpha}) = \check{a}^{(\alpha)}_{p,q} \tag{20.14}$$

We claim that each D_α is open dense in P. Clearly, D_α is open; thus let $p_0 \in P$ be arbitrary and let us find $p \in D_\alpha$ such that $p \leq p_0$. There exist $p_1 \leq p_0$, $q_1 \in Q$ and $a_1 \in A$ such that $(p_1, q_1) \Vdash \dot{f}(\check{\alpha}) = \check{a}_1$. By induction on $\gamma < \lambda^+$, we construct $p_\gamma \in P$, $q_\gamma \in Q$, and $a_\gamma \in A$ such that $p_0 \geq p_1 \geq \cdots \geq p_\gamma \geq \cdots$, that the q_γ are pairwise incompatible and that (p_γ, q_γ) forces $\dot{f}(\check{\alpha}) = \check{a}_\gamma$. If $\{q_\xi : \xi < \gamma\}$ is not maximal incompatible, we can find such p_γ, q_γ, and a_γ since P is λ-closed. By the λ^+-chain condition, there is $\beta < \lambda^+$ such that $W = \{q_\gamma : \gamma < \beta\}$ is maximal incompatible; then we find $p \in P$ stronger than all p_γ, $\gamma < \beta$. Thus D_α is open dense in P.

Since P is λ-closed, it follows that $\bigcap_{\alpha<\lambda} D_\alpha$ is open dense, and so there exists $p \in G$ such that $p \in D_\alpha$ for all $\alpha < \lambda$. We pick (in $\mathfrak{M}$) for each $\alpha < \lambda$ a maximal incompatible $W_\alpha \subseteq Q$ and a family $\{a^{(\alpha)}_{p,q} : q \in W_\alpha\}$ such that (20.14) holds for each $q \in W_\alpha$. By genericity of H, for every α there is a unique $q \in W_\alpha$ such that $q \in H$, and we have, for every $\alpha < \lambda$,

$$f(\alpha) = a^{(\alpha)}_{p,q}, \qquad \text{where } q \text{ is the unique } q \in W_\alpha \cap H \tag{20.15}$$

However, (20.15) defines the function f in $\mathfrak{M}[H]$. ■

Now we can finish the proof of Easton's theorem, that is, at least in the case when F is defined on a set A of regular cardinals.

Let κ be a regular cardinal in $\mathfrak{M}$; we shall show that κ is a regular cardinal in $\mathfrak{M}[G]$. If κ fails to be a regular cardinal, then there exists a function f that maps some $\lambda < \kappa$, regular in $\mathfrak{M}$, cofinally into κ. We consider P as a product: $P = P^{>\lambda} \times P^{\leq\lambda}$. Then $G = G^{>\lambda} \times G^{\leq\lambda}$ and $\mathfrak{M}[G] = \mathfrak{M}[G^{>\lambda}][G^{\leq\lambda}] = \mathfrak{M}[G^{\leq\lambda}][G^{>\lambda}]$. By Lemma 20.5, f is in $\mathfrak{M}[G^{\leq\lambda}]$ and so κ is not a regular cardinal in $\mathfrak{M}[G^{\leq\lambda}]$. However, this is a contradiction since $\operatorname{sat}(P^{\leq\lambda}) \leq \lambda^+ \leq \kappa$ and hence κ is regular in $\mathfrak{M}[G^{\leq\lambda}]$.

It remains to prove that $(2^\lambda)^{\mathfrak{M}[G]} = F(\lambda)$, for each $\lambda \in A$. Again, we regard P as a product $P^{>\lambda} \times P^{\leq\lambda}$ and $G = G^{>\lambda} \times G^{\leq\lambda}$. By Lemma 20.5, every subset of λ in $\mathfrak{M}[G]$ is in $\mathfrak{M}[G^{\leq\lambda}]$ and we have $(2^\lambda)^{\mathfrak{M}[G]} = (2^\lambda)^{\mathfrak{M}[G^{\leq\lambda}]}$. However, an easy computation shows that $|P^{\leq\lambda}| = F(\lambda)$; and since $\operatorname{sat}(P^{\leq\lambda}) = \lambda^+$, we have $|\text{r.o.}(P^{\leq\lambda})| = F(\lambda)$ and hence $(2^\lambda)^{\mathfrak{M}[G]} \leq F(\lambda)$. On the other hand, we have exhibited $F(\lambda)$ subsets of λ for each $\lambda \in A$, and so $\mathfrak{M}[G] \vDash 2^\lambda = F(\lambda)$. ■

Exercise 20.7. The singular cardinals hypothesis holds in Easton's model.

[If κ is singular then every $f\colon \mathrm{cf}(\kappa) \to \kappa$ is in $\mathfrak{N} = \mathfrak{M}[G^{\leq \mathrm{cf}\,\kappa}]$, and so if $F(\mathrm{cf}\,\kappa) < \kappa$ then $(\kappa^{\mathrm{cf}\,\kappa})^{\mathfrak{M}[G]} = (\kappa^{\mathrm{cf}\,\kappa})^{\mathfrak{N}} \leq (2^\kappa)^{\mathfrak{N}} \leq |\mathrm{r.o.}(P^{\leq\,\mathrm{cf}\,\kappa})|^\kappa = (F(\mathrm{cf}\,\kappa))^\kappa = \kappa^+$.]

Forcing with a Class of Conditions

We shall now show how to generalize the preceding construction to prove Theorem 46 in full generality, when the function $\mathbf{F}$ is defined for all regular cardinals (in $\mathfrak{M}$). This generalization involves forcing with a proper class of conditions. Although it is possible to give a general method of forcing with a class, we shall concentrate only on the particular example.

Thus let $\mathfrak{M}$ be a transitive model of ZFC + GCH. Moreover, we assume that $\mathfrak{M}$ has a well-ordering of the universe (e.g., if $\mathfrak{M}$ satisfies $V = L$). Let $\mathbf{F}$ be a function (in $\mathfrak{M}$) defined on all regular cardinals and having the properties (20.8)(i)–(iii). We define a class $\mathbf{P}$ of forcing conditions as follows: $\mathbf{P}$ is the class of all functions p with values 0 and 1, whose domain consists of triples (κ, α, β) where κ is a regular cardinal, $\alpha < \kappa$ and $\beta < \mathbf{F}(\kappa)$, and such that for every regular cardinal γ,

$$(20.11) \qquad |\{(\kappa, \alpha, \beta) \in \mathrm{dom}(p) : \kappa \leq \gamma\}| < \gamma$$

(and p is stronger than q iff $p \supset q$).

As before, we define $P^{\leq \lambda}$ and $\mathbf{P}^{>\lambda}$ for every regular cardinal λ. Note that $P^{\leq \lambda}$ is a set. To define the Boolean-valued model $\mathfrak{M}^{\mathbf{B}}$ and the forcing relation, we use the fact that $\mathbf{P}$ is the Easton product of P_κ, κ a regular cardinal. For each regular λ, we let $B_\lambda = \mathrm{r.o.}(P^{\leq \lambda})$. If $\lambda < \mu$ then the inclusion $P^{\leq \lambda} \subseteq P^{\leq \mu}$ defines an obvious embedding of B_λ into B_μ; thus we arrange the definition of the B_λ so that B_λ is a complete subalgebra of B_μ whenever $\lambda < \mu$. Then we let $\mathbf{B} = \bigcup_\lambda B_\lambda$. $\mathbf{B}$ is a proper class; otherwise it has all the features of a complete Boolean algebra. In particular, $\sum X$ exists for every set $X \subset \mathbf{B}$. Also, $\mathbf{P}$ is dense in $\mathbf{B}$.

To define $\mathfrak{M}^{\mathbf{B}}$, we cannot quite use the inductive definition (18.8) since $\mathbf{B}$ is not a set. However, we simply let $\mathfrak{M}^{\mathbf{B}} = \bigcup_\lambda \mathfrak{M}^{B_\lambda}$; the formal definition of $\mathfrak{M}^{\mathbf{B}}$ does not present any problem. Similarly, to define $\|x \in y\|$ and $\|x = y\|$, we first notice that if $x, y \in \mathfrak{M}^{B_\lambda}$ and $\lambda \leq \mu$, then $\|x \in y\|_{B_\lambda} = \|x \in y\|_{B_\mu}$ and so we let $\|x \in y\| = \|x \in y\|_{B_\lambda}$ where λ is such that $x, y \in \mathfrak{M}^{B_\lambda}$. The same for $\|x = y\|$.

As for the forcing relation in general, we cannot define $\|\varphi\|$ unless φ is restricted; this is because $\sum \mathbf{X}$ does not generally exist if $\mathbf{X} \subseteq \mathbf{B}$ is a class. However, we can still define $p \Vdash \varphi$ using the formulas from Lemma 18.8.

Now, we call $\mathbf{G} \subseteq \mathbf{P}$ *generic* over $\mathfrak{M}$ if (i) $p \subset q$ and $p \in \mathbf{G}$ implies $q \in \mathbf{G}$, (ii) $p, q \in \mathbf{G}$ implies $p \cup q \in \mathbf{G}$, and (iii) if $\mathbf{D}$ is a *class* in $\mathfrak{M}$ and $\mathbf{D}$ is dense in $\mathbf{P}$, then $\mathbf{D} \cap \mathbf{G} \neq \emptyset$.

The question of existence of a generic filter can be settled in a more or less the same way as in case when P is a set. One possible way is to assume that $\mathfrak{M}$ is a countable transitive model. Then there are only countably many classes in $\mathfrak{M}$ and $\mathbf{G}$ exists. Another possible way is to use the canonical generic ultrafilter. It is the class G in $\mathfrak{M}^{\mathbf{B}}$ defined by $G(\check{p}) = p$ for all $p \in \mathbf{P}$ (here we need the assumption that $\mathfrak{M}$ is a class in $\mathfrak{M}^{\mathbf{B}}$).

Thus let $\mathbf{G}$ be an $\mathfrak{M}$-generic filter on $\mathbf{P}$. For every regular λ, $G_\lambda = \mathbf{G} \cap P^{\leq\lambda}$ is generic on $P^{\leq\lambda}$. If $x \in \mathfrak{M}^{B_\lambda}$ and $\lambda \leq \mu$, then $i_{G_\lambda}(x) = i_{G_\mu}(x)$, and so we define $i_{\mathbf{G}}(x) = i_{G_\lambda}(x)$ where λ is such that $x \in \mathfrak{M}^{B_\lambda}$. Then we let $\mathfrak{M}[\mathbf{G}] = i_{\mathbf{G}}[\mathfrak{M}^{\mathbf{B}}]$; it follows that $\mathfrak{M}[\mathbf{G}] = \bigcup_\lambda \mathfrak{M}[G_\lambda]$.

Using genericity of $\mathbf{G}$ and properties of the forcing relation, we get

Forcing Theorem:

$$(20.16) \qquad \mathfrak{M}[\mathbf{G}] \Vdash \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad (\exists p \in \mathbf{G}) p \vdash \varphi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathfrak{M}^{\mathbf{B}}$ *are names for* $x_1, \ldots, x_n$.

The formula (20.16) is proved first for atomic formulas and then by induction on φ; in the induction step involving the quantifiers, we use the fact that $\mathbf{G}$ intersects every dense class of $\mathfrak{M}$.

We shall now show that $\mathfrak{M}[\mathbf{G}]$ is a model of ZFC. The proofs of all axioms of ZFC except power set and replacement go through as when we forced with a set. It is no surprise that the power set and replacement axioms present problems. It is easy to generalize the constructions of Section 19 either to get a class of forcing conditions adding a proper class of Cohen reals, or a class of conditions collapsing *Ord* onto ω. The present proof of the power set and replacement axioms uses the fact that for every regular λ (or at least for arbitrarily large regular λ), $\mathbf{P} = \mathbf{P}_1 \times P_2$ where $\mathbf{P}_1$ is λ-closed and P_2 is a set and $\operatorname{sat}(P_2) \leq \lambda^+$. (Here $\mathbf{P} = \mathbf{P}^{>\lambda} \times P^{\leq\lambda}$.)

Power set. Let λ be a regular cardinal. Lemma 20.5 remains true even when applied to $\mathbf{P}^{>\lambda} \times P^{\leq\lambda}$. It does not matter that each $\mathbf{D}_\alpha$ in (20.14) is a class. The "sequence" of classes $\langle \mathbf{D}_\alpha : \alpha < \lambda \rangle$ can be defined (e.g., as a class of pairs $\{(p, \alpha) : p \in \mathbf{D}_\alpha\}$) and since $\mathbf{P}^{>\lambda}$ is λ-closed, the intersection $\bigcap_{\alpha<\lambda} \mathbf{D}_\alpha$ is dense, and there exists $p \in \mathbf{G} \cap \mathbf{P}^{>\lambda}$ such that $p \in \mathbf{D}_\alpha$ for all $\alpha < \lambda$. The rest of the proof of Lemma 20.5 remains unchanged, and thus we have proved that every subset of λ in $\mathfrak{M}[\mathbf{G}]$ is in $\mathfrak{M}[G_\lambda]$. Since $P^{\leq\lambda}$ is a set, it follows that the power set axiom holds in $\mathfrak{M}[\mathbf{G}]$.

Replacement. To show that the axioms of replacement hold in $\mathfrak{M}[\mathbf{G}]$, we combine the proof for ordinary generic extension with Lemma 20.5. It suffices to prove that if in $\mathfrak{M}[\mathbf{G}]$, $\varphi(\alpha, v)$ defines a function $\mathbf{K}$: *Ord* $\to \mathfrak{M}[\mathbf{G}]$, then $\mathbf{K}[\lambda]$ is a set in $\mathfrak{M}[\mathbf{G}]$ for every regular cardinal λ. Without loss of generality, let us assume that for every $p \in \mathbf{P}$,

$$(20.17) \qquad p \Vdash \text{for every } \alpha \text{ there is a unique } v \text{ such that } \varphi(\alpha, v).$$

Let λ be a regular cardinal, and let us consider again $\mathbf{P} = \mathbf{P}^{>\lambda} \times P^{\leq\lambda}$, and $\mathbf{G} = (\mathbf{G} \cap \mathbf{P}^{>\lambda}) \times G_\lambda$. As in Lemma 20.5, let us define, for each $\alpha < \lambda$, a class

$\mathbf{D}_\alpha \subset \mathbf{P}^{>\lambda}$:

$p \in \mathbf{D}_\alpha$ iff there is a maximal incompatible $W \subseteq P^{\leq\lambda}$ and a family $\{a^{(\alpha)}_{p,q} : q \in W\}$ such that for each $q \in W$,

$$p \cup q \Vdash \varphi(\check{\alpha}, a^{(\alpha)}_{p,q}) \tag{20.18}$$

As in Lemma 20.5, each $\mathbf{D}_\alpha$, $\alpha < \lambda$, is open dense; since $\mathbf{P}^{>\lambda}$ is λ-closed, $\bigcap_{\alpha<\lambda} \mathbf{D}_\alpha$ is dense and there exists $p \in \mathbf{G} \cap P^{>\lambda}$ such that $p \in \mathbf{D}_\alpha$ for all $\alpha < \lambda$. We pick (in $\mathfrak{M}$) for each $\alpha < \lambda$ a maximal incompatible $W_\alpha \subseteq P^{\leq\lambda}$ and a family $\{a^{(\alpha)}_{p,q} : q \in W_\alpha\}$ such that (20.18) holds for each $q \in W_\alpha$. Now if we let $S = \{a^{(\alpha)}_{p,q} : \alpha < \lambda \text{ and } q \in W_\alpha\}$, then it follows that

$$\mathbf{K}[\lambda] \subseteq i_{\mathbf{G}}[S]$$

However, $i_{\mathbf{G}}[S]$ is a set in $\mathfrak{M}[\mathbf{G}]$: there is γ such that $S \subset \mathfrak{M}^{B_\gamma}$, and we have $i_{\mathbf{G}}[S] = i_{G_\gamma}[S] \in \mathfrak{M}[G_\gamma]$.

Thus $\mathfrak{M}[\mathbf{G}]$ is a model of ZFC and it remains to show that $\mathfrak{M}[\mathbf{G}]$ has the same cardinals and cofinalities as $\mathfrak{M}$, and that in $\mathfrak{M}[\mathbf{G}]$, $2^\kappa = \mathbf{F}(\kappa)$ for every regular cardinal κ. However, this is proved in exactly the same way as when we forced with a set of Easton conditions. ■

*Exercise 20.8.** Let $\mathfrak{M}$ be a transitive model of ZFC. There is a generic extension $\mathfrak{M}[\mathbf{G}]$ such that $\mathfrak{M}[\mathbf{G}]$ satisfies the GCH.

[For each α, let P_α be the notion of forcing which collapses $\lambda = \beth_{\alpha+1}$ onto $\kappa = (\beth_\alpha)^+$ (see 19.12). P_α is $\beth_\alpha$-closed and $\text{sat}(P_\alpha) = \lambda^+$. Let P be an Easton product of P_α, $\alpha \in Ord$; namely, we require that $|s(p) \cap \gamma| < \gamma$ for every inaccessible $\gamma = \beth_\alpha$. Show that for each α, $\kappa = (\beth_\alpha)^+$ is a cardinal in $\mathfrak{M}[\mathbf{G}]$, $\kappa = \aleph^{\mathfrak{M}[\mathbf{G}]}_{\alpha+1}$, and $\mathfrak{M}[\mathbf{G}] \vDash 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Apply Lemma 20.5 in two ways: (a) for each α, consider $P^{\leq\alpha} \times \mathbf{P}^{>\alpha}$: $\text{sat}(P^{\leq\alpha}) = \beth^+_{\alpha+1}$ and $\mathbf{P}^{>\alpha}$ is $\beth_{\alpha+1}$-closed; (b) if α is inaccessible and $\alpha = \beth_\alpha$, consider $P^{<\alpha} \times \mathbf{P}^{\geq\alpha}$: $\text{sat}(P^{<\alpha}) = \beth^+_\alpha$ and $P^{\geq\alpha}$ is $\beth_\alpha$-closed.]

We conclude the section with a remark on the Bernays–Gödel axiomatic set theory. At the end of Chapter 1, I made a remark to the effect that if a sentence involving only set variables is provable in BGC = BG + Axiom E, then it is provable in BG + AC. This is a consequence of the following: If $\mathfrak{M}$ is a transitive model of BG + AC, then there is a generic extension $\mathfrak{M}[\mathbf{G}]$ that has the same sets and has a choice function $\mathbf{F}$ defined for all nonempty sets. The forcing conditions $p \in \mathbf{P}$ used in the proof are choice functions whose domain is a set of nonempty sets (and $p < q$ iff $p \supset q$). The proof that $\mathfrak{M}[\mathbf{G}]$ is a model of BG is rather easy since no new sets are added ($\mathbf{P}$ is κ-closed for all κ). The generic filter on $\mathbf{P}$ defines a choice function $\mathbf{F} = \bigcup \mathbf{G}$, and $\mathbf{F}$ is defined for all nonempty sets $X \in \mathfrak{M}[\mathbf{G}]$.

21. SYMMETRIC SUBMODELS OF GENERIC MODELS

In Section 19, we constructed a model of set theory in which the reals cannot be well-ordered, thus showing that the axiom of choice is independent of the axioms of ZF. What follows is a more systematic study of models in which the axiom of choice fails. I shall present a general method of construction of submodels of generic extensions. The construction uses symmetry arguments similar to those used in Theorem 45, and the models obtained are generally models of ZF and do not satisfy the axiom of choice. This method has been widely used to obtain a number of results about the relative strength of various weaker versions and consequences of the axiom of choice.

The main idea of the construction of symmetric models is the use of automorphisms of the Boolean-valued model V^B and the "symmetry lemma" 19.10. In fact, the idea of using automorphisms of the universe to show that the axiom of choice is unprovable dates back into the preforcing era of set theory. I shall describe this older construction first. To do it, we have to modify somewhat the axioms of set theory.

Set Theory with Atoms (ZFA)

In this modified set theory we have not only sets, but also additional objects, *atoms*. These atoms do not have any elements themselves but can be collected into sets. Obviously, we have to modify the axiom of extensionality, for any two atoms have the same elements—none.

The language of ZFA has, in addition to the predicate $\in$, a constant A. The elements of A are called *atoms*; all other objects are *sets*.

Axiom A. *If $a \in A$, then there is no x such that $x \in a$.*

The axiom of extensionality takes this form:

(21.1) If two sets X and Y have the same elements, then $X = Y$.

All the other axioms of ZF remain unchanged. In particular, the axiom of regularity states that every nonempty set has an $\in$-minimal element. This minimal element may be an atom.

The effect of atoms is that the universe is no longer obtained by iterated power set operation from the empty set. In ZFA, the universe is built up from atoms.

Ordinal numbers are defined as usual except that one has to add that an ordinal does not contain any atom. For any set, let us define the following cummulative hierarchy:

$$(21.2)\qquad P^0(S) = S, \qquad P^\alpha(S) = \bigcup_{\beta<\alpha} P^\beta(S) \qquad \text{if } \alpha \text{ is a limit}$$

$$P^{\alpha+1}(S) = P^\alpha(S) \cup P(P^\alpha(S))$$

$$P^\infty(S) = \bigcup_{\alpha \in Ord} P^\alpha(S)$$

It follows that

$$(21.3)\qquad V = P^\infty(A)$$

and that the *kernel*, the class $P^\infty(\emptyset)$ of "hereditary" sets, is a model of ZF. If A is empty, then we have just ZF; right now we are more interested in the case when A is infinite.

Exercise 21.1. The theory ZFA + AC + "A is infinite" is consistent relative to ZFC.

[Construct a model of ZFA. Let C be an infinite set of sets of the same rank (so that $X \notin \mathrm{TC}(Y)$ for any X, $Y \in C$). Consider one $X_0 \in C$ as the empty set, and all other $X \in C$ as atoms. Build up the model from C by iterating the operation $P^*(Z) = P(Z) - \{\emptyset\}$.]

Permutation Models of ZFA

While in ZF, the universe does not admit nontrivial automorphisms, the important feature of ZFA is that every permutation of atoms induces an automorphism of V: If π is a one-to-one mapping of A onto A (a *permutation* of A), then we define for every x (by $\in$-induction):

$$\pi(x) = \pi[x] = \{\pi(t) : t \in x\} \tag{21.4}$$

Clearly, π is an $\in$-automorphism, and we have $\pi(x) = x$ for every x in the kernel $P^\infty(\emptyset)$.

We use these automorphisms to construct transitive models of ZFA. First we point out that the analog of Theorem 31 in Section 11 is true in ZFA: If $\mathfrak{M}$ is a transitive, closed, almost universal class and if $A \in \mathfrak{M}$ (or at least $A \cap \mathfrak{M} \in \mathfrak{M}$), then $\mathfrak{M}$ is a model of ZFA.

Let $\mathcal{G}$ be a group of permutations of a set S. A set $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a *normal filter on* $\mathcal{G}$, if for all subgroups H, K of $\mathcal{G}$:

(21.5) (i) $\mathcal{G} \in \mathcal{F}$;
(ii) if $H \in \mathcal{F}$ and $H \subseteq K$, then $K \in \mathcal{F}$;
(iii) if $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$;
(iv) if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$ then $\pi H \pi^{-1} \in \mathcal{F}$.

For a given group of permutations $\mathcal{G}$ of the set A of atoms and a normal filter $\mathcal{F}$ on $\mathcal{G}$, we say that x is *symmetric* if the group

$$\mathrm{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi(x) = x\} \tag{21.6}$$

belongs to $\mathcal{F}$.

Let us further assume that

$$\mathrm{sym}_{\mathcal{G}}(a) \in \mathcal{F} \qquad \text{for all} \quad a \in A \tag{21.7}$$

that is, that all atoms are symmetric (w.r.t. $\mathcal{G}$ and $\mathcal{F}$) and let $\mathfrak{V}$ be the class of all hereditarily symmetric objects:

$$\mathfrak{V} = \{x : \text{every } z \in \mathrm{TC}(\{x\}) \text{ is symmetric}\} \tag{21.8}$$

The class $\mathfrak{V}$ (a *permutation model*) is a transitive class and obviously includes the kernel (because $\mathrm{sym}(x) = \mathcal{G}$ for all $x \in P^\infty(\emptyset)$), moreover, all atoms are in $\mathfrak{V}$ and $A \in \mathfrak{V}$.

Lemma 21.1. *$\mathfrak{V}$ is a transitive model of* ZFA.

Proof. We have to show that $\mathfrak{V}$ is closed under Gödel operations and almost universal. It is easy to see that $\mathfrak{F}_i(\pi x, \pi y) = \pi(\mathfrak{F}_i(x, y))$ for $i = 1, \ldots, 10$, and therefore

$$\text{(21.9)} \qquad \operatorname{sym}(\mathfrak{F}_i(x, y)) \supseteq \operatorname{sym}(x) \cap \operatorname{sym}(y) \qquad (i = 1, \ldots, 10)$$

It follows that if x and y are hereditarily symmetric, then so is $\mathfrak{F}_i(x, y)$.

To show that $\mathfrak{V}$ is almost universal, it suffices to verify that for each α, $\mathfrak{V} \cap P^\alpha(A)$ is symmetric. For all x and all $\pi \in \mathscr{G}$ we have $\operatorname{rank}(\pi x) = \operatorname{rank} x$. Also,

$$\text{(21.10)} \qquad \operatorname{sym}(\pi x) = \pi \cdot \operatorname{sym}(x) \cdot \pi^{-1}$$

and hence, by property (iv) in (21.5), if x is symmetric and $\pi \in \mathscr{G}$, then $\pi(x)$ is symmetric. Thus for all $\pi \in \mathscr{G}$ we have $\pi[\mathfrak{V} \cap P^\alpha(A)] = \mathfrak{V} \cap P^\alpha(A)$ and therefore, $\operatorname{sym}(\mathfrak{V} \cap P^\alpha(A)) = \mathscr{G}$. ∎

In the following examples we construct permutation models as follows: For every finite $E \subset A$, we let

$$\text{(21.11)} \qquad \operatorname{fix}_{\mathscr{G}}(E) = \{\pi \in \mathscr{G} : \pi a = a \text{ for all } a \in E\}$$

and let $\mathscr{F}$ be the filter on $\mathscr{G}$ generated by $\{\operatorname{fix}(E) : E \subset A \text{ is finite}\}$. $\mathscr{F}$ is a normal filter since $\pi \cdot \operatorname{fix}(E) \cdot \pi^{-1} = \operatorname{fix}(\pi[E])$, and satisfies (21.7). Thus x is symmetric if and only if there exists a finite set of atoms E, a *support* for x, such that $\pi(x) = x$ whenever $\pi \in \mathscr{G}$ and $\pi a = a$ for all $a \in E$.

We shall now give two examples of permutation models.

Example I. Let A be infinite, and let $\mathscr{G}$ be the group of all permutations of A. Let $\mathscr{F}$ be generated by $\{\operatorname{fix}(E) : E \subset A \text{ is finite}\}$, and let $\mathfrak{V}$ be the permutation model.

Lemma 21.2. *In the model $\mathfrak{V}$ from Example I, the set A, although infinite, has no countable subset. Hence the axiom of choice fails in $\mathfrak{V}$.*

Proof. Assume that there exists $f \in \mathfrak{V}$ which is a one-to-one mapping of ω into A. Let E be a finite subset of A such that $\pi f = f$ for every $\pi \in \operatorname{fix}(E)$. Since E is finite, there exists $a \in A - E$ such that $a = f(n)$ for some n; also, let $b \in A - E$ be arbitrary such that $b \neq a$. Now, let π be a permutation of A such that $\pi a = b$ but $\pi x = x$ for all $x \in E$. Then $\pi f = f$, and since n is in the kernel, we have $\pi n = n$. It follows that

$$\pi(f(n)) = (\pi f)(\pi n) = f(n);$$

however, $f(n) = a$ while $\pi(f(n)) = \pi a \neq a$. A contradiction. ∎

Example II. Let A be a disjoint countable union of pairs:

$$\text{(21.12)} \qquad A = \bigcup_{n=0}^{\infty} P_n, \qquad P_n = \{a_n, b_n\}$$

and let $\mathscr{G}$ be the group of all permutations of A such that $\pi(\{a_n, b_n\}) = \{a_n, b_n\}$, for all n. Let $\mathscr{F}$ be generated by $\{\text{fix}(E) : E \subset A \text{ is finite}\}$, and let $\mathfrak{V}$ be the permutation model.

Lemma 21.3. *In the model $\mathfrak{V}$ from Example II, $\{P_n : n \in \omega\}$ is a countable set of pairs and has no choice function.*

Proof. Each P_n is a symmetric set since $\pi(P_n) = P_n$ for all $\pi \in \mathscr{G}$. For the same reason, $\pi\langle P_n : n \in \omega\rangle = \pi(\{(n, P_n) : n \in \omega\}) = \langle P_n : n \in \omega\rangle$, for all $\pi \in \mathscr{G}$, and so $\langle P_n : n \in \omega\rangle \in \mathfrak{V}$. Hence $S = \{P_n : n \in \omega\}$ is a countable set in $\mathfrak{V}$.

We show that there is no function $f \in \mathfrak{V}$ such that $\text{dom}(f) = S$ and $f(P_n) \in P_n$ for all n. Assume that f is such a function and let E be a support of f. There exists n such that neither a_n nor b_n is in E, and we let $\pi \in \mathscr{G}$ be such that $\pi a_n = b_n$ but $\pi x = x$ for all $x \in E$. Then $\pi f = f$, $\pi P_n = P_n$, and so

$$\pi(f(P_n)) = (\pi f)(\pi P_n) = f(P_n)$$

but $\pi(f(P_n)) = b_n$ while $f(P_n) = a_n$; a contradiction. ■

Exercise 21.2. We recall that a set is called D-finite if it has no countable subset. Show that in Example II A is D-finite.

Exercise 21.3. Show that in either Example I or II, A cannot be linearly ordered.

The method of permutation models gives numerous examples of violation of the axiom of choice. One usually uses the set of atoms to produce a counterexample (in the permutation model) to some consequence of the axiom of choice, thus showing the limitations of proofs not using the axiom of choice. (A typical example is a vector space that has no basis, a set that cannot be linearly ordered, etc.) However, these examples do not give any information about the "true" sets, like real numbers, sets of real numbers, etc., since those sets are in the kernel. It is clear that a different method has to be used to investigate the role of the axiom of choice in ZF. We shall now describe such a method and exploit the similarities between it and permutation models.

Symmetric Models

Although in ZF, transitive classes have no nontrivial automorphisms, we can still use automorphisms (symmetries) to construct submodels of generic extensions. As shown in Section 19, every automorphism π of a complete Boolean algebra B induces an automorphism of the Boolean-valued model V^B (see (19.13)). The important property of such an automorphism is the "symmetry lemma" 19.10:

$$\|\varphi(\pi x_1, \ldots, \pi x_n)\| = \pi(\|\varphi(x_1, \ldots, x_n)\|) \tag{19.14}$$

for all names $x_1, \ldots, x_n$.

Let $\mathscr{G}$ be a group of automorphisms of B. A set $\mathscr{F}$ of subgroups of $\mathscr{G}$ is a *normal filter* on $\mathscr{G}$ if it has the properties (21.5). For each name $x \in V^B$ we define its symmetry group

$$\text{sym}_{\mathscr{G}}(\mathbf{x}) = \{\pi \in \mathscr{G} : \pi(\mathbf{x}) = \mathbf{x}\} \tag{21.13}$$

If π is an automorphism of B, then

$$\text{sym}(\pi\mathbf{x}) = \pi \cdot \text{sym}(\mathbf{x}) \cdot \pi^{-1} \tag{21.14}$$

This is because $\sigma(\pi\mathbf{x}) = \pi\mathbf{x}$ if and only if $(\pi^{-1}\sigma\pi)(\mathbf{x}) = \mathbf{x}$. Given a normal filter $\mathscr{F}$ on $\mathscr{G}$, we call $\mathbf{x}$ *symmetric* if $\text{sym}(\mathbf{x}) \in \mathscr{F}$. The class HS of *hereditarily symmetric names* is defined by induction on $\rho(\mathbf{x})$:

if $\text{dom}(\mathbf{x}) \subseteq \text{HS}$ and if $\mathbf{x}$ is symmetric, then $\mathbf{x} \in \text{HS}$.

Note that $\pi(\check{x}) = \check{x}$ for all x and all π, and so all $\check{x}$ are in HS. If a name $\mathbf{x}$ is symmetric, and if $\pi \in \mathscr{G}$, then by (21.14) and since $\mathscr{F}$ is normal, $\pi\mathbf{x}$ is also symmetric. It follows that $\pi\mathbf{x} \in \text{HS}$ whenever $\mathbf{x} \in \text{HS}$ and $\pi \in \mathscr{G}$.

The class HS constitutes a submodel of the Boolean-valued model V^B, and satisfies all axioms of ZF. We shall prove this in the context of generic extensions.

Thus let $\mathfrak{M}$ be the ground model, let B be a complete Boolean algebra in $\mathfrak{M}$, and let $\mathscr{G}$ and $\mathscr{F}$ be respectively (in $\mathfrak{M}$), a group of automorphisms of B and a normal filter on $\mathscr{G}$. Let G be an $\mathfrak{M}$-generic ultrafilter on B. We let

$$\mathfrak{N} = \{i_G(\mathbf{x}) : \mathbf{x} \in \text{HS}\} \tag{21.15}$$

be the class of all elements of $\mathfrak{M}[G]$ that have a hereditarily symmetric name. $\mathfrak{N}$ is called a *symmetric submodel* of $\mathfrak{M}[G]$. We will prove that $\mathfrak{N}$ is a transitive model of ZF. Before we do so, we notice that HS is a Boolean-valued model (with the same $\|x \in y\|$ and $\|x = y\|$ as $\mathfrak{M}^B$). Thus we can define $\|\varphi\|_{\text{HS}}$ for every formula φ. Note that

$$\|\exists x\, \varphi(x)\|_{\text{HS}} = \sum_{\mathbf{x} \in \text{HS}} \|\varphi(\mathbf{x})\| \tag{21.16}$$

and that $\|\varphi\|_{\text{HS}} = \|\varphi\|$ whenever φ is a restricted formula. We also have a forcing theorem for the model $\mathfrak{N}$:

$$\mathfrak{N} \vDash \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad \|\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n\|_{\text{HS}} \in G \tag{21.17}$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \text{HS}$ are names for $x_1, \ldots, x_n$. Finally, since $\pi[\text{HS}] = \text{HS}$ for all $\pi \in \mathscr{G}$, we have the symmetry lemma for $\|\ \|_{\text{HS}}$: If $\pi \in \mathscr{G}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \text{HS}$, then

$$\|\varphi(\pi\mathbf{x}_1, \ldots, \pi\mathbf{x}_n)\|_{\text{HS}} = \pi(\|\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)\|_{\text{HS}}) \tag{21.18}$$

Lemma 21.4. *A symmetric submodel* $\mathfrak{N}$ *of* $\mathfrak{M}[\mathscr{G}]$ *is a transitive model of* ZF, *and* $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{M}[G]$.

Proof. Since $\check{x} \in \text{HS}$ for every $x \in \mathfrak{M}$, we have $\mathfrak{M} \subseteq \mathfrak{N}$. The heredity of HS implies that $\mathfrak{N}$ is transitive. To verify that the axioms of ZF hold in $\mathfrak{N}$, we follow closely the proof of the generic model theorem. As there, we have to show that certain sets exist in the model by exhibiting names for the sets; here we have to find such names in HS.

A. *Extensionality, Regularity, Infinity.* These axioms hold in $\mathfrak{N}$ since $\mathfrak{N}$ is transitive and $\mathfrak{N} \supseteq \mathfrak{M}$.

B. *Separation.* Let φ be a formula and let

$$Y = \{x \in X : \mathfrak{N} \vDash \varphi(x, p)\} \tag{21.19}$$

where $X, p \in \mathfrak{N}$. Let $\boldsymbol{X}, \boldsymbol{p} \in \mathrm{HS}$ be names for X, p. We let $\boldsymbol{Y} \in \mathfrak{M}^B$ as follows:

$$\operatorname{dom}(\boldsymbol{Y}) = \operatorname{dom}(\boldsymbol{X}), \qquad \boldsymbol{Y}(\boldsymbol{t}) = \boldsymbol{X}(\boldsymbol{t}) \cdot \|\varphi(\boldsymbol{t}, \boldsymbol{p})\|_{\mathrm{HS}} \tag{21.20}$$

A routine argument shows that $\boldsymbol{Y}$ is a name for Y; it remains to show that $\boldsymbol{Y}$ is symmetric.

We shall show that $\operatorname{sym}(\boldsymbol{Y}) \supseteq \operatorname{sym}(\boldsymbol{X}) \cap \operatorname{sym}(\boldsymbol{p})$. Thus let $\pi \in \mathscr{G}$ be such that $\pi\boldsymbol{X} = \boldsymbol{X}$ and $\pi\boldsymbol{p} = \boldsymbol{p}$. For every $\boldsymbol{t} \in \operatorname{dom}(\boldsymbol{X})$ we have $\pi\boldsymbol{t} \in \operatorname{dom}(\pi\boldsymbol{x}) = \operatorname{dom} \boldsymbol{x}$ and

$$\boldsymbol{X}(\pi\boldsymbol{t}) = (\pi\boldsymbol{X})(\pi\boldsymbol{t}) = \pi(\boldsymbol{X}(\boldsymbol{t})) \qquad \|\varphi(\pi\boldsymbol{t}, \boldsymbol{p})\|_{\mathrm{HS}} = \pi\|\varphi(\boldsymbol{t}, \boldsymbol{p})\|_{\mathrm{HS}}$$

and so $\boldsymbol{Y}(\pi\boldsymbol{t}) = \pi(\boldsymbol{Y}(\boldsymbol{t}))$. Therefore, $\pi\boldsymbol{Y} = \boldsymbol{Y}$.

C. *Pairing, Union, Power set.* Let $X \in \mathfrak{N}$ and let $\boldsymbol{X} \in \mathrm{HS}$ be a name for X. For the union, we let

$$S = \bigcup \{\operatorname{dom}(\boldsymbol{y}) : \boldsymbol{y} \in \operatorname{dom}(\boldsymbol{X})\}$$

If $\pi \in \operatorname{sym}(\boldsymbol{X})$ then $\pi[S] = S$ and so the set $Y = i_G[S]$ has a hereditarily symmetric name $\boldsymbol{Y}$: $\boldsymbol{Y}(\boldsymbol{t}) = 1$ for all $\boldsymbol{t} \in S$. Moreover, $Y \supseteq \bigcup X$.

Pairing and Power set are handled similarly.

D. *Replacement.* We show that if $X \in \mathfrak{N}$, then there exists $Y \in \mathfrak{N}$ such that for all $u \in X$, $\mathfrak{N}$ satisfies

$$\exists v\, \varphi(u, v) \to (\exists v \in Y)\, \varphi(u, v)$$

We proceed as in (18.37) except that (we deal with $\|\ \|_{\mathrm{HS}}$ instead of $\|\ \|$ and that) we look for $S \subset \mathrm{HS}$ such that $\pi[S] = S$ for all $\pi \in \mathscr{G}$ (for then $Y = i_G[S]$ has a name in HS). This is accomplished by taking for S the set $\mathrm{HS} \cap \mathfrak{M}^B_\alpha$ for large enough α. Since every π preserves the rank and since each $\pi \in \mathscr{G}$ preserves HS, we have $\pi[S] = S$ for all $\pi \in \mathscr{G}$. ∎

In general, the set G is not a member of $\mathfrak{N}$, and $\mathfrak{N}$ does not satisfy the axiom of choice. We shall use symmetry arguments to show that AC fails in the forthcoming examples. The models in Examples I and II are due to Cohen.

Example I. This example is an analog of the first example of a permutation model and in fact, describes the same model that was used in Chapter 19 in the proof of Theorem 45. We let $\mathfrak{M}[G]$ be the generic extension adjoining countably many Cohen reals: P is the set of all finite 0–1-functions p with domain $\mathrm{dom}(p) \subset \omega \times \omega$ (cf. (19.18)). We define a_n, $n \in \omega$, and $A = \{a_n : n \in \omega\}$, as well as their canonical names as in (19.18)–(19.21).

Every permutation π of ω induces an automorphism of P (and in turn an automorphism of B) via (19.24). We can view such permutations as permutations of the set $\{\boldsymbol{a}_n : n \in \omega\}$. Let $\mathscr{G}$ be the group of all such automorphisms of B that are induced by permutations of ω. For every finite $E \subset \omega$, let

$$\mathrm{fix}(E) = \{\pi \in \mathscr{G} : \pi n = n \text{ for each } n \in E\} \tag{21.21}$$

Let $\mathscr{F}$ be the filter on $\mathscr{G}$ generated by $\{\mathrm{fix}(E) : E \subseteq \omega \text{ is finite}\}$; $\mathscr{F}$ is a normal filter.

Now let HS be the class of all hereditarily symmetric names, and let $\mathfrak{N}$ be the corresponding symmetric submodel of $\mathfrak{M}[G]$. It is easy to see that all $\boldsymbol{a}_n$ are in HS and so is $\boldsymbol{A}$. Moreover, the a_n are distinct subsets of ω (see Lemma 19.11), and so A is an infinite set of reals in $\mathfrak{N}$.

We claim that

$$\mathfrak{N} \vDash A \text{ is D-finite} \tag{21.22}$$

To prove (21.22), let us show that there is no $f \in \mathfrak{N}$ that maps ω onto A. Assuming that there is such f, let $\boldsymbol{f} \in$ HS and let $p_0 \in G$ be such that

$$p_0 \Vdash \boldsymbol{f} \text{ maps } \check{\omega} \text{ onto } \boldsymbol{A}$$

The contradiction is obtained as in Lemma 19.12. We let E be a support of $\boldsymbol{f}$, i.e., a finite subset of ω such that $\mathrm{sym}(\boldsymbol{f}) \supseteq \mathrm{fix}(E)$. We pick $i \in \omega$ such that $i \notin E$, and find $p \leq p_0$ and $n \in \omega$ such that

$$p \Vdash \boldsymbol{f}(\check{n}) = \boldsymbol{a}_i$$

Then we find a permutation $\pi \in \mathscr{G}$ (see Fig. 19.1) such that:

(21.23) (i) πp and p are compatible;
(ii) $\pi \in \mathrm{fix}(E)$;
(iii) $\pi i = j \neq i$.

Then $\pi \boldsymbol{f} = \boldsymbol{f}$, $\pi(\check{n}) = \check{n}$, and we have

$$p \cup \pi p \Vdash \boldsymbol{f}(\check{n}) = \boldsymbol{a}_i \qquad \text{and} \qquad p \cup \pi p \Vdash \boldsymbol{f}(\check{n}) = \boldsymbol{a}_j$$

a contradiction. ■

Exercise 21.4. Show that if there is an infinite, D-finite set of reals, then the statements in Exercises 5.5 and 5.6 are false.

The similarity between the model HOD(A) used in Theorem 45 and the above symmetric model $\mathfrak{N}$ is no accident: If the ground model is L, then these models are the same. (If $\mathfrak{M}$ is arbitrary, then $\mathfrak{N}$ is (in $\mathfrak{M}[G]$), the class of all sets hereditarily $\mathfrak{M}$-definable over A.)

Example II. This example is an analog of the second example of a permutation model. We construct a symmetric model $\mathfrak{N}$ such that

(21.24) $\mathfrak{N} \vDash$ there exists a countable set $\{P_n : n \in \omega\}$ of pairs without a choice function.

Unlike Example I, where we used generic reals to play the role of the atoms in the corresponding permutation model, here we use *sets of reals* to be the indistinguishable objects. The reason for this is that the natural ordering of real numbers implies existence of a choice function on any family of pairs of reals.

We first construct a generic model $\mathfrak{M}[G]$ that has countably many pairs

$$P_n = \{a_n, b_n\} \qquad (n \in \omega) \tag{21.25}$$

where each a_n and each b_n is an infinite set of generic reals:

$$a_n = \{a_{nm} : m \in \omega\}, \qquad b_{nm} = \{b_{nm} : m \in \omega\} \tag{21.26}$$

Thus we let $(P, \supseteq)$ consist of all finite 0–1-functions p such that

$$\mathrm{dom}(p) \subseteq (\{0, 1\} \times \omega \times \omega) \times \omega \tag{21.27}$$

Let G be a generic filter on P. For each n and each m, let

$$\begin{aligned} a_{nm} &= \{i \in \omega : p(0, n, m; i) = 1 \text{ for some } p \in G\} \\ b_{nm} &= \{i \in \omega : p(1, n, m; i) = 1 \text{ for some } p \in G\} \end{aligned} \tag{21.28}$$

Then we define a_n, b_n, and P_n for each n by (21.26) and (21.25). The sets a_{nm}, b_{nm}, a_n, b_n, P_n have obvious canonical names $\boldsymbol{a}_{nm}$, $\boldsymbol{b}_{nm}$, $\boldsymbol{a}_n$, $\boldsymbol{b}_n$, $\boldsymbol{P}_n$:

$$\boldsymbol{a}_{nm}(\check{i}) = \sum \{p : p(0, n, m; i) = 1\}$$

$$\boldsymbol{a}_n(\boldsymbol{a}_{nm}) = 1 \qquad (\text{all } m \in \omega)$$

etc. See Fig. 21.1.

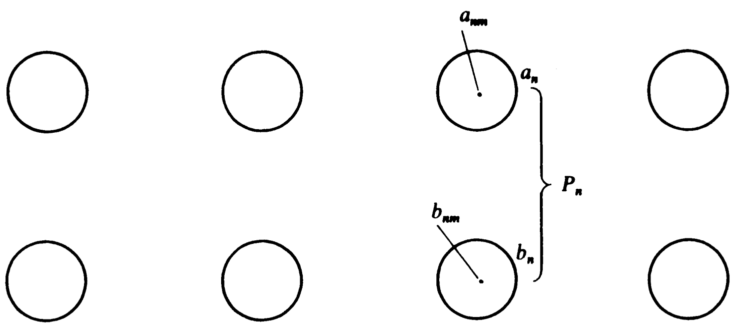

FIGURE 21.1

A familiar argument (see Lemma 19.11) shows that every condition $p \in P$ forces $\boldsymbol{a}_{nm} \neq \boldsymbol{b}_{kl}$ for any n, m, k, l, and $\boldsymbol{a}_{nm} \neq \boldsymbol{a}_{n'm'}$, $\boldsymbol{b}_{nm} \neq \boldsymbol{b}_{n'm'}$ whenever $(n, m) \neq (n', m')$. It follows that every condition forces

$$(21.29) \qquad \boldsymbol{a}_n \neq \boldsymbol{b}_m, \qquad \boldsymbol{a}_n \neq \boldsymbol{a}_{n'}, \qquad \boldsymbol{b}_n \neq \boldsymbol{b}_{n'} \qquad (\text{all } n, m, n \neq n')$$

To construct the symmetric model, we want to consider automorphisms π which have the property

$$(21.30) \qquad \pi(\boldsymbol{P}_n) = \boldsymbol{P}_n \qquad (\text{all } n \in \omega)$$

Thus we consider all permutations π of $\{0, 1\} \times \omega \times \omega$ such that if $\pi(\varepsilon, n, m) = (\varepsilon^*, n^*, m^*)$, then:

(21.31) (i) $n^* = n$ for all n;
(ii) for every n, either $\pi(\varepsilon, n, m) = (\varepsilon, n, m^*)$ for all m
or $\pi(\varepsilon, n, m) = (1 - \varepsilon, n, m^*)$ for all m.

Every permutation of this kind induces an automorphism of P (and in turn, an automorphism of r.o.P):

$$(21.32) \qquad (\pi p)(\pi(\varepsilon, n, m); i) = p(\varepsilon, n, m; i)$$

Let $\mathscr{G}$ be the group of all such automorphisms. Note that for every $\pi \in \mathscr{G}$, we have

$$(21.33) \qquad \begin{aligned} \pi(\boldsymbol{a}_{nm}) &= \boldsymbol{a}_{nm^*} \text{ or } \boldsymbol{b}_{nm^*} \\ \pi(\boldsymbol{a}_n) &= \boldsymbol{a}_n \text{ or } \boldsymbol{b}_n \\ \pi(\boldsymbol{P}_n) &= \boldsymbol{P}_n \end{aligned}$$

For every finite $E \subset \{0, 1\} \times \omega \times \omega$, let

$$(21.34) \qquad \operatorname{fix}(E) = \{\pi \in \mathscr{G} : \pi(\varepsilon, n, m) = (\varepsilon, n, m) \text{ for all } (\varepsilon, n, m) \in E\}$$

Let $\mathscr{F}$ be the filter on $\mathscr{G}$ generated by $\{\operatorname{fix}(E) : E \subset \{0, 1\} \times \omega \times \omega \text{ finite}\}$; $\mathscr{F}$ is a normal filter.

Now let HS be the class of all hereditarily symmetric names, and let $\mathfrak{N}$ be the symmetric submodel of $\mathfrak{M}[G]$. It is easy to see that all $\boldsymbol{a}_{nm}, \boldsymbol{b}_{nm}, \boldsymbol{a}_n, \boldsymbol{b}_n, \boldsymbol{P}_n$ are symmetric. Moreover, the set $\{\boldsymbol{P}_n : n \in \omega\}$ is countable in $\mathfrak{N}$.

We let $\boldsymbol{f}$ be the following name (for the function f: $n \mapsto P_n$)

$$(21.35) \qquad \operatorname{dom} \boldsymbol{f} = \{(\check{n}, \boldsymbol{P}_n)^B : n \in \omega\}, \boldsymbol{f}((\check{n}, \boldsymbol{P}_n)^B) = 1 \qquad (\text{all } n)$$

It follows from (21.30) that $\pi(\boldsymbol{f}) = \boldsymbol{f}$ for all $\pi \in \mathscr{G}$ and so $\boldsymbol{f} \in$ HS. Hence f: $n \mapsto P_n$ is in $\mathfrak{N}$.

The following lemma completes the proof of (21.24):

Lemma 21.5. *There is no function $f \in \mathfrak{N}$ such that $f(n) \in P_n$ for all $n \in \omega$.*

Proof. Assuming that there is such f, let $f \in \mathrm{HS}$ and let $p_0 \in G$ be such that

$$p_0 \Vdash \quad \text{for every } n \in \omega, f(n) \in \boldsymbol{P}_n$$

We let $E \subset \{0, 1\} \times \omega \times \omega$ be a support of f: $\mathrm{sym}(f) \supseteq \mathrm{fix}(E)$. Let n be such that $(\varepsilon, n, m) \notin E$ for any ε, m. There exists $p \le p_0$ such that (for instance)

$$p \Vdash f(\check{n}) = \boldsymbol{a}_n$$

We shall find $\pi \in \mathscr{G}$ such that:

$$\text{(21.36)} \qquad \begin{aligned} &\text{(i)} && \pi p \text{ and } p \text{ are compatible;} \\ &\text{(ii)} && \pi \in \mathrm{fix}(E); \\ &\text{(iii)} && \pi(\boldsymbol{a}_n) = \boldsymbol{b}_n. \end{aligned}$$

Then $\pi(f) = f$, $\pi(\check{n}) = \check{n}$, and we shall have

$$p \cup \pi p \Vdash f(\check{n}) = \boldsymbol{a}_n, \qquad p \cup \pi p \Vdash f(\check{n}) = \boldsymbol{b}_n$$

a contradiction.

To find π with properties (21.36), let m_0 be large enough so that $(\varepsilon, n, m, i) \notin \mathrm{dom}(p)$ for any $m \ge m_0$ and any ε, i. See Fig. 21.2.

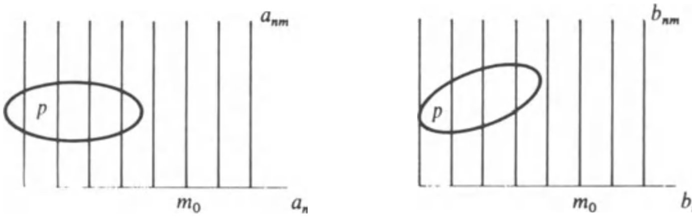

FIGURE 21.2

We let π be the permutation of $\{0, 1\} \times \omega \times \omega$ defined as follows:

$$\text{(21.37)} \qquad \pi(\varepsilon, n', m) = (\varepsilon, n', m) \qquad \text{if } n' \ne n$$

$$\pi(0, n, m) = \begin{cases} (1, n, m + m_0) & \text{if } m < m_0 \\ (1, n, m - m_0) & \text{if } m_0 \le m < 2m_0 \\ (1, n, m) & \text{if } m \ge 2m_0 \end{cases}$$

$$\pi(1, n, m) \ldots \text{ correspondingly}$$

The permutation π is in $\mathscr{G}$, and $\pi\boldsymbol{a}_n = \boldsymbol{b}_n$. Since n does not occur in E, we have $\pi \in \mathrm{fix}(E)$. It follows from (21.37) that p and πp are compatible. Thus π satisfies (21.36).

Exercise 21.5. The set $\bigcup_{n=0}^{\infty} P_n$ cannot be linearly ordered in the model $\mathfrak{N}$ of Example II.

On the other hand, Lévy has shown that every set can be linearly ordered in the symmetric model of Example I. In fact, Halpern and Lévy proved that the model even satisfies the prime ideal theorem, thus establishing the independence of the axiom of choice from the prime ideal theorem. We note that numerous consequences of the axiom of choice in mathematics can be proved using the prime ideal theorem instead—among others the Hahn–Banach theorem, compactification theorems, the completeness theorem, the Tychonoff theorem for Hausdorff spaces, etc.

The Jech–Sochor Embedding Theorem

The above examples suggest that there is a similarity between permutation models and the symmetric submodels of generic models. We shall present a general theorem which shows that any permutation model can be embedded in a symmetric model of ZF, "with a prescribed degree of accuracy."

Theorem 47 (*Jech–Sochor Embedding Theorem*). *Let $\mathfrak{V}$ be a permutation model, let A be its set of atoms, and let α be an ordinal (in $\mathfrak{V}$). There exists a symmetric model $\mathfrak{N}$ of* ZF *and an embedding $x \mapsto \tilde{x}$ of $\mathfrak{V}$ in $\mathfrak{N}$ such that*

$$(P^\alpha(A))^{\mathfrak{V}} \text{ is } \in\text{-isomorphic to } (P^\alpha(\tilde{A}))^{\mathfrak{N}}.$$

Proof. See Fig. 21.3. We work in the set theory with atoms, ZFA, plus the axiom of choice. We denote by A the set of all atoms, and let $\mathfrak{M}$ be the kernel, $\mathfrak{M} = P^\infty(\emptyset)$. We consider a group $\mathcal{G}$ of permutations of A, and a normal filter $\mathcal{F}$ on $\mathcal{G}$, and let $\mathfrak{V}$ be the permutation model given by $\mathcal{G}$ and $\mathcal{F}$. Let α be an ordinal number.

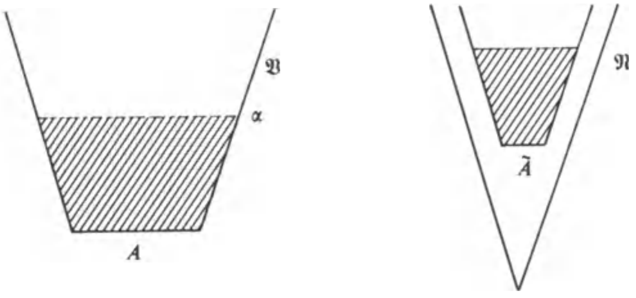

FIGURE 21.3

We shall construct a generic extension $\mathfrak{M}[G]$ of the kernel, and then the model $\mathfrak{N}$ as a symmetric submodel of $\mathfrak{M}[G]$. We construct $\mathfrak{M}[G]$ by adjoining to $\mathfrak{M}$ a number of subsets of a regular cardinal κ, κ of them for each $a \in A$. We use these to embed $\mathfrak{V}$ in $\mathfrak{M}[G]$.

Let κ be a regular cardinal such that $\kappa > |P^\alpha(A)|$. The set P of forcing conditions consists of functions p with values 0, 1 such that $|\mathrm{dom}(p)| < \kappa$ and $\mathrm{dom}(p) \subseteq (A \times \kappa) \times \kappa$; as usual, $p < q$ iff $p \supset q$.

Let G be an $\mathfrak{M}$-generic filter on P. For each $a \in A$ and each $\xi < \kappa$, we let

$$x_{a\xi} = \{\eta \in \kappa : p(a, \xi; \eta) = 1 \text{ for some } p \in G\} \tag{21.38}$$

Each $x_{a\xi}$ has a canonical name $\boldsymbol{x}_{a\xi}$:

$$\boldsymbol{x}_{a\xi}(\check{\eta}) = \sum \{p \in P : p(a, \xi; \eta) = 1\} \qquad (\eta \in \kappa) \tag{21.39}$$

Then we define, for every $a \in A$,

$$\tilde{a} = \{x_{a\xi} : \xi < \kappa\} \tag{21.40}$$

and let $\tilde{A} = \{\tilde{a} : a \in A\}$ (see Fig. 21.4). The sets $\tilde{a}$ and $\tilde{A}$ have obvious canonical names.

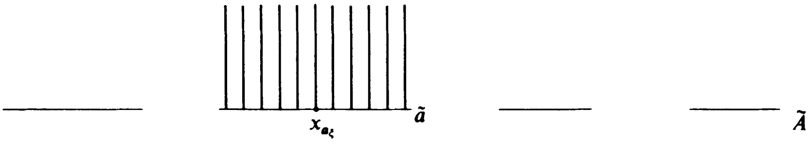

FIGURE 21.4

Having defined $\tilde{a}$ for each $a \in A$, we can define $\tilde{x}$ (and its canonical name $\tilde{\boldsymbol{x}}$) for each x by $\in$-induction:

$$\tilde{x} = \{\tilde{y} : y \in x\}, \qquad \tilde{\boldsymbol{x}}(\tilde{\boldsymbol{y}}) = 1 \qquad (\text{all } y \in x) \tag{21.41}$$

We shall show that the function $x \mapsto \tilde{x}$ is an $\in$-isomorphism.

Lemma 21.6. *For all* x, y,

$$x \in y \ \textit{ iff } \ \tilde{x} \in \tilde{y}, \qquad x = y \ \textit{ iff } \ \tilde{x} = \tilde{y}$$

Proof. First we note (cf. Lemma 19.11) that $\|\boldsymbol{x}_{a\xi} = \boldsymbol{x}_{a'\xi'}\| = 0$ whenever $(a, \xi) \neq (a', \xi')$, and that $\|\boldsymbol{x}_{a\xi} = \check{z}\| = 0$ for all $z \in \mathfrak{M}$. Consequently, we have $\tilde{a} \neq \tilde{b}$ whenever $a \neq b$ are atoms. We claim that for all $x, \tilde{x} \neq x_{a\xi}$ for any a, ξ. If $x \in \mathfrak{M}$, then $\tilde{x} = x$ and so $\tilde{x} \neq x_{a\xi}$. If $x \notin \mathfrak{M}$, then $\tilde{x}$ is of higher rank than any $x_{a\xi}$: $x_{a\xi}$ is a subset of κ, while the transitive closure of $\tilde{x}$ contains some of the $x_{a\xi}$.

Now we can prove the lemma, simultaneously for $\in$ and $=$, by induction on rank:

(a) If $x \in y$, then $\tilde{x} \in \tilde{y}$ follows from the definition (21.41). If $\tilde{x} \in \tilde{y}$, then y cannot be an atom because then we would have $\tilde{x} = x_{a\xi}$ for some a, ξ, which is impossible. Hence $\tilde{x} = \tilde{z}$ for some $z \in y$ and we have $x = z$ by the induction hypothesis; thus $x \in y$.

(b) If $x = y$, then $\tilde{x} = \tilde{y}$. Conversely, if $x \neq y$, then either both x and y are atoms and then $\tilde{x} \neq \tilde{y}$; or, e.g., x contains some z that is not in y, and then, by the induction hypothesis, $\tilde{z} \in \tilde{x}$ and $\tilde{z} \notin \tilde{y}$; thus $\tilde{x} \neq \tilde{y}$. ■

Note that the proof of Lemma 21.6 does not depend on the particular G and so in fact we have proved

$$(21.42) \qquad x = y \quad \text{iff} \quad \|\tilde{x} = \tilde{y}\| \neq 0 \quad \text{iff} \quad \|\tilde{x} = \tilde{y}\| = 1$$

and similarly for $\in$.

Now we shall construct a symmetric submodel $\mathfrak{N}$ of $\mathfrak{M}[G]$. We construct $\mathfrak{N}$ so that for every $x \in \mathfrak{V}$, $\tilde{x}$ is in $\mathfrak{N}$ and that $(P^\alpha(A))^{\mathfrak{V}}$ is isomorphic to $(P^\alpha(\tilde{A}))^{\mathfrak{N}}$. For every permutation σ of A, let $\bar{\sigma}$ be the group of all permutations π of $A \times \kappa$ such that for all a, ξ,

$$(21.43) \qquad \pi(a, \xi) = (\sigma a, \xi') \quad \text{for some } \xi'$$

(π permutes the blocks in Fig. 21.4 as prescribed by σ). We let

$$(21.44) \qquad \bar{H} = \{\bar{\sigma} : \sigma \in H\}$$

for every subgroup H of $\mathcal{G}$. Since every permutation π of $A \times \kappa$ induces an automorphism of P (and hence of r.o.P) via

$$(\pi p)(\pi(a, \xi), \eta) = p(a, \xi, \eta) \qquad (\text{all } a, \xi, \eta)$$

we consider $\bar{\mathcal{G}}$ as a group of automorphisms of $B = \text{r.o.}P$. For every finite $E \subseteq A \times \kappa$ we let

$$(21.45) \qquad \text{fix}(E) = \{\pi \in \bar{\mathcal{G}} : \pi(a, \xi) = (a, \xi) \text{ for all } (a, \xi) \in E\}$$

and we let $\bar{\mathcal{F}}$ be the filter on $\bar{\mathcal{G}}$ generated by the set

$$(21.46) \qquad \{\bar{H} : H \in \mathcal{F}\} \cup \{\text{fix}(E) : E \subseteq A \times \kappa \text{ finite}\}$$

It is easy to verify that $\bar{\mathcal{F}}$ is a normal filter.

Let HS be the class of all hereditarily symmetric names and let $\mathfrak{N}$ be the corresponding symmetric submodel of $\mathfrak{M}[G]$. It is an immediate consequence of the definition of $\bar{\mathcal{F}}$ (21.46) that all $\boldsymbol{x}_{a\xi}$, all $\tilde{\boldsymbol{a}}$ ($a \in A$), and $\tilde{\boldsymbol{A}}$ are symmetric, and so $\tilde{A}$ is in $\mathfrak{N}$. The following two lemmas show that for any x, $\tilde{x}$ is in $\mathfrak{N}$ if and only if $x \in \mathfrak{V}$.

Lemma 21.7. *For all x,*

$$x \in \mathfrak{V} \qquad \textit{iff} \qquad \tilde{x} \in \text{HS}$$

Proof. It suffices to show that $\text{sym}_{\mathcal{G}}(x) \in \mathcal{F}$ iff $\text{sym}_{\bar{\mathcal{G}}}(\tilde{x}) \in \bar{\mathcal{F}}$. If $\sigma \in \mathcal{G}$ and $\pi \in \bar{\sigma}$, then $\pi\tilde{x}$ is the canonical name for $(\sigma x)^\sim$, and so $\text{sym}_{\bar{\mathcal{G}}}(\tilde{x}) = \overline{\text{sym}_{\mathcal{G}}(x)}$; thus if $\text{sym}_{\mathcal{G}}(x) \in \mathcal{F}$, then $\text{sym}_{\bar{\mathcal{G}}}(\tilde{x}) \in \bar{\mathcal{F}}$. On the other hand, if $\text{sym}_{\bar{\mathcal{G}}}(\tilde{x}) \in \bar{\mathcal{F}}$, then $\overline{\text{sym}_{\mathcal{G}}(x)} \supseteq \bar{H} \cap \text{fix}(E)$ for some $H \in \mathcal{F}$ and a finite $E \subseteq A \times \kappa$. If $e = \{a \in A : (a, \xi) \subseteq E$ for some $\xi\}$, then $\text{sym}_{\bar{\mathcal{G}}}(x) \supseteq H \cap \text{fix}(e)$, and since $\text{fix}(e) \in \mathcal{F}$ by (21.7), we have $\text{sym}_{\mathcal{G}}(x) \in \mathcal{F}$.

Lemma 21.8. *For all* x,

$$x \in \mathfrak{V} \quad \textit{iff} \quad \tilde{x} \in \mathfrak{N}$$

Proof. By Lemma 21.7, it suffices to show that if $\tilde{x} \in \mathfrak{N}$, then $x \in \mathfrak{V}$. Assume otherwise, and let x be of least rank such that $\tilde{x} \in \mathfrak{N}$ and $x \notin \mathfrak{V}$. Thus $x \subset \mathfrak{V}$, and since $\tilde{x} \in \mathfrak{N}$, there exists $z \in \mathrm{HS}$ and $p \in G$ such that $p \Vdash z = \tilde{x}$. Since $\mathrm{sym}_{\bar{\mathscr{G}}}(z) \in \bar{\mathscr{F}}$, we have $\mathrm{sym}(z) \supseteq \bar{H} \cap \mathrm{fix}(E)$ for some $H \in \mathscr{F}$, and a finite $E \subseteq A \times \kappa$. We shall find $\sigma \in \mathscr{G}$ and $\pi \in \bar{\sigma}$ such that:

(21.47) (i) πp and p are compatible;
(ii) $\pi \in \bar{H} \cap \mathrm{fix}(E)$;
(iii) $\sigma x \neq x$.

Then we shall have $\pi z = z$ by (ii), $\|\pi\tilde{x} = \tilde{x}\| = 0$ by (iii) and (21.42); and since $\pi p \Vdash \pi z = \pi\tilde{x}$, we shall have

$$\pi p \cup p \Vdash z = \tilde{x} \qquad \pi p \cup p \Vdash z = \pi\tilde{x} \tag{21.48}$$

a contradiction.

To find π, note that x is not symmetric, so that there is $\sigma \in \mathscr{G}$ such that $\sigma x \neq x$ and $\sigma \in H \cap \mathrm{fix}(e)$, where $e = \{a \in A : (a, \xi) \in E \text{ for some } \xi\}$. Since $|p| < \kappa$, there exists $\gamma < \kappa$ such that $(a, \xi) \notin \mathrm{dom}(p)$ for any $a \in A$ and $\xi > \gamma$. Thus we define $\pi \in \bar{\sigma}$ as follows:

$$\begin{array}{lll} \text{if } a \in e, \text{ then } \pi(a, \xi) = (a, \xi) & & (\text{all } \xi) \\ \text{if } a \notin e, \text{ then } \pi(a, \xi) = (\sigma a, \gamma + \xi) & \multirow{2}{*}{$\left.\begin{array}{l}\\ \\ \end{array}\right\}$} & \text{if } \xi < \gamma \\ \qquad\qquad \pi(a, \gamma + \xi) = (\sigma a, \xi) & & \\ \qquad\qquad \pi(a, \xi) = (\sigma a, \xi) & & \text{if } \xi > \gamma \cdot 2 \end{array} \tag{21.49}$$

It follows that $\pi \in \bar{H} \cap \mathrm{fix}(E)$ and that p and πp are compatible. ■

We complete the proof of Theorem 47 by showing that

$$((P^\alpha(A))^{\mathfrak{A}})^\sim = (P^\alpha(\tilde{A}))^{\mathfrak{N}}$$

The left-hand side is clearly included in the right-hand side; we prove the converse by induction. Thus let $x \in P^\alpha(A) \cap \mathfrak{V}$ and let $y \in \mathfrak{N}$ be a subset of x; we shall show that $y = \tilde{z}$ for some $z \in \mathfrak{V}$. Let y be a name for y. The notion of forcing that we are using here is κ-closed; and since we have chosen κ large, it follows (see Lemmas 19.5 and 19.6) that there is $p \in G$ that decides $\tilde{t} \in y$ for all $t \in x$. Hence $y = \tilde{z}$, where $z = \{t \in x : p \Vdash \tilde{t} \in y\}$, and by Lemma 21.8 we have $z \in \mathfrak{V}$. ■

As for applications of Theorem 47, consider a formula $\varphi(X, \gamma)$ such that the only quantifiers in φ are $\exists u \in P^\gamma(X)$ and $\forall u \in P^\gamma(X)$. Let $\mathfrak{V}$ be a permutation model such that

$$\mathfrak{V} \models \exists X\, \varphi(X, \gamma) \tag{21.50}$$

Let $X \in \mathfrak{V}$ be such that $\mathfrak{V} \vDash \varphi(X, y)$; let α be such that (in $\mathfrak{V}$) $P^{\gamma}(X) \subseteq P^{\alpha}(A)$. By the embedding theorem, $\mathfrak{V}$ can be embedded in a model $\mathfrak{N}$ of ZF such that $(P^{\alpha}(A))$ is isomorphic to $(P^{\alpha}(\tilde{A}))^{\mathfrak{N}}$. Since the quantifiers in φ are restricted to $P^{\gamma}(X)$, it follows that $\mathfrak{N} \vDash \varphi(\tilde{X}, y)$, and so

$$\mathfrak{N} \vDash \exists X\ \varphi(X, y) \tag{21.51}$$

Therefore, if we wish to prove consistency (with ZF) of an existential statement of the kind just described, it suffices to construct a permutation model (of ZFA).

Note that " X cannot be well-ordered," " X cannot be linearly ordered " are formulas of the above type, and so is " X is a countable set of pairs without a choice function."

Exercise 21.6. Show that the set A of atoms in the first example of a permutation model has the following properties, and by Theorem 47 it is consistent with ZF that such A exists:

(i) A is infinite;
(ii) A is not the union of two disjoint infinite sets;
(iii) $P(A)$ is D-finite.

The embedding theorem, in conjunction with the construction of permutation models, has interesting applications in algebra. One can construct various abstract counterexamples to theorems whose proofs use the axiom of choice. For example, one can construct a vector space that has no basis, etc.

We shall give one example of an application of Theorem 47: Without the axiom of choice, the partial ordering by cardinality is quite arbitrary:

Example III. Let $\mathfrak{M}$ be a transitive model of ZFC and let $(I, \precsim)$ be a partially ordered set in $\mathfrak{M}$. There exists a model $\mathfrak{N} \supseteq \mathfrak{M}$ of ZFC that contains a family of sets $\{S_i : i \in I\}$ such that

$$i \precsim j \quad \text{iff} \quad |S_i| \le |S_j| \tag{21.52}$$

for all $i, j \in I$.

It suffices to construct a permutation model $\mathfrak{V}$ with kernel $\mathfrak{M}$ such that (21.52) holds in $\mathfrak{V}$ for some $\{S_i : i \in I\}$. Then one can embed $\mathfrak{V}$ in a model $\mathfrak{N}$ of ZF such that $\mathfrak{N}$ will satisfy (21.52).

Since $(I, \precsim)$ is embedded in $(P(I), \subseteq)$ by the mapping $i \mapsto \{j \in I : j \precsim i\}$, it suffices to find in $\mathfrak{V}$ a family $\{S_p : p \subseteq I\}$ such that

$$p \subseteq q \quad \text{iff} \quad \mathfrak{V} \vDash |S_p| \le |S_q| \tag{21.53}$$

Let A be the set of all atoms; let us assume that $|A| = |I| \cdot \aleph_0$ and let $\{a_{in} : i \in I, n \in \omega\}$ be an enumeration of A. For each $p \subseteq I$, let $S_p = \{a_{in} : i \in p, n \in \omega\}$. We shall construct a permutation model $\mathfrak{V}$ such that the mapping $h\colon p \mapsto S_p$ is in $\mathfrak{V}$, and that (21.53) holds.

Let $\mathscr{G}$ be the group of all permutations π of A such that $\pi(S_{\{i\}}) = S_{\{i\}}$ for every $i \in I$; i.e., $\pi(a_{in}) = a_{im}$ for some m. Let $\mathscr{F}$ be the filter on $\mathscr{G}$ generated by $\{\text{fix}(E) : E \subseteq A \text{ finite}\}$, and let $\mathfrak{B}$ be the permutation model.

Clearly, each S_p ($p \subseteq I$) as well as the function h: $p \mapsto S_p$ are preserved by all permutations $\pi \in \mathscr{G}$ and thus belong to $\mathfrak{B}$. Also, if $p \subseteq q$, then $S_p \subseteq S_q$ and so $|S_p| \leq |S_q|$; it remains to show that if $p \nsubseteq q$, then $\mathfrak{B} \vDash |S_p| \nleq |S_q|$.

Let $p \nsubseteq q$, let $i \in p - q$ and assume that $g \in \mathfrak{B}$ is a one-to-one function of S_p into S_q; let E be a finite support of g. Let n, m be such that $a_{in}, a_{im} \notin E$ and let π be a permutation of A such that $\pi(a_{in}) = a_{im}$, $\pi(a_{im}) = a_{in}$, and $\pi(a) = a$ otherwise. Clearly $\pi \in \text{fix}(E)$ and so $\pi(g) = g$. Since $i \notin q$, we have $g(a_{in}) = a_{jk}$ where $j \neq i$; hence $\pi(a_{jk}) = a_{jk}$, and we have

$$g(a_{in}) = a_{jk} = \pi(a_{jk}) = \pi(g(a_{in})) = (\pi g)(\pi a_{in}) = g(a_{im})$$

Hence g is not one-to-one, a contradiction. ■

Two More Examples of Symmetric Models

We conclude this section by sketching two examples of models of ZF in which the axiom of choice fails. The first model was constructed by Feferman and Lévy, the other by Feferman.

Example IV. Let $\mathfrak{M}$ be a transitive model of ZFC. There is a model $\mathfrak{N} \supseteq \mathfrak{M}$ such that $(\aleph_1)^{\mathfrak{N}} = (\aleph_\omega)^{\mathfrak{M}}$; hence $\aleph_1$ is singular in $\mathfrak{N}$.

First we construct a generic extension $\mathfrak{M}[G]$ by adjoining collapsing maps f_n: $\omega \to \omega_n$, for all $n \in \omega$: We let $(P, \supseteq)$ consist of finite functions with domain $\subseteq \omega \times \omega$, such that $p(n, i) < \omega_n$ for all $(n, i) \in \text{dom}(p)$. If G is a generic filter on P, then $f = \bigcup G$ is a function on $\omega \times \omega$, and for every n, the function f_n defined on ω by $f_n(i) = f(n, i)$ maps ω onto ω_n. We shall construct a symmetric model $\mathfrak{N} \subset \mathfrak{M}[G]$ such that each f_n is in $\mathfrak{N}$ but $\aleph_\omega$ is a cardinal in $\mathfrak{N}$.

Let $\mathscr{G}$ be the group of all permutations π of $\omega \times \omega$ such that for every n, $\pi(n, i) = (n, j)$, for some j. Every π induces an automorphism of P via

$$\text{(21.54)} \quad \text{dom}(\pi p) = \{\pi(n, i) : (n, i) \in \text{dom } p\}, \qquad (\pi p)(\pi(n, i)) = p(\pi(n, i))$$

Let $\mathscr{F}$ be the (normal) filter on $\mathscr{G}$ generated by $\{H_n : n \in \omega\}$, where H_n consists of all π such that $\pi(k, i) = (k, i)$ for all $k \leq n$, all $i \in \omega$. Let HS be the class of all hereditarily symmetric names and let $\mathfrak{N}$ be the symmetric model.

It is easy to verify that for each n, the canonical name f_n of f_n is symmetric and so $f_n \in \mathfrak{N}$. To show that $\aleph_\omega$ remains a cardinal in $\mathfrak{N}$, we shall use the following lemma:

Lemma 21.9. *If* $\text{sym}(x) \supseteq H_n$ *and* $p \Vdash \varphi(x)$, *then* $p{:}n \Vdash \varphi(x)$, *where* $p{:}n$ *is the restriction of* p *to* $\{(k, i) : k \leq n\}$.

Proof. Let us assume that $p{:}n$ does not force $\varphi(x)$ and let $q \supseteq p{:}n$ be such that $q \Vdash \neg\varphi(x)$. It is easy to find $\pi \in H_n$ such that πp and q are compatible; since $\pi p \Vdash \varphi(\pi x)$, and $\pi x = x$, we get a contradiction. ■

Now let us assume that $g \in \mathfrak{N}$ is a function of ω onto $\aleph_\omega$, and let $\boldsymbol{g}$ be a symmetric name for g. Let $p_0 \in G$ be such that p_0 forces "$\boldsymbol{g}$ is a function from $\check{\omega}$ onto $\check{\aleph}_\omega$." Let n be such that $p_0 : n = p_0$ and that $\operatorname{sym}(\boldsymbol{g}) \supseteq H_n$. Since g takes $\aleph_\omega$ values, it follows that for some $k \in \omega$, there exists an incompatible set W of conditions $p \supseteq p_0$ such that $|W| \geq \aleph_{n+1}$, and distinct ordinals α_p, $p \in W$, such that for each $p \in W$, $p \Vdash \boldsymbol{g}(\check{k}) = \check{\alpha}_p$. By Lemma 21.9, we have $p : n \Vdash \boldsymbol{g}(\check{k}) = \check{\alpha}_p$, for each $p \in W$, which is a contradiction: On the one hand, the conditions $p : n$, $p \in W$, must be mutually incompatible, and on the other hand, the set $\{(p : n) : p \in P\}$ has size only $\aleph_n$. ■

If the ground model $\mathfrak{M}$ in the above example satisfies GCH, then one can show that in $\mathfrak{N}$, the set of all reals is the countable union of countable sets.

To show that a successor cardinal $\aleph_{\alpha+1}$ is regular, one uses the axiom of choice. It is an open problem to construct a model of ZF in which *every* infinite cardinal has cofinality ω. Somewhat surprisingly, this problem is related to large cardinal axioms: It follows from a recent result of Jensen that if both $\aleph_1$ and $\aleph_2$ are singular, then $0^\#$ exists (see Exercise 30.7).

Example V. Let $\mathfrak{M}$ be a transitive model of ZFC. There is a model $\mathfrak{N} \supseteq \mathfrak{M}$ such that in $\mathfrak{N}$, there is no nonprincipal ultrafilter over ω.

The model $\mathfrak{N}$ is obtained by adjoining to $\mathfrak{M}$ infinitely many generic reals a_n, $n < \omega$, without putting in $\mathfrak{N}$ the set $\{a_n : n \in \omega\}$ (unlike in Example I where $\{a_n : n \in \omega\}$ is in $\mathfrak{N}$). First we construct $\mathfrak{M}[G]$ as in Example I: $(P, \supseteq)$ is the set of all finite 0–1-functions with domain $\subseteq \omega \times \omega$. Let G be generic and let $a_n = \{m : p(n, m) = 1 \text{ for some } p \in G\}$, for each $n \in \omega$.

Now let $\mathfrak{N}$ be as follows. Every $X \subseteq \omega \times \omega$ induces a *symmetry* σ_X, an automorphism of P defined by

$$(\sigma_X p)(n, m) = \begin{cases} p(n, m) & \text{if } (n, m) \notin X \\ 1 - p(n, m) & \text{if } (n, m) \in X \end{cases} \tag{21.55}$$

Let $\mathscr{G}$ be the group of all σ_X, $X \subseteq \omega \times \omega$, and let $\mathscr{F}$ be the normal filter on $\mathscr{G}$ generated by $\{\operatorname{fix}(E) : E \subset \omega \text{ finite}\}$, where $\operatorname{fix}(E) = \{\sigma_X : X \cap (E \times \omega) = \emptyset\}$. Let $\mathfrak{N}$ be the symmetric model.

Let $D \in \mathfrak{N}$ be an ultrafilter over ω; we shall show that D is principal. Let $\boldsymbol{D} \in \mathrm{HS}$ be a name for D and let $p \in G$ be such that p forces "$\boldsymbol{D}$ is an ultrafilter over $\check{\omega}$." Let $E \subset \omega$ be finite, such that $\operatorname{sym}(\boldsymbol{D}) \supseteq \operatorname{fix}(E)$, and let $n \notin E$. Then there is $q \leq p$, $q \in G$, which decides $\boldsymbol{a}_n \in \boldsymbol{D}$ (where $\boldsymbol{a}_n$ is the canonical name for a_n). For example, assume that $q \Vdash \boldsymbol{a}_n \in \boldsymbol{D}$ (the proof is similar if $q \Vdash \boldsymbol{a}_n \notin \boldsymbol{D}$).

Let m_0 be such that for all $m \geq m_0$, $(n, m) \notin \operatorname{dom}(q)$, and let $X = \{(n, m) : m \geq m_0\}$. Let $\boldsymbol{b}_n = \sigma_X(\boldsymbol{a}_n)$, and $b_n = i_G(\boldsymbol{b}_n)$. Since for each $m \geq m_0$, $\|\check{m} \in \boldsymbol{b}_n\| = -\|\check{m} \in \boldsymbol{a}_n\|$, it follows that $a_n \cap b_n$ is a finite set. However, $\sigma_X q \Vdash \sigma_X \boldsymbol{a}_n \in \sigma_X \boldsymbol{D}$; it is fairly obvious that $\sigma_X q = q$ and since $\sigma_X \in \operatorname{fix}(E)$, we have $\sigma_X \boldsymbol{D} = \boldsymbol{D}$. Thus $q \Vdash \boldsymbol{b}_n \in \boldsymbol{D}$ and hence $a_n \cap b_n \in D$. Consequently, D is principal. ■

A modification of the construction in Example V gives a model $\mathfrak{N}$ in which all ultrafilters are principal.

It can be shown that Feferman's model (Example V) and Cohen's model (Example I), although different, have the same sets of ordinals. Compare this fact with Lemma 15.1.

Finally, let me mention the following open problem. We have used the axiom of choice to show that there exist disjoint stationary subsets of ω_1 (see Lemma 7.6). Is there a model of ZF which satisfies the following?

(21.56) The closed unbounded filter over $\aleph_1$ is an ultrafilter.

It is known that (21.56) implies (assuming the countable axiom of choice) that there exists a model of ZFC that has a measurable cardinal; thus this problem is related to large cardinal axioms. The problem is particularly interesting because (21.56) is a consequence of the axiom of determinacy.

CHAPTER 4

SOME APPLICATIONS OF FORCING

22. SUSLIN'S PROBLEM

The real line is, up to isomorphism, the unique linearly ordered set that is dense, unbounded, complete, and separable. In 1920 Suslin raised the question as to whether "separable" can be replaced by a weaker condition: Each collection of disjoint open intervals is at most countable (the *countable chain condition*, c.c.c.).

Suslin's Problem. *Let S be a linearly ordered set such that:*

(i) *S is dense and unbounded;*
(ii) *S is complete;*
(iii) *S satisfies the countable chain condition.*

Is S isomorphic to the real line?

This problem cannot be decided in the axiomatic set theory. In this and the following section, we shall show that Suslin's problem is independent of the axioms of ZFC.

A simple argument using Dedekind completions and embedding of countable ordered sets in the rationals shows that Suslin's problem can be formulated as follows (see Section 4): *Is there a dense linearly ordered set S that satisfies the countable chain condition and is not separable?*

We shall call such an ordered set S a *Suslin line*. Upon completion and removal of endpoints if necessary, a Suslin line provides a counterexample to the problem of Suslin. We shall therefore investigate whether Suslin lines exist.

The first step toward solution of Suslin's problem is its reformulation in terms of trees. Trees (which can be considered a generalization of ordinal numbers) are often useful in point set topology and infinitary combinatorics and make an interesting object of investigation in combinatorial set theory.

A *tree* is a partially ordered set $(T, <)$ with the property that for each $x \in T$, the set $\{y : y < x\}$ of all predecessors of x is well-ordered by $<$. See Fig.

22.1. The αth *level* of T consists of all $x \in T$ such that $\{y : y < x\}$ has order-type α. The *height* of T is the least α such that the αth level of T is empty; in other words, it is the length of the well-founded relation $<$:

$$\text{(22.1)} \qquad \begin{aligned} \mathrm{o}(x) &= \text{order-type of } \{y : y < x\} \\ \alpha\text{th level} &= \{x : \mathrm{o}(x) = \alpha\} \\ \text{height}(T) &= \sup\{\mathrm{o}(x) + 1 : x \in T\} \end{aligned}$$

A *branch* in T is a maximal linearly ordered subset of T. The *length* of a branch b is the order-type of b. An *α-branch* is a branch of length α.

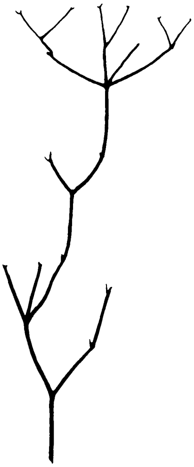

FIGURE 22.1 Example of a (finite) tree.

Exercise 22.1. If T is a tree of height ω such that each level of T is finite, then T has an infinite branch.

[To construct a branch $\{x_0, x_1, \ldots, x_n, \ldots\}$ in T, pick x_0 at level 0 such that $\{y : y > x_0\}$ is infinite. Then pick $x_1, x_2, \ldots$ similarly.]

An *antichain* in T is a set $A \subseteq T$ such that any two distinct elements x, y of A are incomparable, i.e., neither $x < y$ nor $y < x$. Let us call a tree T a *Suslin tree* if:

(22.2) (i) the height of T is ω_1;
(ii) every branch in T is at most countable;
(iii) every antichain in T is at most countable.

Lemma 22.1. *There exists a Suslin line if and only if there exists a Suslin tree.*

Proof. (a) Let S be a Suslin line. We shall construct a Suslin tree. The tree will consist of closed (nondegenerate) intervals on the Suslin line S. The partial ordering of T is by inverse inclusion: If $I, J \in T$, then $I \leq J$ if and only if $I \supseteq J$.

The collection T of intervals is constructed by induction on $\alpha < \omega_1$. We let $I_0 = [a_0, b_0]$ be arbitrary (such that $a_0 < b_0$). Having constructed I_β, $\beta < \alpha$, we consider the countable set $C = \{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$ of endpoints of the intervals I_β, $\beta < \alpha$. Since S is a Suslin line, C is not dense in S and so there exists an interval $[a, b]$ disjoint from C; we pick some such $[a_\alpha, b_\alpha] = I_\alpha$. The set $T = \{I_\alpha : \alpha < \omega_1\}$ is uncountable and partially ordered by $\supseteq$. If $\alpha < \beta$, then either $I_\alpha \supset I_\beta$ or I_α is disjoint from I_β. It follows that for each α, $\{I \in T : I \supset I_\alpha\}$ is well-ordered by $\supseteq$ and thus T is a tree.

We shall show that T has no uncountable branches and no uncountable antichains. Then it is immediate that the height of T is at most ω_1; and since every level is an antichain and T is uncountable, we have height$(T) = \omega_1$.

If $I, J \in T$ are incomparable, then they are disjoint intervals of S; and since S satisfies the countable chain condition, every antichain in T is at most countable. To show that T has no uncountable branch, we note first that if b is a branch of length ω_1, then the left endpoints of the intervals $I \in b$ form an increasing sequence $\{x_\alpha : \alpha < \omega_1\}$ of points of S. It is clear that the intervals $(x_\alpha, x_{\alpha+1})$, $\alpha < \omega_1$, form a disjoint uncountable collection of open intervals in S, contrary to the assumption that S satisfies the c.c.c.

(b) We shall now assume that there exists a Suslin tree and construct a Suslin line. The idea is to let S consist of branches of the tree. However, since the set S so constructed might not generally be dense, we shall first normalize the tree.

Let α be an ordinal number, $\alpha \leq \omega_1$. A *normal α-tree* is a tree T with the following properties:

(22.3) (i) height$(T) = \alpha$;
(ii) T has a unique least point (the root);
(iii) each level of T is at most countable;
(iv) if x is not maximal in T, then there are infinitely many $y > x$ at the next level (immediate successors of x);
(v) for each $x \in T$ there is some $y > x$ at each greater level (less than α);
(vi) if $\beta < \alpha$ is a limit ordinal and x, y are both at level β and if $\{z : z < x\} = \{z : z < y\}$, then $x = y$.

See the following exercise for justification of conditions (ii) and (vi):

Exercise **22.2.** If T is a normal α-tree, then T is isomorphic to a tree $\bar{T}$ whose elements are β-sequences ($\beta < \alpha$), ordered by extension; if $t \subseteq s$ and $s \in \bar{T}$, then $t \in T$, and the βth level of $\bar{T}$ is the set $\{t \in \bar{T} : \text{dom}(t) = \beta\}$.

A *normal Suslin tree* is a Suslin tree that satisfies the normality conditions (22.3). We shall show that given a Suslin tree, we can obtain a normal Suslin tree. Let T be a Suslin tree. T has height ω_1, and each level of T is countable. We first discard all points $x \in T$ such that $T_x = \{y \in T : y \geq x\}$ is at most countable, and let $T_1 = \{x \in T : T_x \text{ is uncountable}\}$. Note that if $x \in T_1$ and $\alpha > o(x)$, then $|T_y| = \aleph_1$ for some $y > x$ at level α. Hence T_1 satisfies condition (v). Next, we consider only branching points of the tree, i.e., points that have at least two immediate successors. For each $x \in T_1$, there are uncountably many branching points $z > x$ in T_1 (otherwise, all branching points $z > x$ would be below a certain level and then there would exist an uncountable branch). Thus $T_2 = \{\text{the branching points of } T_1\}$ is a Suslin tree with properties (i), (iii), (v), and (iv*), where (iv*) stands for: each x has at least two immediate successors. Next we satisfy property (vi): We consider all chains C in T_2 with the property that $C = \{z : z < x\} = \{z : z < y\}$ for some $x \neq y$. For every such C we add to T_2 an extra node a_C and stipulate that $z < a_C$ for all $z \in C$, and $a_C < x$ for each x such that $x > z$ for all $z \in C$. The resulting tree T_3 satisfies (i), (iii), (iv*), (v), and (vi). To get property (iv), let T_4 consist of all $x \in T_3$ at limit levels of T_3. The tree T_4 satisfies (i), (iii), (iv), and (v); and then $T_5 \subset T_4$ satisfying (ii) as well is easily obtained.

Now we can finish the proof of Lemma 22.1. Let T be a normal Suslin tree. The line S will consist of all branches of T (which are all countable). Each $x \in T$ has countably many immediate successors, and we order these successors as rational numbers. Then we order the elements of S lexicographically: If α is the least level where two branches $a, b \in S$ differ, then α is a successor ordinal and the points $a_\alpha \in a$ and $b_\alpha \in b$ at level α are both successors of the same point at level $\alpha - 1$; we let $a < b$ or $b < a$ according to whether $a_\alpha < b_\alpha$ or $b_\alpha < a_\alpha$.

It is easy to see that S is linearly ordered and dense. If (a, b) is an interval in S, then one can find $x \in T$ such that $I_x \subset (a, b)$, where I_x is the interval $I_x = \{c \in S : x \in c\}$. And if I_x and I_y are disjoint, then x and y are incomparable points of T. Thus every disjoint collection of open intervals of S must be at most countable, and so S satisfies the countable chain condition.

The line S is not separable: If C is a countable set of branches of T, let α be a countable ordinal bigger than the length of any branch $b \in C$. Then if x is any point at level α, the interval I_x does not contain any $b \in C$, and so C is not dense in S. ■

Before going further, let me make the following observation: If T is a normal ω_1-tree and if T has a branch of length ω_1, then T has an uncountable antichain. For, if b is an uncountable branch, pick for each $x \in b$ a successor z_x of x such that $z_x \notin b$. Then $A = \{z_x : x \in b\}$ is an antichain. Thus in order that a normal ω_1-tree T be Suslin, it suffices that T has no uncountable antichain.

Suslin lines have often been used in point set topology, mostly to construct nonseparable examples of various types of spaces; it turns out that since existence of Suslin lines cannot be proved in ZFC, the same is true for many of these topological examples: Their existence cannot be proved in ZFC.

If T is a tree, let P_T denote the partially ordered set that is obtained from $(T, \leq)$ by reversing the order. Note that any two elements of P_T are either comparable or incompatible; they are incompatible just in case they are incomparable as points of the tree T. Thus if T is a Suslin tree, then P_T satisfies the countable chain condition.

Lemma 22.2. *If T is a normal Suslin tree, then P_T is $\aleph_0$-distributive.*

Corollary. *If T is a normal Suslin tree, then $B = \text{r.o. } P_T$ is an $\aleph_0$-distributive, c.c.c., atomless, complete Boolean algebra.*

Proof. Let D_n, $n = 0, 1, 2, \ldots$, be open dense subsets of P_T. We shall prove that $\bigcap_{n=0}^{\infty} D_n$ is dense in P_T. First I claim that if $D \subseteq P_T$ is open dense, then there is $\alpha < \omega_1$ such that D contains all levels of T above α. To prove this, let A be a maximal incompatible subset of D. A is an antichain in T and hence countable. Thus pick $\alpha < \omega_1$ such that all $a \in A$ are below level α. Now if $x \in T$ is at level $\geq \alpha$, there is $d \in D$ such that $a \leq_T d$ (by density), and by maximality of A, d is comparable with some $a \in A$. Hence $a \leq_T d$ (draw a picture), and consequently $a \leq_T x \leq_T d$. Since D is open, we have $x \in D$. End of claim.

Now if D_n, $n = 0, 1, \ldots$, are open dense, we pick countable ordinals α_n such that D_n contains all levels above α_n, and let $\alpha = \sup_n \alpha_n$. The intersection $\bigcap_{n=0}^{\infty} D_n$ contains all levels of T above α; and since T is normal (cf. 22.3.v), this implies that $\bigcap_{n=0}^{\infty} D_n$ is dense in P_T.

Finally, it is an easy consequence of normality of T that the algebra $B = \text{r.o. } P_T$ is atomless. ■

A complete Boolean algebra that is atomless, $\aleph_0$-distributive, and satisfies the countable chain condition is called a *Suslin algebra*. Thus if there exists a Suslin tree, there exists a Suslin algebra.

Exercise 22.3. If there exists a Suslin algebra, then there exists a Suslin tree.

[If $(B, \leq)$ is a Suslin algebra, construct a Suslin tree $T \subset B - \{0\}$, ordered by $\geq$, by induction on levels. To get successors of $t \in T$, pick an infinite partition of t. At limit levels take all possible nonzero Boolean products along branches of the tree constructed so far. By $\aleph_0$-distributivity, each level of T is nonempty (and is a partition of B).]

As I mentioned earlier, it is independent of ZFC whether the countable chain condition of partially ordered sets is preserved under products. A Suslin tree provides an easy counterexample.

Exercise 22.4. If T is a normal Suslin tree then $P_T \times P_T$ is not c.c.c.

[For each $x \in T$, pick two immediate successors p_x and q_x of x. The set $\{(p_x, q_x) : x \in T\} \subset P_T \times P_T$ is pairwise incompatible.]

So far, we have reduced the original problem of Suslin to an equivalent problem on trees. It is worth mentioning—and an interesting result on its own—that if the definition of Suslin tree is somewhat weakened, then trees with the weaker properties exist. An *Aronszajn tree* is a tree of height ω_1 all of whose levels are at most countable and which has no uncountable branches.

Lemma 22.3. *There exists an Aronszajn tree.*

Proof. We construct a tree T whose elements are some bounded increasing transfinite sequences of rational numbers. If x, $y \in T$ are two such sequences, then we let $x \leq y$ just in case y extends x, i.e., $x \subseteq y$. Also, if $y \in T$ and x is an initial segment of y, then we let $x \in T$; thus the αth level of T will consist of all those $x \in T$ whose length is α.

Whatever the construction is, it is clear that an uncountable branch in T yields an increasing ω_1-sequence of rational numbers, which is impossible. Thus T will be an Aronszajn tree, provided we arrange that T has $\aleph_1$ levels, all of them at most countable. We construct T by induction on levels. For each $\alpha < \omega_1$, we construct a set U_α of increasing α-sequences of rationals; U_α will be the αth level of T. We construct the U_α so that for each α, $|U_\alpha| \leq \aleph_0$, and that:

(22.4) For each $\beta < \alpha$, each $x \in U_\beta$ and each $q > \sup(x)$ there is $y \in U_\alpha$, such that $x \subset y$ and $q \geq \sup(y)$.

Condition (22.4) enables us to continue the construction at limit steps.

To start, we let $U_0 = \{\emptyset\}$. The successor steps of the construction are also fairly easy. Given U_α, we let $U_{\alpha+1}$ be the set of all extensions $x^\frown r$ of sequences in U_α such that $r > \sup(x)$. It is clear that since U_α satisfies the condition (22.4), $U_{\alpha+1}$ satisfies it also (for $\alpha + 1$), and it is equally clear that $U_{\alpha+1}$ is at most countable.

Thus let α be a limit ordinal ($\alpha < \omega_1$) and assume that we have constructed all levels U_γ, $\gamma < \alpha$, of T below α, and that all the U_γ satisfy (22.4); we shall construct U_α. The points $x \in T$ below level α form a (normal) tree T_α of length α. I claim that T_α has the following property:

(22.5) For each $x \in T_\alpha$ and each $q > \sup(x)$ there is an increasing α-sequence of rationals y such that $x \subset y$ and $q \geq \sup(y)$ and that $y | \beta \in T_\alpha$ for all $\beta < \alpha$.

The last condition means that $\{y | \beta : \beta < \alpha\}$ is a branch in T_α. To prove the claim, we let α_n, $n = 0, 1, \ldots$, be an increasing sequence of ordinals such that $x \in U_{\alpha_0}$ and $\lim \alpha_n = \alpha$, and let $\{q_n\}_{n=0}^\infty$ be an increasing sequence of rational numbers such that $q_0 > \sup(x)$ and $\lim q_n \leq q$. Using repeatedly condition (22.4), for all α_n ($n = 0, 1, \ldots$), we can construct a sequence $y_0 \subset y_1 \subset \cdots \subset y_n \subset \cdots$ such that $y_0 = x$, $y_n \in U_{\alpha_n}$, and $\sup(y_n) \leq q_n$ for each n. Then we let $y = \bigcup_{n=0}^\infty y_n$; clearly, y satisfies (22.5).

Now we construct U_α as follows: For each $x \in T_\alpha$ and each rational number q such that $q > \sup(x)$, we choose a branch y in T_α that satisfies (22.5), and let U_α consist of all these y: $\alpha \to Q$. The set U_α so constructed is countable and satisfies the condition (22.4).

Then $T = \bigcup_{\alpha<\omega_1} U_\alpha$ is an Aronszajn tree. ∎

We shall now show that existence of a Suslin line is consistent with ZFC.

Theorem 48 (Tennenbaum, Jech). *There is a model of* ZFC *in which a Suslin tree exists. Thus existence of Suslin trees is consistent with the axioms of* ZFC.

The forcing construction given below uses countable trees as forcing conditions. An alternative forcing construction, using finite trees, is described in Exercises 9 and 10. And Theorem 49 to be proved later in this section shows that Suslin trees exist in the constructible universe L.

Let P be the collection of all T of the following form:

(22.6) There is $\alpha < \omega_1$ such that:

(i) each $t \in T$ is a function $t\colon \beta \to \omega$ for some $\beta < \alpha$;

(ii) if $t \in T$ and s is an initial segment of t (i.e., $s = t \mid \gamma$ for some γ), then $s \in T$;

(iii) if $\beta + 1 < \alpha$ and $t\colon \beta \to \omega$ is in T, then $t^\frown n \in T$ for all $n \in \omega$;

(iv) if $\beta < \alpha$ and $t\colon \beta \to \omega$ is in T, then for every γ such that $\beta \leq \gamma < \alpha$ there exists $s\colon \gamma \to \omega$ in T such that $t \subseteq s$;

(v) $T \cap {}^{\beta}\omega$ is at most countable for all $\beta < \alpha$.

Note that each $T \in P$ is a normal α-tree (for some $\alpha < \omega_1$) under partial ordering by $\subseteq$, and in fact every normal α-tree is isomorphic to some $T \in P$.

We shall construct a generic model of ZFC in which there is a Suslin tree. Given a ground model $\mathfrak{M}$, we extend it to a model $\mathfrak{M}[G]$ such that $\omega_1^{\mathfrak{M}[G]} = \omega_1^{\mathfrak{M}}$, and $\mathfrak{M}[G]$ contains a normal Suslin tree.

Let P be the set of all countable normal trees defined in (22.6), and let T_1 be stronger than T_2 if T_1 is an *extension* of T_2, i.e.,

$$(22.7) \quad T_1 < T_2 \quad \text{iff} \quad \exists\alpha < \text{height}(T_1) \quad \text{such that} \quad T_2 = \{t \mid \alpha : t \in T_1\}$$

(in other words, T_1 is obtained from T_2 by adding more levels on top of T_2).

Let G be an $\mathfrak{M}$-generic set of conditions and let $\mathcal{T} = \bigcup \{T : T \in G\}$. We shall show that in $\mathfrak{M}[G]$, $\mathcal{T}$ is a normal Suslin tree.

First we note that if T_1 and T_2 are two conditions, then either one is an extension of the other, or T_1 and T_2 are incompatible. Thus G consists of pairwise comparable trees and one can easily verify that $\mathcal{T}$ is a normal tree (of height $\leq \omega_1$).

If $T_0, T_1, \ldots, T_n, \ldots$ is a sequence of conditions such that for each n, T_{n+1} is an extension of T_n, then $\bigcup_{n=0}^{\infty} T_n$ is a normal countable tree (and extends each T_n). Hence P is $\aleph_0$-closed, and consequently, the cardinal $\aleph_1$ is preserved (and $\mathfrak{M}[G]$ has the same countable sequences in $\mathfrak{M}$ as $\mathfrak{M}$).

To show that the height of $\mathcal{T}$ is ω_1, we verify that for every $\alpha < \omega_1$, G contains a condition T of height at least α. We show that the set $\{T \in P : \text{height}(T) \geq \alpha\}$ is dense in P, for any $\alpha < \omega_1$. In other words, we show that for each $T_0 \in P$ and each $\alpha < \omega_1$, there is an extension $T \in P$ of T_0, of height at least α. It suffices to show that each $T_0 \in P$ has an extension $T \in P$ that has one more level; for then we can proceed by induction and take unions at limit steps.

If height(T_0) is a successor ordinal, then an extension of T_0 is easily obtained. If height(T_0) is a limit ordinal, then we first observe that for each $t \in T_0$ there exists a branch b of length α in T_0 such that $t \in b$: Using an increasing sequence $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots$ with limit α, we use the normality condition (22.6.iv) (or 22.3.v) to obtain such a branch. Now we construct an extension T of T_0, of height $\alpha + 1$, as follows: For each $t \in T_0$, we pick a branch b_t of length α in T_0 such that $t \in b$, and let $T = T_0 \cup \{s : s = \bigcup b_t \text{ for some } t \in T\}$; we "extend all the branches b_t, $t \in T_0$, of T_0." Since T_0 is countable, the added level is countable, and one can verify that $T \in P$.

It remains to show that $\mathcal{T}$ has no uncountable antichain. If T is a tree and A is an antichain in T, then A is called a *maximal antichain* if there is no antichain A' in T such that $A' \supset A$: Each $t \in T$ is comparable with some $a \in A$. If A a maximal antichain in T and if T' is an extension of T, then A is not necessarily maximal in T'. Let us call a set $S \subseteq T$ *bounded* in T if there is $\alpha <$ height(T) such that all elements of S are at levels $\leq \alpha$. (If the height of T is a successor ordinal, then every $S \subseteq T$ is bounded.)

Lemma 22.4. *If A is a maximal antichain in a normal tree T and if A is bounded in T (in particular, if the height of T is a successor ordinal), then A is maximal in every extension of T.*

Proof. Let T' be an extension of T. Let $\alpha <$ height(T) be such that each $a \in A$ is at level $\leq \alpha$. If $t' \in T' - T$, then there exists $t \in T$ at level α such that $t \subset t'$; in turn, there exists $a \in A$ such that $a \subseteq t$. Hence t' is comparable with some $a \in A$. ■

The key element of the proof is the following lemma:

Lemma 22.5. *Let α be a countable limit ordinal, let $T \in P$ be a normal α-tree and let A be a maximal antichain in T. Then there exists an extension $T' \in P$ of T of height $\alpha + 1$ such that A is a maximal antichain in T' (and hence A is a bounded maximal antichain in T').*

Proof. For each $t \in T$ there exists $a \in A$ such that either $t \subseteq a$ or $a \subseteq t$. In either case, there exists a branch $b = b_t$ of length α in T such that $t \in b$ and $a \in b$. Let T' be the extension of T obtained by extending the branches b_t, for all $t \in T$: $T' = T \cup \{\bigcup b_t : t \in T\}$. The tree T' is a normal $(\alpha + 1)$-tree and extends T; moreover, since every $s \in T'$ is comparable with some $a \in A$, A is maximal in T'. ■

Now I can finish the proof of Theorem 48 by showing that

$$\mathfrak{M}[G] \models \text{every antichain in } \mathcal{T} \text{ is countable} \tag{22.8}$$

Since every antichain can be extended to a maximal antichain, it suffices to show that every *maximal* antichain is countable. Thus let A be a maximal antichain in $\mathscr{T}$. There is a name A for A and a condition $T \in G$ such that

$$(22.9) \qquad T \Vdash A \text{ is a maximal antichain in } \mathscr{T}$$

(here I use $\mathscr{T}$ rather than its canonical name). We intend to show that the following set of conditions is dense below T:

$$(22.10) \quad D = \{T' \leq T : \text{there is a bounded maximal antichain } A' \text{ in } T' \text{ such that } T' \Vdash A' \subseteq A\}$$

Then some $T' \in D$ is in G and there is a bounded maximal antichain A' in T' such that $A' \subseteq A$. However, $\mathscr{T}$ is an extension of T', and by Lemma 22.4, A' is maximal in $\mathscr{T}$. Consequently, $A = A'$, and since A' is countable, we are done.

To show that D is dense below T let $T_0 \leq T$ be arbitrary. We shall construct $T' \leq T_0$ such that $T' \in D$. Since $T_0 \Vdash$ (A is a maximal antichain in $\mathscr{T}$ and $\mathscr{T}$ is an extension of $\check{T}_0$), there exists for each $s \in T_0$ an extension T_0' of T_0, and some $t_s \in T_0'$ such that

$$(22.11) \qquad s \text{ and } t_s \text{ are comparable} \quad \text{and} \quad T_0' \Vdash \check{t}_s \in A$$

Since T_0 is countable, we repeat this countably many times and obtain an extension $T_0' < T_0$ such that (22.11) holds for every $s \in T_0$. Let $T_1 = T_0'$. Then we proceed by induction and construct a sequence of trees $T_0 \geq T_1 \geq \cdots \geq T_n \geq \cdots$ such that for each n, T_{n+1} extends T_n and

$$(22.12) \quad (\forall s \in T_n)\,(\exists t_s \in T_{n+1})\,[s \text{ and } t_s \text{ are comparable and } T_{n+1} \Vdash \check{t}_s \in A]$$

We let $T_\infty = \bigcup_{n=0}^{\infty} T_n$, and let

$$(22.13) \qquad A' = \{t_s : s \in T_\infty\}$$

By (22.12), A' is a maximal antichain in T_∞, and

$$T_\infty \Vdash A' \subseteq A$$

Now we apply Lemma 22.5 and get an extension T' of T such that A' is a bounded maximal antichain in T'. Clearly, $T' \Vdash A' \subseteq A$, and hence $T' \in D$. ∎

Exercise 22.5. The notion of forcing $(P, <)$ used in the proof of Theorem 48 is isomorphic to the partially ordered set $(Q', <)$ used in Exercise 19.7. Thus (a) the model $\mathfrak{M}[G]$ is obtained by adjoining a generic subset of ω_1 via (19.7), and (b) the continuum hypothesis holds in $\mathfrak{M}[G]$.

If T_1 and T_2 are trees, then an *isomorphism* $\pi: T_1 \to T_2$ between T_1 and T_2 is a one-to-one mapping of T_1 onto T_2 such that $x < y$ if and only if $\pi(x) < \pi(y)$. An isomorphism maps level α of T_1 onto level α of T_2 (for all α); and if b is a branch in T_1, then $\pi[b]$ is a branch in T_2. An *automorphism* of T is an isomorphism of T_1 onto T_2. A tree T is *rigid* if it has no nontrivial automorphism, i.e., the only automorphism of T is the identity mapping. T is *homogeneous* if for any x, y at the same level of T, there exists an automorphism π of T such that $\pi(x) = y$.

Exercise 22.6. If T is a normal α-tree where $\alpha < \omega_1$ is a limit ordinal and if π is a nontrivial automorphism of T, then T has an extension $T' \in P$ of height $\alpha + 1$ such that π cannot be extended to an automorphism of T'.

[Construct T' so that for some branch b in T, b is extended while $\pi[b]$ is not.]

*Exercise 22.7.** The generic Suslin tree constructed above is rigid.

[If $T \Vdash \mathbf{p}$ is a nontrivial automorphism of $\mathscr{T}$, then the set $\{T' \le T : \exists$ automorphism π of an initial segment of T' that cannot be extended to an automorphism of T' and $T' \Vdash \pi \subseteq \mathbf{p}\}$ is dense below T; a contradiction.]

If $s\colon \alpha \to \omega$ and $t\colon \alpha \to \omega$, let $s \sim t$ just in case $s(\xi) = t(\xi)$ for all but finitely many $\xi < \alpha$.

*Exercise 22.8.** There is a generic model $\mathfrak{M}[G]$ in which there exists a homogeneous Suslin tree.

[Let the forcing conditions be normal countable trees satisfying (22.6) with the additional properties: (vi) if $t \in T$ and $s \sim t$, then $s \in T$; and (vii) if $s \in T$ and $t \in T$ are at the same level, then $s \sim t$.]

Similar, albeit somewhat more complicated constructions give examples of rigid or homogeneous Suslin lines.

Suslin trees can also be obtained by forcing with finite conditions. The construction is described below ((22.14) and Exercises 22.9 and 22.10). In the proof one uses the following lemma that is often used in topology and combinatorics. The collection Z in Lemma 22.6 is called a Δ-*system* by combinatorists.

Lemma 22.6 (Δ-lemma). *Let W be an uncountable collection of finite sets. Then there exists an uncountable $Z \subset W$, and a finite set S such that $X \cap Y = S$ for any distinct elements X, $Y \in Z$.*

Proof. Since W is uncountable, it is clear that uncountably many $X \in W$ have the same size; thus we may assume that for some n, $|X| = n$ for all $X \in W$. We prove the lemma by induction on n. If $n = 1$, the lemma is trivial. Thus assume that the lemma holds for n, and let W be such that $|X| = n + 1$ for all $X \in W$.

If there is some element a that belongs to uncountably many $X \in W$, we apply the induction hypothesis to the collection $\{X - \{a\} : X \in W \text{ and } a \in X\}$, and obtain $Z \subset W$ with the required properties.

Otherwise, each a belongs to at most countably many $X \in W$, and we construct a disjoint collection $Z = \{X_\alpha : \alpha < \omega_1\}$ as follows, by induction on α. Given X_ξ, $\xi < \alpha$, we find $X = X_\alpha \in W$ that is disjoint from all X_ξ, $\xi < \alpha$. ■

Let $(P, <)$ be the following notion of forcing:

(22.14) A condition is a finite tree $(T, <)$ such that $T \subset \omega_1$, and if $\alpha <_T \beta$, then $\alpha < \beta$;

$(T_1, <_{T_1})$ is stronger than $(T_2, <_{T_2})$ if
$$T_1 \supseteq T_2 \text{ and } <_{T_2} = <_{T_1} \cap (T_2 \times T_2).$$

If G is a generic set of conditions, then $\mathscr{T} = \bigcup \{T : T \in G\}$ is a Suslin tree. The crucial properties to verify are: (a) $(P, <)$ satisfies the countable chain condition, and (b) $\mathscr{T}$ has no uncountable antichain.

*Exercise 22.9.** $(P, <)$ satisfies the c.c.c.

[Given an uncountable set W of conditions, use the Δ-lemma to find an uncountable $Z \subset W$ such that any $X, Y \in Z$ are compatible.]

*Exercise 22.10.** $\mathscr{T}$ has no uncountable antichain.

[If $T_0 \Vdash \dot{A}$ is uncountable, we first find an uncountable set W of pairs (T, α_T) such that $T \leq T_0$ and $T \Vdash \check{\alpha}_T \in \dot{A}$. By the Δ-lemma, find an uncountable $Z \subset W$ with the property that if $T_1, T_2 \in Z$, then there is T stronger than both T_1 and T_2 such that $T \Vdash \check{\alpha}_{T_1}$ is comparable with $\check{\alpha}_{T_2}$. Then some $T \leq T_0$ forces that $\dot{A}$ is not an antichain.]

Since the notion of forcing $(P, <)$ satisfies the c.c.c., the cardinals of $\mathfrak{M}$ remain cardinals in $\mathfrak{M}[G]$. In particular, if the continuum hypothesis fails in $\mathfrak{M}$, then it fails in $\mathfrak{M}[G]$. Thus existence of Suslin trees is consistent with $2^{\aleph_0} > \aleph_1$.

We shall now turn our attention to the constructible universe.

Theorem 49 (Jensen). *The axiom of constructibility implies that Suslin trees exist.*

We break the proof into two parts. First we formulate an important combinatorial principle and use it to construct a Suslin tree. Then we prove that the principle holds in L.

(22.15) **Jensen's Principle** ($\diamondsuit$). There exists a sequence of sets S_α, $\alpha < \omega_1$, such that for every $X \subseteq \omega_1$, the set

$$\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$$

is a stationary subset of ω_1.

Notice that the diamond principle implies the continuum hypothesis: If $X \subseteq \omega$, then $X = S_\alpha$ for some $\alpha < \omega_1$; thus $2^{\aleph_0} = \aleph_1$.

Exercise 22.11. If $\diamondsuit$ holds, then there exists a family $\mathscr{F}$ of stationary subsets of ω_1 such that $|\mathscr{F}| = 2^{\aleph_1}$ and $|S_1 \cap S_2| \leq \aleph_0$ whenever S_1 and S_2 are distinct elements of $\mathscr{F}$.

[Let $\mathscr{F} = \{S_X : X \subseteq \omega_1\}$, where $S_X = \{\alpha : X \cap \alpha = S_\alpha\}$.]

We shall prove that $\diamondsuit$ implies existence of Suslin trees.

First we make a useful observation. If T is a normal ω_1-tree, let T_α, for each $\alpha < \omega_1$, consist of all $x \in T$ at levels below α: $T_\alpha = \{x \in T : \mathrm{o}(x) < \alpha\}$.

Lemma 22.7. *If A is a maximal antichain in T, then the set*

$$C = \{\alpha : A \cap T_\alpha \textit{ is a maximal antichain in } T_\alpha\}$$

is closed unbounded.

Proof. It is easy to see that C is closed. To show that C is unbounded, let $\alpha_0 < \omega_1$ be arbitrary. Since T_{α_0} is countable, there exists a countable ordinal

$\alpha_1 > \alpha_0$ such that every $t \in T_{\alpha_0}$ is comparable with some $a \in A \cap T_{\alpha_1}$. Then there is $\alpha_2 > \alpha_1$ such that each $t \in T_{\alpha_1}$ is comparable with some $a \in A \cap T_{\alpha_2}$, etc. If $\alpha_0 < \alpha_1 < \alpha_2 < \cdots \alpha_n < \cdots$ is constructed in this way and if $\alpha = \lim_n \alpha_n$, then $A \cap T_\alpha$ is a maximal antichain in T_α. ■

Now we are ready to construct a Suslin tree, using $\diamondsuit$. Again, the key element (besides $\diamondsuit$) is Lemma 22.5.

Lemma 22.8. *If $\diamondsuit$ holds, then there exists a Suslin tree.*

Proof. We construct a normal Suslin tree $(T, <_T)$ by induction on levels. To facilitate the use of $\diamondsuit$, we let points of T be countable ordinals, $T = \omega_1$, and in fact each T_α (the first α levels of T) is an initial segment of ω_1.

We construct T_α, $\alpha < \omega_1$, such that each T_α is a normal α-tree and such that T_β extends T_α whenever $\beta > \alpha$. T_1 consists of one point. If α is a limit ordinal, then $(T_\alpha, <_T)$ is the union of the trees $(T_\beta, <_T)$, $\beta < \alpha$. If α is a successor ordinal, then $(T_{\alpha+1}, <_T)$ is an extension of $(T_\alpha, <_T)$ obtained by adjoining infinitely many immediate successors to each x at the top level of T_α.

It remains to describe the construction of $T_{\alpha+1}$ if α is a limit ordinal. By $\diamondsuit$, we are given a sequence S_α, $\alpha < \omega_1$, with property (22.15). If S_α happens to be a maximal antichain in $(T_\alpha, <_T)$, then we use Lemma 22.5 and find an extension $(T_{\alpha+1}, <_T)$ of T_α such that S_α is maximal in $T_{\alpha+1}$. Otherwise, we let $T_{\alpha+1}$ be any extension of T_α that is a normal $(\alpha + 1)$-tree. (In either case, we let the set $T_{\alpha+1}$ be an initial segment of countable ordinals.)

We shall now show that the tree $T = \bigcup_{\alpha<\omega_1} T_\alpha$ is a normal Suslin tree. It suffices to verify that T has no uncountable antichain. If $A \subseteq T$ $(= \omega_1)$ is a maximal antichain in T, then by Lemma 22.7, $A \cap T_\alpha$ is a maximal antichain in T_α, for a closed unbounded set of α's. It follows easily from the construction that for a closed unbounded set of α's, $T_\alpha = \alpha$. Thus using the diamond principle, we find a limit ordinal α such that $A \cap \alpha = S_\alpha$ and $A \cap \alpha$ is a maximal antichain in T_α. However, we constructed $T_{\alpha+1}$ in such a way that $A \cap \alpha$ is maximal in $T_{\alpha+1}$, and therefore in T. It follows that $A = A \cap \alpha$ and so A is countable. ■

Consistency of $\diamondsuit$ is easily obtained by forcing. In fact, if $\mathfrak{M}[G]$ is obtained by adjoining a generic subset of ω_1 via (19.7), then $\mathfrak{M}[G] \models \diamondsuit$:

Let P consist of all countable sequences $p = \langle S_\xi : \xi < \alpha \rangle$ $(\alpha < \omega_1)$ where $S_\xi \subseteq \xi$ for all $\xi < \alpha$; let $p \leq q$ just in case p extends q. P is clearly $\aleph_0$-closed.

*Exercise 22.12.** Let G be P-generic. Then $\mathfrak{M}[G] \models \diamondsuit$.

[If $p \Vdash (C$ is a closed unbounded set and $X \subseteq \omega_1)$, find $q \leq p$ such that $q = \langle S_\xi : \xi \leq \alpha \rangle$ and $q \Vdash (\alpha \in C$ and $X \cap \alpha = S_\alpha)$.]

Exercise 22.13. $(P, <)$ is isomorphic to the partially ordered set $(Q', <)$ in Exercise 19.7. Thus $\mathfrak{M}[G]$ is obtained by adjoining a generic subset of ω_1 via (19.7).

Lemma 22.9. *Assume that $V = L$. Then $\diamondsuit$ holds.*

Proof. The proof uses an argument very similar to the proof of the continuum hypothesis (Lemma 13.1). I shall use the notation and terminology of Section 13.

By induction on $\alpha < \omega_1$, we define a sequence of pairs (S_α, C_α), $\alpha < \omega_1$, such that $S_\alpha \subseteq \alpha$ and C_α is a closed unbounded subset of α (in the obvious meaning of "closed unbounded"). We let $S_0 = C_0 = \emptyset$ and $S_{\alpha+1} = C_{\alpha+1} = \alpha + 1$ for all α. If α is a limit ordinal, we define:

(22.16) (S_α, C_α) is the $<_L$-least pair (in the canonical well ordering of L) such that $S_\alpha \subseteq \alpha$, C_α is a closed unbounded subset of α, and $S_\alpha \cap \xi \neq S_\xi$ for all $\xi \in C_\alpha$; if no such pair exists, let $S_\alpha = C_\alpha = \alpha$.

We are going to show that the sequence S_α, $\alpha < \omega_1$, defined above satisfies (22.15). Thus assume that the sequence does not satisfy (22.15); then for some $X \subseteq \omega_1$, the set $\{\alpha : X \cap \alpha = S_\alpha\}$ is thin, and hence there is a closed unbounded set C such that

$$X \cap \alpha \neq S_\alpha \quad \text{for all} \quad \alpha \in C \tag{22.17}$$

Let (X, C) be the $<_L$-least pair such that $X \subseteq \omega_1$, C is a closed unbounded subset of ω_1, and such that (22.17) holds.

Since $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1 \rangle$ is a ω_1-sequence of pairs of subsets of ω_1, it belongs to L_{ω_2}, and moreover, it satisfies the same definition (22.16) in the model $\langle L_{\omega_2}, \in \rangle$ (since L_{ω_2} is a model of ZF$^-$). Also, $(X, C) \in L_{\omega_2}$, and (X, C) is, in $\langle L_{\omega_2}, \in \rangle$, the $<_L$-least pair such that $X \subseteq \omega_1$, C is a closed unbounded subset of ω_1, and such that (22.17) holds.

Let $\langle N, \in \rangle$ be an elementary submodel of $\langle L_{\omega_2}, \in \rangle$ such that:

(22.18) (i) N is countable;
(ii) if $\xi < \eta < \omega_1$ and $\eta \in N$, then $\xi \in N$;
(iii) if $\xi \in N$, then $(S_\xi, C_\xi) \in N$;
(iv) $\langle (S_\xi, C_\xi) : \xi < \omega_1 \rangle \in N$, and $(X, C) \in N$.

In fact, since $\langle N, \in \rangle \prec \langle L_{\omega_2}, \in \rangle$, and $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1 \rangle$ and (X, C) are definable elements of L_{ω_2}, conditions (iii) and (iv) are satisfied automatically; and the argument in Exercise 13.15 shows that (ii) holds also automatically.

There exists an isomorphism π of N onto a transitive set $M = \pi[N]$; in fact, $M = L_\beta$ for some $\beta < \omega_1$. Because of condition (ii), if α is the least ordinal not in N, then $\alpha = \pi(\omega_1)$ and $\pi(\xi) = \xi$ for all $\xi < \alpha$. Consequently, $\pi(S_\xi, C_\xi) = (S_\xi, C_\xi)$ for all $\xi < \alpha$, and $\pi\langle (S_\xi, C_\xi) : \xi < \omega_1 \rangle = \langle (S_\xi, C_\xi) : \xi < \alpha \rangle$. Also, $\pi(X) = X \cap \alpha$ and $\pi(C) = C \cap \alpha$, and by the definition of X and C, the model M satisfies:

(22.19) $(X \cap \alpha, C \cap \alpha)$ is the $<_L$-least pair (Z, D) s.t. $Z \subseteq \alpha$, $D \subseteq \alpha$ is closed unbounded and $Z \cap \xi \neq S_\xi$ for all $\xi \in D$.

By our choice of the model N, the definition (22.16) is absolute for M and therefore (22.19) holds (in the universe L); comparing (22.19) with (22.16), we have $X \cap \alpha = S_\alpha$ (and $C \cap \alpha = C_\alpha$). Since $C \cap \alpha$ is unbounded in α, and C is closed, it follows that $\alpha \in C$. Thus we have $X \cap \alpha = S_\alpha$ for some $\alpha \in C$, contrary to the assumption (22.17). ■

A purely combinatorial argument can be used to show that $\diamondsuit$ is equivalent to the following statement:

(22.20) There exists a sequence of functions h_α, $\alpha < \omega_1$, such that for every f: $\omega_1 \to \omega_1$, the set

$$\{\alpha < \omega_1 : f \restriction \alpha = h_\alpha\}$$

is stationary.

Exercise 22.14. Show that the version (22.20) of $\diamondsuit$ holds in L.

Exercise 22.15.* Use (22.20) and $\diamondsuit$ to construct a rigid Suslin tree in L.
[Use Exercise 22.6.]

Exercise 22.16.* If $\diamondsuit$ holds, then there exists a homogeneous Suslin tree.
[Modify the construction in Lemma 22.8., as in Exercise 22.8.]

It should be noted that the construction of a Suslin tree in L given above is Δ_1 over L_{ω_1}:

Exercise 22.17.* ($V = L$) There is a function $\alpha \mapsto S_\alpha$ ($\alpha < \omega_1$), Δ_1 over L_{ω_1}, such that the S_α satisfy the diamond principle.

Exercise 22.18.* ($V = L$) There is a normal Suslin tree T such that the function $\alpha \mapsto T_\alpha$ (where $T_\alpha = \{x \in T : o(x) < \alpha\}$) is Δ_1 over L_{ω_1}.

I shall conclude this section with the statement of the following theorem, which will be proved in the next section. Together with Theorem 48, the theorem establishes the independence of Suslin's problem on the axioms of ZFC: By Theorem 48, there are Suslin trees in some models; and by Theorem 50, there are none in other models.

Theorem 50 (Solovay–Tennenbaum). *There is a model of* ZFC *in which there are no Suslin trees. Thus existence of Suslin trees is independent of* ZFC.

23. MARTIN'S AXIOM AND ITERATED FORCING

Martin's Axiom

Rather than giving a direct forcing proof of Theorem 50, we shall formulate an axiom that easily implies nonexistence of Suslin trees, and then give a consistency proof of the axiom. The axiom, known as *Martin's axiom*, is an interesting alternative to the continuum hypothesis, and is an important tool in point set topology and combinatorics.

In Section 16 I proved the following lemma (Lemma 16.1): If $(P, <)$ is a partially ordered set and $\mathscr{D}$ is a countable collection of dense subsets in P, then there exists a $\mathscr{D}$-generic filter on P. (Lemma 17.5 gives a Boolean algebraic version: If $\mathscr{X}$ is a countable family of subsets of a Boolean algebra B, then there exists an $\mathscr{X}$-complete ultrafilter on B.)

The generalization of Lemma 16.1 for uncountable $\mathscr{D}$ is false:

Exercise 23.1. Let $(P, <)$ be the notion of forcing that collapses ω_1 (see (18.45)). There is an uncountable collection $\mathscr{D}$ of dense subsets of P such that there is no $\mathscr{D}$-generic filter on P.

[For each $\alpha < \omega_1$, let $D_\alpha = \{p \in P : \alpha \in \operatorname{ran}(p)\}$.]

Thus consider the following generalization of Lemma 16.1. Let κ be an infinite cardinal:

(23.1) MA_κ: If $(P, <)$ is a partially ordered set that satisfies the countable chain condition, and if $\mathscr{D}$ is a collection of at most κ dense subsets of P, then there exists a $\mathscr{D}$-generic filter on P.

$\mathrm{MA}_{\aleph_0}$ is just Lemma 16.1 and is provable in ZFC. Note that MA_κ implies that $\kappa < 2^{\aleph_0}$:

Exercise 23.2. Let $(P, <)$ be the notion of forcing used to obtain a Cohen-generic real. There is a collection $\mathscr{D}$ of size $2^{\aleph_0}$ of dense subsets of P such that there is no $\mathscr{D}$-generic filter on P.

[For each $g: \omega \to \{0, 1\}$, let $D_g = \{p \in P : p \not\subset g\}$.]

Martin's Axiom states that MA_κ holds for every $\kappa < 2^{\aleph_0}$:

Martin's Axiom (MA). *If $(P, <)$ is partially ordered set that satisfies the countable chain condition and if $\mathscr{D}$ is a collection of less than $2^{\aleph_0}$ dense subsets of P, then there exists a $\mathscr{D}$-generic filter on P.*

Since $\mathrm{MA}_{\aleph_0}$ is always true, it follows that Martin's axiom is a consequence of the continuum hypothesis. We shall see later that MA can replace the continuum hypothesis in numerous proofs that use CH; this is important since the axiom can hold even when CH fails.

Martin's axiom plus $2^{\aleph_0} > \aleph_1$ settles Suslin's problem, rather easily in fact:

Lemma 23.1. *If $\mathrm{MA}_{\aleph_1}$ holds, then there is no Suslin tree.*

Proof. Let us assume that T is a normal Suslin tree and let P_T be the partially ordered set obtained from T by reversing the order. P_T satisfies the countable chain condition. For each $\alpha < \omega_1$, let D_α be the union of all levels above α: $D_\alpha = \{x \in T : \mathrm{o}(x) > \alpha\}$. Each D_α is dense in P_T; if we let $\mathscr{D} = \{D_\alpha : \alpha < \omega_1\}$ and if G is a $\mathscr{D}$-generic filter on P, then G is a branch in T of length ω_1. A contradiction. ■

There are several, more or less natural, ways of reformulating Martin's axiom. The most obvious reformulation is in terms of Boolean algebras (just as

Lemma 17.5 is a reformulation of Lemma 16.1 and just as Lemma 17.4 relates the two definitions of genericity). The formulation (ii) below will be useful in the forthcoming consistency proof.

Lemma 23.2. *The following are equivalent, for any infinite cardinal* κ:

(i) MA_κ

(ii) *If* $(P, <)$ *is a partially ordered set satisfying the c.c.c. such that* $|P| \leq \kappa$ *and if* $\mathscr{D}$ *is a collection of at most* κ *dense subsets of* κ, *then there is a* $\mathscr{D}$*-generic filter on* P.

(iii) *If* B *is a complete, c.c.c. Boolean algebra and if* $\mathscr{X}$ *is a family of at most* κ *subsets of* B, *then there exists an* $\mathscr{X}$*-complete ultrafilter on* B.

Proof. (i) → (iii): We follow the proof of Lemma 17.4b. Let us assume MA_κ and let B be a complete Boolean algebra satisfying the c.c.c. Let $\mathscr{X}$ be a collection of at most κ subsets of B. Let $P = B - \{0\}$, and let for each $X \in \mathscr{X}$, $D_X = \{u \neq 0 : \text{either } u \leq \prod X, \text{ or for some } x \in X, u \cdot x = 0\}$. Each D_X is dense and by MA_κ there is a filter F on B that meets each D_X. It is easy to see that any ultrafilter G that extends F is an $\mathscr{X}$-complete ultrafilter on B.

(iii) → (ii): We follow the proof of Lemma 17.4a. Let us assume (iii) and let P be a c.c.c. partially ordered set of size at most κ. Let $B = \text{r.o. } P$ and let $e: P \to B$ be the canonical embedding. First we notice that if (iii) holds, then for any collection $\mathscr{D}$ of at most κ dense subsets of B there is an ultrafilter on B that meets all $D \in \mathscr{D}$ (for each D consider $X_D = \{-u : u \in D\}$). If $\mathscr{D}$ is a collection of dense sets in P, $|\mathscr{D}| \leq \kappa$, we consider the following dense sets in B: (a) all $e[D]$, $D \in \mathscr{D}$; and (b) all $e[E_{p,q}]$ for $p, q \in P$, where $E_{p,q} = \{r \in P : \text{either } r \leq p \text{ and } r \leq q, \text{ or } r \text{ is incompatible with } p, \text{ or } r \text{ is incompatible with } q\}$. Since $|P| \leq \kappa$, (iii) gives an ultrafilter G on B that meets all $e[D]$ and all $e[E_{p,q}]$, and then $G' = \{p \in P : e(p) \in G\}$ is a $\mathscr{D}$-generic filter on P.

(ii) → (i): Let P be a c.c.c. partially ordered set and let us assume that (ii) holds. Let $\mathscr{D}$ be a family of at most κ dense subsets of P. For each $D \in \mathscr{D}$, we let W_D be a maximal incompatible subset of D. Since each W_D is countable, there exists a set $Q \subseteq P$ of size at most κ such that (a) $W_D \subseteq Q$ for all $D \in \mathscr{D}$, and (b) if $p, q \in Q$ are compatible, then there exists $r \in Q$ such that $r \leq p$ and $r \leq q$. (Condition (b) guarantees that $p, q \in Q$ are compatible in $(Q, <)$ if and only if they are compatible in $(P, <)$.) Each W_D is a maximal incompatible subset of Q; let $E_D = \{q \in Q : q \leq w \text{ for some } w \in W_D\}$. Each E_D is dense in Q.

The partially ordered set Q has size at most κ and satisfies the c.c.c. By (ii), there is a filter G on Q that meets every E_D. It is easily seen that G generates a $\mathscr{D}$-generic filter on P. ■

Let me state (without proof) another equivalent of Martin's axiom in terms of point set topology:

In every compact Hausdorff c.c.c. space, the intersection of fewer than $2^{\aleph_0}$ dense open sets is dense.

Theorem 51 (Solovay–Tennenbaum). *There is a model of* ZFC *in which Martin's axiom holds and* $2^{\aleph_0} > \aleph_1$.

By Lemma 23.1, there are no Suslin trees in this model. Hence Suslin's problem is independent of the axioms of ZFC (Theorem 50). Since the continuum hypothesis fails in the model too, one could still ask whether existence of Suslin trees follows just from $2^{\aleph_0} = \aleph_1$. This has been resolved too: R. Jensen constructed a model that satisfies GCH and in which there are no Suslin trees.

The proof of Theorem 51 consists in construction of the model in stages: One successively destroys counterexamples (or potential counterexamples) to Martin's axiom. The method, known as *iterated forcing*, deserves a detailed exposition.

Iterated Forcing: Two-Step Iteration

Let P be a notion of forcing in the ground model $\mathfrak{M}$, and let $G_1 \subset P$ be generic over $\mathfrak{M}$. In $\mathfrak{M}[G_1]$, let Q be a partially ordered set and let $G_2 \subset Q$ be $\mathfrak{M}[G_1]$-generic. Then $\mathfrak{M}[G_1][G_2]$ is a model of ZFC extending $\mathfrak{M}$ and the natural question to ask is whether $\mathfrak{M}[G_1][G_2]$ is a generic extension of $\mathfrak{M}$, and what is the corresponding notion of forcing in $\mathfrak{M}$. The answer to the first question is yes, and the two-step generic extension can be described uniformly for all generic filters on P. That is, if $\underset{\sim}{Q}$ is a name for Q, then there is a notion of forcing $P * \underset{\sim}{Q}$ in $\mathfrak{M}$ that gives the same model $\mathfrak{M}[G_1][G_2]$ as the two step iteration of forcing: first P and then $i_{G_1}(\underset{\sim}{Q})$.

We shall first describe the two-step iteration in terms of complete Boolean algebras. Let B be a complete Boolean algebra in $\mathfrak{M}$, and let $C \in \mathfrak{M}^B$ be such that

$$\|C \text{ is a complete Boolean algebra}\|_B = 1 \tag{23.2}$$

We shall define a complete Boolean algebra $D = B * C$ in $\mathfrak{M}$ such that the D-valued model $\mathfrak{M}^D$ "represents" the C-valued model $(\mathfrak{M}^B)^C$ ("constructed in $\mathfrak{M}^B$").

Let us consider the class of all $c \in \mathfrak{M}^B$ such that $\|c \in C\| = 1$ and the equivalence relation $c_1 \sim c_2$ defined by $\|c_1 = c_2\| = 1$. There is a set D that contains exactly one element c in each $\sim$-equivalence class; thus D is a maximal subset of $\mathfrak{M}^B$ such that:

(23.3) (i) $\|c \in C\| = 1$ for each $c \in D$;

(ii) $\|c_1 = c_2\| < 1$ whenever $c_1, c_2 \in D$ and $c_1 \neq c_2$.

We make D a Boolean algebra as follows: If $c_1, c_2 \in D$, then there is a unique $c \in D$ such that

$$\|c = c_1 +_C c_2\| = 1$$

we let $c = c_1 +_D c_2$. The operations $\cdot_D$ and $-_D$ are defined similarly.

It is easily verified that D is a Boolean algebra; I leave this to the reader. Note that the partial ordering $\leq_D$ of D induced by the Boolean operations relates naturally to the partial ordering $\leq_C$ of C (in $\mathfrak{M}^B$):

$$c_1 \leq_D c_2 \quad \text{iff} \quad \|c_1 \leq_C c_2\| = 1 \tag{23.4}$$

Also, we see that if $c_1, c_2 \in D - \{0_D\}$ then c_1, c_2 are incompatible (in D) if and only if

$$\|c_1 \cdot c_2 = 0\| = 1 \tag{23.5}$$

Let me show that D is a complete Boolean algebra. Let X be a subset of D; we show that $\sum_D X$ exists. Let $\boldsymbol{X} \in \mathfrak{M}^B$ be as follows: $\operatorname{dom}(\boldsymbol{X}) = X$, and $\boldsymbol{X}(c) = 1$ for all $c \in X$. We have $\|\boldsymbol{X} \subseteq C\| = 1$, and since C is a complete Boolean algebra in $\mathfrak{M}^B$ and since $\mathfrak{M}^B$ is full (Lemma 18.6), there exists c such that

$$\|c = \textstyle\sum_C \boldsymbol{X}\| = 1 \tag{23.6}$$

Therefore there is $c \in D$ that satisfies (23.6), and clearly $c = \sum_D X$.

For each $b \in B$, there is a unique $c \in D$ such that

$$\|c = 1_C\| = b \quad \text{and} \quad \|c = 0_C\| = -b \tag{23.7}$$

Let $c = e(b)$.

Lemma 23.3. *The mapping e: $B \to D$ is an embedding (a complete one-to-one homomorphism) of B in D.*

Proof. By easy calculation. ■

Thus we can assume, for all practical purposes, that B is a complete subalgebra of the algebra $D = B * C$.

Remark. If C_1 and C_2 are complete Boolean algebras in $\mathfrak{M}^B$ and if $\|C_1$ is a complete subalgebra of $C_2\|_B = 1$, then $B * C_1$ is a complete subalgebra of $B * C_2$. In fact, if $\boldsymbol{h} \in \mathfrak{M}^B$ is such that

$$\|\boldsymbol{h}\colon C_1 \to C_2 \text{ is an embedding}\|_B = 1 \tag{23.8}$$

then there is an embedding k: $B * C_1 \to B * C_2$ such that $k(b) = b$ for all $b \in B$. Namely, if $c_1 \in B * C_1$, we let

$$k(c_1) = \text{the unique } c_2 \in B * C_2 \text{ such that } \|\boldsymbol{h}(c_1) = c_2\| = 1 \tag{23.9}$$

We recall that if G is a generic ultrafilter on B, then i_G is the G-interpretation of $\mathfrak{M}^B$ (see (18.28)).

Lemma 23.4. *Let B be a complete Boolean algebra in $\mathfrak{M}$, let $C \in \mathfrak{M}^B$ be such that $\|C$ is a complete Boolean algebra$\|_B = 1$, and let $D = B * C$ be such that B is a complete subalgebra of D.*

(a) *If* G_1 *is an* $\mathfrak{M}$*-generic ultrafilter on* B*, if* $C = i_{G_1}(C)$*, and if* G_2 *is an* $\mathfrak{M}[G_1]$*-generic ultrafilter on* C*, then there is an* $\mathfrak{M}$*-generic ultrafilter* G *on* $B * C$ *such that*

$$\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2]$$

(b) *If* G *is an* $\mathfrak{M}$*-generic ultrafilter on* $B * C$*, if* $G_1 = G \cap B$ *and* $C = i_{G_1}(C)$*, then there is an* $\mathfrak{M}[G_1]$*-generic ultrafilter* G_2 *on* C *such that*

$$\mathfrak{M}[G_1][G_2] = \mathfrak{M}[G]$$

Proof. (a) Let $D = B * C$, and let G_1, C and G_2 be as assumed and let us define $G \subseteq B * C$ as follows: for each $c \in D$,

$$c \in G \quad \text{iff} \quad i_{G_1}(c) \in G_2 \tag{23.10}$$

I claim that G is an $\mathfrak{M}$-generic ultrafilter on D. I shall show that if $\Delta \in \mathfrak{M}$ is a dense subset of D, then $\Delta \cap G \neq \varnothing$, and leave verification of the other properties to the reader. If Δ is dense in D and $\Delta \in \mathfrak{M}$, then

$$\Delta_1 = i_{G_1}[\Delta] = \{i_{G_1}(c) : c \in \Delta\} \tag{23.11}$$

is a dense subset of $C = i_{G_1}[C]$ (here we use (23.4)), and $\Delta_1 \in \mathfrak{M}[G_1]$. Since G_2 is $\mathfrak{M}[G_1]$-generic, there is $c \in D$ such that $i_{G_1}(c) \in \Delta_1 \cap G_2$. By (23.11) we may assume that $c \in \Delta$ and hence $c \in \Delta \cap G$.

The definition (23.10) of G can be done in $\mathfrak{M}[G_1][G_2]$; thus $G \in \mathfrak{M}[G_1][G_2]$ and $\mathfrak{M}[G] \subseteq \mathfrak{M}[G_1][G_2]$. On the other hand, it follows from (23.10) (and the embedding (23.7) of B in D) that $G_1 = G \cap B$, and so $G_1 \in \mathfrak{M}[G]$; furthermore, G_2 satisfies in $\mathfrak{M}[G]$ the formula (23.10) and we have $G_2 \in \mathfrak{M}[G]$. Thus $\mathfrak{M}[G_1][G_2] \subseteq \mathfrak{M}[G]$.

(b) If G, G_1 and C are as assumed, then we define $G_2 \subseteq C$ by the formula (23.10). I claim that G_2 is $\mathfrak{M}[G_1]$-generic, and again verify only the following property: If $\Delta_1 \in \mathfrak{M}[G_1]$ is dense in C, then $\Delta_1 \cap G_2 \neq \varnothing$. Let $\mathit{\Delta}_1 \in \mathfrak{M}^B$ be a name for Δ_1 and let $b_0 \in G_1$ be such that

$$b_0 \Vdash \mathit{\Delta}_1 \text{ is dense in } C - \{0_C\} \tag{23.12}$$

We let

$$\Delta = \{c \in D - \{0_D\} : c \leq \|c \in \mathit{\Delta}_1\|\} \tag{23.13}$$

and show that Δ is dense in D below b_0. Thus let $d \in D - \{0_D\}$ be arbitrary such that $d \leq b_0$. There is some (nonzero) $b \leq b_0$ such that $b \Vdash d \neq 0_C$, and by (23.12) we have

$$b \Vdash (\exists x \in \mathit{\Delta}_1)\, x \leq_C d \tag{23.14}$$

Since $\mathfrak{M}^B$ is full, there is x such that $b \Vdash (x \in \mathit{\Delta}_1 \text{ and } x \leq_C d)$. We may assume that $x \in D$. Now we let $c = x \cdot_D b$. Note that by (23.7), $b \Vdash b = 1$, and so

$b \Vdash \mathbf{x} = \mathbf{c}$; hence $b \Vdash \mathbf{c} \in \mathit{\Delta}_1$. Thus we have $\mathbf{c} \leq \|\mathbf{c} \in \mathit{\Delta}_1\|$; i.e., $\mathbf{c} \in \Delta$. Since $-b \Vdash \mathbf{c} = 0$, we have $\|\mathbf{c} \leq_C \mathbf{d}\| = 1$ and so $\mathbf{c} \leq_D \mathbf{d}$. We have proved that Δ is dense in D below b_0.

Therefore, there exists $\mathbf{c} \in \Delta$ such that $\mathbf{c} \in G$. By (23.13), $\|\mathbf{c} \in \mathit{\Delta}_1\|$ is in G and hence in $G_1 = G \cap B$. Thus $i_{G_1}(\mathbf{c}) \in \Delta_1$, and by the definition of G_2, $i_{G_1}(\mathbf{c})$ also belongs to G_2. ∎

Let B, C, D, G_1, G_2, and G have the same meaning as above. Let P be a dense subset of B and let $\mathbf{Q} \in \mathfrak{M}^B$ be such that $\|\mathbf{Q}$ is a dense subset of $C\| = 1$. Let us define a subset $P * \mathbf{Q}$ of D as follows:

(23.15) $\quad c \in P * \mathbf{Q} \quad$ iff $\quad c = p \cdot_D \mathbf{q}\quad$ where $p \in P$ and $\|\mathbf{q} \in \mathbf{Q}\| = 1$

Clearly, $P * \mathbf{Q}$ is a dense subset of $B * C$.

Lemma 23.5. *If $p_1 \cdot \mathbf{q}_1$ and $p_2 \cdot \mathbf{q}_2$ are elements of $P * \mathbf{Q}$, then $p_1 \cdot \mathbf{q}_1 \leq_D p_2 \cdot \mathbf{q}_2$ if and only if $p_1 \leq p_2$ and $p_1 \Vdash \mathbf{q}_1 \leq_C \mathbf{q}_2$.*

Proof. Routine. ∎

If $P \in \mathfrak{M}$ is a notion of forcing and $\mathbf{Q}$ is a notion of forcing in $\mathfrak{M}^{\text{r.o.}(P)}$, then Lemma 23.5 gives an alternative definition of the two-step forcing $P * \mathbf{Q}$. (If P is not separative, then the p's have to be replaced by their images under the natural embedding of P into r.o. P; similarly for $\mathbf{Q}$.) Then we have r.o.$(P * \mathbf{Q}) = (\text{r.o. } P) * (\text{r.o. } \mathbf{Q})^{\mathfrak{M}^{\text{r.o. } P}}$.

The two-step iteration is a generalization of the product $P \times Q$:

Exercise 23.3. If $P, Q \in \mathfrak{M}$, then $P \times Q$ is densely embedded in $P * \check{Q}$.

[Consider the elements $p \cdot \check{q}$ where $(p, q) \in P \times Q$.]

In this sense, Lemma 23.4 is a generalization of the product Lemma 20.1.

Lemma 23.6. *Let $D = B * C$.* (a) *If B satisfies the c.c.c. and if $\|C$ satisfies the c.c.c.$\| = 1$, then D satisfies the c.c.c.* (b) *On the other hand, if D satisfies the c.c.c., then B satisfies the c.c.c. and $\|C$ satisfies the c.c.c.$\| = 1$.*

Proof. (a) Let B and C satisfy the c.c.c. and let W be a pairwise disjoint set of nonzero elements of D. Let $\mathbf{W} \in \mathfrak{M}^B$ be such that $\operatorname{dom}(\mathbf{W}) = W$ and $\mathbf{W}(\mathbf{c}) = 1$ for all $\mathbf{c} \in W$; clearly, $\|\mathbf{W} \subseteq C\| = 1$. It follows from (23.5) that

$$\|\text{for any distinct } x, y \in \mathbf{W},\ x \cdot y = 0\| = 1$$

Since C satisfies the c.c.c., we have

$$\|\mathbf{W} \text{ is countable}\| = 1$$

Now we use the countable chain condition in B, in a manner similar to the proof of Lemma 19.2. There is $\mathbf{f} \in \mathfrak{M}^B$ such that

$$\|\mathbf{f} \text{ is a function of } \check{\omega} \text{ onto } \mathbf{W}\| = 1$$

For every n, there is a partition U_n of B and a set $Z_n = \{c_u : u \in U_n\} \subseteq W$ such that for every $u \in U_n$,

$$\|f(\check{n}) = c_u\| \geq u$$

Since B satisfies the c.c.c., each U_n is countable, and so there is a countable set $Z \subseteq W$ such that for every $c \in W$,

$$\sum \{\|c = z\| : z \in Z\} = 1 \tag{23.16}$$

Now we use the fact that the elements of W are pairwise disjoint in D. If c, d are distinct elements of W, then $\|c \cdot d = 0\| = 1$, and since $\|c = d\| \leq \|c \cdot d = c\|$, we have

$$\|c = d\| \leq \|c = 0\| \tag{23.17}$$

Thus if $c \in W - Z$, we have by (23.16) and (23.17): $\|c = 0\| = 1$. Since we assumed that $c \neq 0_D$ for each $c \in W$, we conclude that $W = Z$. Thus W is countable.

(b) Let D satisfy the c.c.c. Since B is a complete subalgebra of D, B also satisfies the c.c.c. To show that $\|C$ satisfies the c.c.c.$\| = 1$, let $W \in \mathfrak{M}^B$, and let $b_0 \in B - \{0\}$ be such that

(23.18) $\quad b_0 \Vdash W$ is an uncountable set of nonzero elements of C

We shall show that there exists a nonzero $b \leq b_0$ such that

(23.19) $\quad b \Vdash W$ is not pairwise incompatible

There is $f \in \mathfrak{M}^B$ such that

$b_0 \Vdash f$ is a one-to-one function of $\check{\omega}_1$ into W

For every $\alpha < \omega_1$, $b_0 \Vdash (\exists x \in W)\, x = f(\check{\alpha})$; and since $\mathfrak{M}^B$ is full, there exists $c_\alpha \in D$ such that $b_0 \Vdash (c_\alpha \in W$ and $c_\alpha = f(\check{\alpha}))$. Let $d_\alpha = b_0 \cdot c_\alpha$. Since $b_0 \Vdash c_\alpha \neq c_\beta$, for all $\alpha \neq \beta$, the set $\{d_\alpha : \alpha < \omega_1\}$ is an uncountable subset of D. Since D satisfies the c.c.c., there exist α and β such that d_α and d_β are compatible. Hence there exists a nonzero $d \in D$ such that $d \leq d_\alpha \cdot d_\beta$; moreover, we can find d such that $d = b \cdot c$, where $0 \neq b \leq b_0$ and $b \Vdash (c \neq 0$ and $c \leq c_\alpha \cdot c_\beta)$. Now (23.19) follows. ■

As a consequence of the above lemma, we observe that if P and Q are two c.c.c. notions of forcing, then $P \times Q$ satisfies the c.c.c. if and only if Q is c.c.c. in $\mathfrak{M}^{\text{r.o.}\,P}$ if and only if P is c.c.c. in $\mathfrak{M}^{\text{r.o.}\,Q}$.

Exercise 23.4. Let T be a Suslin tree in $\mathfrak{M}$ and let P be the notion of forcing that adjoins κ generic reals (19.1). Let $G \subset P$ be $\mathfrak{M}$-generic. Then T is a Suslin tree in $\mathfrak{M}[G]$.

[Let P_T be the notion of forcing associated with the Suslin tree T. P satisfies the c.c.c. in any model $\mathfrak{N} \supseteq \mathfrak{M}$, in particular in any $\mathfrak{M}[G_1]$ where G_1 is a generic filter on P_T. Thus $P_T \times P$ is c.c.c., and so P_T is c.c.c. in $\mathfrak{M}[G]$.]

The above argument gives another consistency proof for "there exists a Suslin tree and $2^{\aleph_0} > \aleph_1$."

Exercise 23.5. If P is λ-closed and $\|Q$ is $\check{\lambda}$-closed$\|_{\text{r.o.}P} = 1$, then $P * Q$ is λ-closed.

One can define the inverse operation to the operation $B * C$. If B is a complete subalgebra of the complete Boolean algebra D (in $\mathfrak{M}$), let $C \in \mathfrak{M}^B$ be the Boolean algebra that is (in $\mathfrak{M}^B$) the quotient of $\check{D}$ modulo the filter generated by the canonical generic ultrafilter on B. In other words, $C = \check{D}/\equiv$ where the (B-valued) equivalence relation $\equiv$ is defined as follows:

$$\|\check{d}_1 \equiv \check{d}_2\| = \sum \{b \in B : b \text{ is disjoint from } d_1 \triangle d_2\}$$

(where $d_1 \triangle d_2 = (d_1 - d_2) + (d_2 - d_1)$. Let us denote $C = D:B$.

Exercise 23.6. $\|D:B$ is a complete Boolean algebra$\|_B = 1$, and D is isomorphic to $B * (D:B)$.

It follows that if (in $\mathfrak{M}$) B is a complete subalgebra of a complete Boolean algebra D and if G is a generic ultrafilter on D, and $G_1 = G \cap B$, then $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}[G_1]$.

*Exercise 23.7.** Let B, D, G, G_1 be as above and let P be a dense subset of D. There exists $Q \in \mathfrak{M}[G_1]$, $Q \subset P$, such that $G \cap Q = G \cap P$ is an $\mathfrak{M}[G_1]$-generic filter on $(Q, \leq)$.

[For each $p \in P$, let $h(p) = \prod \{b \in B : b \geq p\}$, and let $Q = \{p \in P : h(p) \in G_1\}$.]

Direct Limits of Sequences of Complete Boolean Algebras

Let α be a limit ordinal number, and let for each $i < \alpha$, B_i be a complete Boolean algebra such that if $i < j$, then B_i is a complete subalgebra of B_j. Then

$$C = \bigcup_{i<\alpha} B_i$$

is a Boolean algebra (not necessarily complete), and we call the completion B of C the *direct limit* of $\{B_i : i < \alpha\}$:

$$B = \lim_{i \to \alpha} \operatorname{dir} B_i \tag{23.20}$$

The direct limit of $\{B_i : i < \alpha\}$ is the unique (up to isomorphism) complete Boolean algebra B such that $\bigcup_{i<\alpha} B_i$ is dense in B. More generally, if B_i, $i < \alpha$, are complete Boolean algebras and if for each $i, j < \alpha$, e_{ij} is a complete embedding of B_i into B_j such that the e_{ij} commute (i.e., $e_{jk} \circ e_{ij} = e_{ik}$), then there is a unique (up to isomorphism) complete Boolean algebra B and complete embeddings $e_{i\alpha}$ of B_i into B such that the $e_{i\alpha}$ commute with the e_{ij} and that $\bigcup_{i<\alpha} e_{i\alpha}[B_i]$ is dense in B. [Even more generally, if I is a directed partially ordered set, i.e., for any $i_1, i_2 \in I$ there is $j \in I$ such that $i_1 \leq j$ and $i_2 \leq j$, and if $\{B_i, e_{ij}\}_{i,j \in I}$ is a directed system of complete Boolean algebras and complete embeddings, then there is a unique complete Boolean algebra B, and embeddings $e_{i\infty}$, $i \in I$, etc.; $B = \lim \operatorname{dir}_{i \in I} B_i$.]

We shall prove that direct limits of sequences of complete Boolean algebras preserve the chain condition.

Before giving the proof, let me state a few facts about *canonical projections* of complete Boolean algebras. Let B be a complete subalgebra of a complete Boolean algebra D. For each $d \in D$, let us define

$$h(d) = \prod \{b \in B : d \leq b\} \tag{23.21}$$

$h(d)$ is the least $b \in B$ such that $d \leq b$. The function h maps D onto B and it is easy to see that h has the following properties:

(23.22)
(i) $h(d) \geq d$;
(ii) $h(b) = b$ for all $b \in B$;
(iii) $h(d_1 + d_2) = h(d_1) + h(d_2)$;
(iv) $h(d_1 \cdot d_2) \leq h(d_1) \cdot h(d_2)$.

Regarding (iv), we might have $d_1 \cdot d_2 = 0$ while $h(d_1) \cdot h(d_2) \neq 0$. However, the following follows easily from the definition of $h(d)$:

$$\text{if} \quad b \in B \text{ and } b \cdot d = 0, \qquad \text{then} \quad b \cdot h(d) = 0 \tag{23.23}$$

Lemma 23.7. *Let κ be a regular uncountable cardinal. Let α be a limit ordinal, and let*

$$B_0 \subseteq B_1 \subseteq \cdots \subseteq B_i \subseteq \cdots \qquad (i < \alpha)$$

be a sequence of complete Boolean algebras such that for any $i < j$, B_i is a complete subalgebra of B_j and such that for each limit ordinal $\gamma < \alpha$, $B_\gamma = \lim \operatorname{dir}_{i<\gamma} B_i$. Let B be the direct limit of B_i, $i < \alpha$. If each B_i is κ-saturated, then B is also κ-saturated. In particular, if each B_i satisfies the c.c.c., then B satisfies the c.c.c.

Proof. Let us assume that B is not κ-saturated. Let $C = \bigcup_{i<\alpha} B_i$; C is dense in B. Let W be a maximal pairwise incompatible subset of C, and assume that $|W| \geq \kappa$. If $\operatorname{cf} \alpha \neq \kappa$, then some B_i contains a subset $Z \subset W$ of size κ, a contradiction. Thus $\operatorname{cf} \alpha = \kappa$. If $\langle i_\xi : \xi < \kappa \rangle$ is any normal κ-sequence with limit α, then $B = \lim \operatorname{dir}_{\xi \to \kappa} B_{i_\xi}$. Hence we may assume that $\alpha = \kappa$.

For each $\xi < \kappa$, let h_ξ be the canonical projection of B onto B_ξ; $h_\xi(x) = \prod \{u \in B_\xi : u \geq x\}$. Since $\sum W = 1$, we have for each ξ,

$$\sum \{h_\xi(w) : w \in W\} = 1$$

Each B_ξ is κ-saturated and so there exists a set $W_\xi \subset W$ of size $<\kappa$ such that

$$\sum \{h_\xi(w) : w \in W_\xi\} = 1 \tag{23.24}$$

For each ξ, let $\eta(\xi)$ be an ordinal $\eta < \kappa$ such that $W_\xi \subset B_\eta$ (such an η exists because $|W_\xi| < \kappa$).

Now, there exists a limit ordinal $\gamma < \kappa$ such that for every $\xi < \gamma$, $\eta(\xi) < \gamma$. I claim that $W \subseteq B_\gamma$ and therefore $|W| < \kappa$, contrary to our assumption.

Let $w \in W$ be arbitrary and consider $h_\gamma(w) \in B_\gamma$. Since $\bigcup_{\xi<\gamma} B_\xi$ is dense in B_γ, there exists $\xi < \gamma$ and $b \in B_\xi$ such that $0 \neq b \leq h_\gamma(w)$. By (23.24), there exists $u \in W_\xi$ such that b and $h_\xi(u)$ are compatible. By the choice of γ, we have $u \in B_\gamma$. By (23.23) (applied to h_ξ), we have that b and u are compatible. Since $b \leq h_\gamma(w)$, u and $h_\gamma(w)$ are compatible. By (23.23) again (applied to h_γ), u and w are compatible. Since u and w are both in W, it follows that $w = u$, and we have $w \in B_\gamma$, as claimed. ■

Exercise 23.8. In Lemma 23.7 let $\alpha = \kappa$ and assume that $B_\gamma = \lim \operatorname{dir}_{i \to \gamma} B_i$ for a stationary set of γ's (rather than for all $\gamma < \kappa$). The conclusion of Lemma 23.7 remains valid.

Proof of Theorem 51

Let $\mathfrak{M}$ be a transitive model of ZFC + GCH. We shall construct (in $\mathfrak{M}$) a complete Boolean algebra B such that if $G \subseteq B$ is generic, then $\mathfrak{M}[G]$ satisfies $2^{\aleph_0} \leq \aleph_2$ and $\mathrm{MA}_{\aleph_1}$. Thus $\mathfrak{M}[G]$ will be a model of MA + $2^{\aleph_0} = \aleph_2$.

The algebra B will satisfy the countable chain condition and so cardinals will be preserved. Also, we shall have $|B| = \aleph_2$ and so $(2^{\aleph_0})^{\mathfrak{M}[G]} = \aleph_2$ will follow by Lemma 19.4.

We shall construct a sequence B_α, $\alpha < \omega_2$, of complete Boolean algebras such that:

(23.25)
(i) B_α is a complete subalgebra of B_β whenever $\alpha < \beta$;
(ii) if α is a limit ordinal, then $B_\alpha = \lim \operatorname{dir}_{i \to \alpha} B_i$;
(iii) each B_α satisfies the c.c.c.;
(iv) $|B_\alpha| \leq \aleph_2$ for all $\alpha < \omega_2$.

Then we let $B = \lim \operatorname{dir}_{\alpha \to \omega_2} B_\alpha$. By Lemma 23.7, B satisfies the c.c.c., and since it has a dense subset $\bigcup_{\alpha<\omega_2} B_\alpha$ of size $\leq \aleph_2$, it follows that $|B| \leq \aleph_2^{\aleph_0} = \aleph_2$.

At successor stages, each $B_{\alpha+1}$ will be constructed by two-step forcing: $B_{\alpha+1} = B_\alpha * C$, where $C \in \mathfrak{M}^{B_\alpha}$ is such that

$$\|C \text{ is a complete Boolean algebra, satisfies the c.c.c. and has a dense subset of size } \leq \aleph_1 \|_{B_\alpha} = 1 \tag{23.26}$$

Then by Lemmas 23.6a and 23.7, each B_α satisfies the countable chain condition. Let me show, by induction on α, that the condition (23.26) also guarantees that $|B_\alpha| \leq \aleph_2$, for each $\alpha < \omega_2$. If α is a limit ordinal, then B_α has a dense subset of size $\leq \aleph_2$ and so $|B_\alpha| \leq \aleph_2$. If $|B_\alpha| \leq \aleph_2$ and C satisfies (23.26), then let $Q \in \mathfrak{M}^{B_\alpha}$ be such that $\| \, |Q| \leq \aleph_1$ and Q is dense in $C \| = 1$. Then the following set is dense in the algebra $B_\alpha * C$:

$$B_\alpha * Q = \{b \cdot q : 0 \neq b \in B_\alpha,\ q \in B_\alpha * C, \text{ and } \|q \in Q\| = 1\} \tag{23.27}$$

The number of B_α-valued elements of Q is at most equal to the number of functions from $\aleph_1$ to B_α; since $|B_\alpha| \leq \aleph_2$, this number is $\leq \aleph_2^{\aleph_1} = \aleph_2$. Thus there are at most $\aleph_2$ q's in (23.27) and so the set $B_\alpha * \boldsymbol{Q}$ has size $\leq\aleph_2$. Since $B_{\alpha+1}$ satisfies the c.c.c. and has a dense subset of size $\leq\aleph_2$, we have $|B_{\alpha+1}| \leq \aleph_2^{\aleph_0} = \aleph_2$.

One more preliminary step before the construction of the B_α starts. If D is any complete Boolean algebra of size $\leq\aleph_2$ that satisfies the c.c.c., then the number of D-valued binary relations on $\check{\omega}_1$ is $\aleph_2$ (the number of functions from $\omega_1 \times \omega_1$ into D, which is $\aleph_2^{\aleph_1} = \aleph_2$). That is, there are $\aleph_2$ names $\boldsymbol{R}_\alpha$, $\alpha < \omega_2$, such that whenever $\boldsymbol{R}$ is a name for a binary relation on ω_1, then $\|\boldsymbol{R} = \boldsymbol{R}_\alpha\| = 1$ for some α. For each such complete Boolean algebra D, let us fix this sequence $\boldsymbol{R}_\alpha^D$, $\alpha < \omega_2$, of D-valued relations on $\check{\omega}_1$.

Now let me describe the construction of the sequence B_α, $\alpha < \omega_2$. We construct B_α by induction on α such that the conditions (23.25) are satisfied. Let $\alpha \mapsto (\beta_\alpha, \gamma_\alpha)$ be the canonical mapping of ω_2 onto $\omega_2 \times \omega_2$; in fact, any onto mapping will do, provided $\beta_\alpha \leq \alpha$ for all α. We let B_0 be the trivial algebra $\{0, 1\}$; and if α is a limit ordinal, we let B_α be the direct limit of B_i, $i < \alpha$.

The crucial step is the construction of $B_{\alpha+1}$ from B_α. We look at the algebra $D = B_{\beta_\alpha}$, and in D we consider the γ_αth D-valued relation $\boldsymbol{R} = \boldsymbol{R}_{\gamma_\alpha}^D$ on $\check{\omega}_1$. Since D is a complete subalgebra of B_α, we have $\boldsymbol{R} \in \mathfrak{M}^{B_\alpha}$. Let

$$(23.28)\qquad b = \|\boldsymbol{R} \text{ is a partial ordering of } \check{\omega}_1 \text{ and } (\check{\omega}_1, \boldsymbol{R}) \text{ satisfies the c.c.c.}\|_{B_\alpha}$$

Let $\boldsymbol{C} \in \mathfrak{M}^{B_\alpha}$ be the (B_α-valued) complete Boolean algebra defined in $\mathfrak{M}^{B_\alpha}$ as follows:

$$(23.29)\qquad \begin{aligned} &\|\boldsymbol{C} = \text{r.o.}(\check{\omega}_1, \boldsymbol{R})\| = b \\ &\|\boldsymbol{C} \text{ is the trivial algebra } \{0, 1\}\| = -b \end{aligned}$$

Let $B_{\alpha+1} = B_\alpha * \boldsymbol{C}$.

It follows from the definition of $\boldsymbol{C}$ that $\boldsymbol{C}$ has property (23.26), and hence $B_{\alpha+1}$ satisfies the c.c.c. and has size at most $\aleph_2$.

Finally, we let $B = \lim \operatorname{dir}_{\alpha\to\omega_2} B_\alpha$.

Let G be a generic ultrafilter on B. Since B satisfies the countable chain condition, $\mathfrak{M}[G]$ has the same cardinals as $\mathfrak{M}$. Since $|B| = \aleph_2$, we have $\mathfrak{M}[G] \vDash 2^{\aleph_0} \leq \aleph_2$. We shall show that $\mathfrak{M}[G]$ satisfies $\text{MA}_{\aleph_1}$. By Exercise 23.2, $\text{MA}_{\aleph_1}$ implies $2^{\aleph_0} \geq \aleph_2$, and it follows that Martin's Axiom holds in $\mathfrak{M}[G]$.

For each $\alpha < \omega_2$, let $G_\alpha = G \cap B_\alpha$.

Lemma 23.8. *If $X \in \mathfrak{M}[G]$ is a subset of ω_1, then $X \in \mathfrak{M}[G_\alpha]$ for some $\alpha < \omega_2$.*

Proof. Let $\boldsymbol{X} \in \mathfrak{M}^B$ be a name for X. The set X is determined by the B-values $b_\xi = \|\check{\xi} \in \boldsymbol{X}\|$, $\xi < \omega_1$. Since $\bigcup_{\alpha<\omega_2} B_\alpha$ is dense in B, and B satisfies the c.c.c., there exist, for each $\xi < \omega_1$, countably many $b_{\xi_n} \in \bigcup_{\alpha<\omega_2} B_\alpha$, $n < \omega$, such that $b_\xi = \sum_{n=0}^\infty b_{\xi_n}$. It follows that there is $\alpha < \omega_2$ such that all the b_ξ belong to B_α, and so X has a B_α-valued name. ■

To prove that $\text{MA}_{\aleph_1}$ holds in $\mathfrak{M}[G]$, let $(P, <)$ be a partially ordered set in $\mathfrak{M}[G]$ such that $\mathfrak{M}[G] \vDash (P, <)$ satisfies the c.c.c., and let $\mathscr{D} \in \mathfrak{M}[G]$ be a collection of at most $\aleph_1$ dense subsets of P. By Lemma 23.2(ii), we may assume that $|P| = \aleph_1$; there is a binary relation R on ω_1 isomorphic to $(P, <)$, and so we may as well assume that $(P, <) = (\omega_1, R)$.

Now we use Lemma 23.8, which of course applies also to each $X \subseteq \omega_1 \times \omega_1$. $\mathscr{D}$ is a set of at most $\aleph_1$ subsets of ω_1 and as such can be encoded by a subset of $\omega_1 \times \omega_1$; therefore Lemma 23.8 applies to $\mathscr{D}$ as well. Thus there is $\beta < \omega_2$ such that both R and $\mathscr{D}$ belong to $\mathfrak{M}[G_\beta]$. Let $\boldsymbol{R} \in \mathfrak{M}^{B_\beta}$ be a B_β-valued name for R. Because of the preparations we made before the construction, there is $\gamma < \omega_2$ such that $\boldsymbol{R}$ is in fact the γth B_β-valued relation on $\check{\omega}_1$.

Let $\alpha \geq \beta$ be such that $(\beta, \gamma) = (\beta_\alpha, \gamma_\alpha)$. Since $\mathfrak{M}[G_\alpha]$ is a submodel of $\mathfrak{M}[G]$, and $\mathfrak{M}[G] \vDash (\omega_1, R)$ satisfies the c.c.c., it follows that $\mathfrak{M}[G_\alpha] \vDash (\omega_1, R)$ satisfies the c.c.c.

Thus

$$b = \|(\check{\omega}_1, \boldsymbol{R}) \text{ satisfies the c.c.c.}\|_{B_\alpha} \in G_\alpha \tag{23.30}$$

We have constructed $B_{\alpha+1}$ to be $B_\alpha * C$ where $\|C = \text{r.o.}(\check{\omega}_1, \boldsymbol{R}\| = b$ (see (23.29)). Hence (by Lemma 23.4b), $\mathfrak{M}[G_{\alpha+1}] = \mathfrak{M}[G_\alpha][H]$ where H is an $\mathfrak{M}[G_\alpha]$-generic filter on (ω_1, R). Now because $\mathscr{D} \in \mathfrak{M}[G_\alpha]$, we have $H \cap D \neq \emptyset$ for each $D \in \mathscr{D}$. Hence H is $\mathscr{D}$-generic. Therefore $\text{MA}_{\aleph_1}$ holds in $\mathfrak{M}[G]$, and the proof of Theorem 51 is complete. ■

*Exercise 23.9.** If $\mathfrak{M}$ is a model of ZFC + GCH, and if κ is a regular uncountable cardinal in $\mathfrak{M}$, then there is a generic extension $\mathfrak{M}[G]$ that satisfies MA and $2^{\aleph_0} = \kappa$.

[Iterate to κ instead to ω_2 and make sure that MA_λ holds for all $\lambda < \kappa$.]

More about Martin's Axiom

I shall prove two more consequences of Martin's axiom. One is that $2^\kappa = 2^{\aleph_0}$ for every $\kappa < 2^{\aleph_0}$, and the other is that if $\text{MA}_{\aleph_1}$, then products of partially ordered sets (or topological spaces) preserve the countable chain condition. There will be more applications of Martin's axiom in Section 24, and more in Chapter 7 (concerning real numbers, particularly Lebesgue measure). Let me also mention that $\text{MA}_{\aleph_1}$ has a number of consequences in point set topology, usually involving separability and the countable chain condition. A typical theorem using $\text{MA}_{\aleph_1}$ is, e.g.: Every compact first countable c.c.c. Hausdorff space is separable (while a Suslin line can be used to produce a counterexample).

Theorem 52 (Martin–Solovay). *Martin's axiom implies that* $2^\kappa = 2^{\aleph_0}$ *for all* $\kappa < 2^{\aleph_0}$.

Corollary. $2^{\aleph_0}$ *is a regular cardinal.*

[Proof of Corollary: cf $2^{\aleph_0} = \text{cf } 2^\kappa > \kappa$, for all $\kappa < 2^{\aleph_0}$.]

The proof uses almost disjoint subsets of ω. If A_1, A_2 are subsets of ω then we say that A_1 and A_2 are *almost disjoint* if $A_1 \cap A_2$ is finite.

Lemma 23.9. *There exists a family of* $2^{\aleph_0}$ *pairwise almost disjoint subsets of* ω.

Proof. Let S be the set of all finite $\{0, 1\}$-sequences: $S = \bigcup_{n=0}^{\infty} {}^{n}\{0, 1\}$. For every $f: \omega \to \{0, 1\}$, let $A_f \subseteq S$ be the set $A_f = \{s \in S : s \subset f\} = \{f \mid n : n \in \omega\}$. Clearly, $A_f \cap A_g$ is finite if $f \neq g$; thus $\{A_f : f \in {}^{\omega}\{0, 1\}\}$ is a family of $2^{\aleph_0}$ almost disjoint subsets of the countable set S, and the lemma follows. ■

Let $\kappa < 2^{\aleph_0}$ and let us assume that MA_κ holds. We fix a family $\{A_\alpha : \alpha < \kappa\}$ of almost disjoint subsets of ω.

Lemma 23.10. *For every* $X \subseteq \kappa$ *there exists* $A \subseteq \omega$ *such that for all* $\alpha < \kappa$,

$$\alpha \in X \quad \text{iff} \quad A_\alpha \cap A \text{ is infinite} \tag{23.31}$$

Granted Lemma 23.10, the mapping that sends each $X \subseteq \kappa$ to some $A \subseteq \omega$ such that (23.31) holds is one-to-one, and maps $P(\kappa)$ into $P(\omega)$. Hence $2^\kappa \leq 2^{\aleph_0}$.

Proof. Let $X \subseteq \kappa$. We let $(P, <)$ be the following notion of forcing: A condition is a function p from a subset of ω into $\{0, 1\}$ such that:

(23.32) (i) $\mathrm{dom}(p) \cap A_\alpha$ is finite for every $\alpha \in X$;
(ii) $\{n : p(n) = 1\}$ is finite.

The set P is partially ordered by inverse inclusion: $p \leq q$ iff p extends q.

We first show that P satisfies the countable chain condition. If p and q are incompatible, then $\{n : p(n) = 1\} \neq \{n : q(n) = 1\}$ and since there are only countably many finite subsets of ω, it follows that P satisfies the c.c.c.

For each $\beta \in \kappa - X$, let $D_\beta = \{p \in P : A_\beta \subseteq \mathrm{dom}(p)\}$. Any $q \in P$ can be extended to some $p \in D_\beta$: Simply let $p(n) = 0$ for all $n \in A_\beta - \mathrm{dom}(p)$. Since A_β is almost disjoint from all A_α, $\alpha \in X$, p has property (i) in (23.32) and hence is a condition. Thus each D_β is dense.

For each $\alpha \in X$ and each $k \in \omega$, let

$$E_{\alpha,k} = \{p \in P : \{n \in A_\alpha : p(n) = 1\} \text{ has size at least } k\}$$

It is easy to see that each $E_{\alpha,k}$ is dense in P.

Let $\mathscr{D}$ be the collection of all D_β ($\beta \in \kappa - X$) and all $E_{\alpha,k}$($\alpha \in X$, $k \in \omega$). By MA_κ, there exists a $\mathscr{D}$-generic filter G on P. Note that $f = \bigcup G$ is a function on a subset of ω. We let

$$A = \{n : f(n) = 1\} = \{n : p(n) = 1 \text{ for some } p \in G\}$$

If $\alpha \in X$, then $A \cap A_\alpha$ is infinite because for each k there is some $p \in G \cap E_{\alpha,k}$. If $\beta \in \kappa - X$, then $A \cap A_\beta$ is finite because for some $p \in G$, $A_\beta \subset \mathrm{dom}(p)$ and $\{n : p(n) = 1\}$ is finite. ■

Theorem 53. $\mathrm{MA}_{\aleph_1}$ *implies that if* P_i, $i \in I$, *are partially ordered sets, all satisfying the countable chain condition, then* $\prod_{i \in I} P_i$ *also satisfies the countable chain condition.*

We first observe that it is enough to show that (assuming $\mathrm{MA}_{\aleph_1}$), the product of two partially ordered sets preserves the c.c.c.

Lemma 23.11. *If $\prod_{i \in I} P_i$ does not satisfy the c.c.c., then there is a finite $S \subseteq I$ such that $\prod_{i \in S} P_i$ does not satisfy the c.c.c.*

Proof. Let W be an uncountable pairwise incompatible subset of the product P. For each $p \in P$, the support of p, $\mathrm{s}(p) = \{i \in I : p_i \neq 1\}$ is finite. By the Δ-lemma (Lemma 22.6), there exists an uncountable $Z \subset W$, and a finite set $S \subset I$ such that $\mathrm{s}(p) \cap \mathrm{s}(q) = S$ for any distinct elements p, $q \in Z$. Then $\{p \mid S : p \in Z\}$ is an uncountable incompatible subset of $\prod_{i \in S} P_i$. ■

If $(P, <)$ is a partially ordered set, we say that P satisfies the *strong countable chain condition* if every uncountable $W \subseteq P$ has an uncountable subset Z such that the elements of Z are pairwise compatible.

Lemma 23.12. *If P and Q are partially ordered sets such that P satisfies the strong c.c.c. and Q satisfies the c.c.c., then $P \times Q$ satisfies the c.c.c.*

Proof. Let W be an uncountable subset of $P \times Q$. If the set

$$W_0 = \{p \in P : (p, q) \in W \text{ for some } q \in Q\}$$

is at most countable, then there exists some p such that

$$W_p = \{q \in Q : (p, q) \in W\}$$

is uncountable. Since Q satisfies the c.c.c., there are $q_1, q_2 \in W_p$ that are compatible. Then (p, q_1), (p, q_2) are compatible elements of W.

If W_0 is uncountable, then since P satisfies the strong c.c.c., there exists an uncountable $Z \subseteq W_0$ such that any two elements of Z are compatible. For each $p \in Z$ pick $q_p \in Q$ such that $(p, q_p) \in W$. If $q_{p_1} = q_{p_2}$ for some $p_1, p_2 \in Z$, then (p_1, q_{p_1}), (p_2, q_{p_1}) are compatible elements of W. Otherwise, the set $\{q_p : p \in Z\}$ is uncountable, and hence some q_{p_1} and q_{p_2} are compatible. Then (p_1, q_{p_1}), (p_2, q_{p_2}) are compatible. ■

Exercise 23.10. If P and Q satisfy the strong c.c.c., then $P \times Q$ satisfies the strong c.c.c.

Exercise 23.11. If $P = \prod_{i \in I} P_i$ and if all P_i, $i \in I$, satisfy the strong c.c.c., then P satisfies the strong c.c.c.

In particular, if P is the notion of forcing that adjoins κ generic reals (19.1), then P satisfies the strong c.c.c. (compare with Exercise 23.4).

Theorem 53 will now follow from this lemma:

Lemma 23.13. $\mathrm{MA}_{\aleph_1}$ *implies that every partially ordered set that satisfies the c.c.c. satisfies the strong c.c.c.*

Proof. Let P be a partially ordered set that satisfies the c.c.c. and let $W = \{w_\alpha : \alpha < \omega_1\}$ be an uncountable subset of P. We will use $\mathrm{MA}_{\aleph_1}$ to find a filter G such that $Z = G \cap W$ is uncountable.

First I claim that there is some $p_0 \in W$ such that every $p \leq p_0$ is compatible with uncountably many w_α. Otherwise, for each $\alpha < \omega_1$ there is $\beta > \alpha$ and some $v_\alpha \leq w_\alpha$ which is incompatible with all w_γ, $\gamma \geq \beta$; then we can construct an ω_1-sequence $\{v_{\alpha_i} : i < \omega_1\}$ of pairwise incompatible elements.

For each $\alpha < \omega_1$, let

$$D_\alpha = \{p \leq p_0 : p \leq w_\gamma \text{ for some } \gamma \geq \alpha\}$$

By the above claim, each D_α is dense below p_0. By $\text{MA}_{\aleph_1}$, there exists a filter G on P such that $p_0 \in G$ and $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. It follows that $G \cap W$ is uncountable. ■

24. SOME COMBINATORIAL PROBLEMS

In this section I shall discuss various combinatorial problems that are related to the topics covered in the two preceding sections. I shall prove several theorems dealing with trees, almost disjoint sets and functions, and ultrafilters, and state several related results without proof.

Trees

First let us consider Aronszajn trees (see Section 22), that is, trees of height ω_1 such that each level is at most countable. The Aronszajn tree constructed in Lemma 22.3 has the following property: Its nodes (elements of T) are one-to-one functions from some countable ordinal into a fixed countable set, and $x \leq y$ just in case $x \subseteq y$. Let us call an Aronszajn tree that can be represented this way a *special* Aronszajn tree. [Special Aronszajn trees (and their generalization to greater heights) have application in model theory; this is because "there is a special Aronszajn tree" can be expressed as: "a certain sentence has a model of type $(\aleph_1, \aleph_0)$."]

Now let us consider a stronger property: An Aronszajn tree T is *regular* if there exists a function $f \colon T \to \omega$ such that

(24.1) $\quad f(x) \neq f(y)$ whenever x and y are comparable

Exercise 24.1. A regular Aronszajn tree is special.

[If $x \in T$ is at level α, represent x by the $(\alpha + 1)$-sequence $\langle f(x_0), f(x_1), \ldots, f(x_\xi), \ldots, f(x) \rangle$, where $x_0 < x_1 < \cdots < x_\xi < \cdots$ are the predecessors of x.]

Exercise 24.2. If T is a special Aronszajn tree, then the set of all successor nodes of T is a regular Aronszajn tree.

Lemma 24.1. *If T is an Aronszajn tree, then the following are equivalent:*

(i) *T is regular*

(ii) *T is the union of countably many antichains.*

(iii) *there exists a function* π *from* T *into the set* Q *of all rationals, such that*

$$x < y \quad \textit{implies} \quad \pi(x) < \pi(y) \tag{24.2}$$

Proof. (i) → (ii): Let $f\colon T \to \omega$ be a function that satisfies (24.1). For each natural number n, we let $A_n = \{x \in T : f(x) = n\}$. Then clearly each A_n is antichain and $T = \bigcup_{n=0}^{\infty} A_n$.

(ii) → (iii): Let $T = \bigcup_{n=0}^{\infty} A_n$, where each A_n is an antichain. We define $\pi\colon T \to Q$ by induction on the A_n. First we let $\pi(x) = 0$ for all $x \in A_0$. Having defined $\pi(x)$ for all $x \in \bigcup_{k=0}^{k=n} A_n$, such that π is order-preserving, we can easily define $\pi(x)$ for all $x \in A_{n+1}$ such that π stays order-preserving.

(iii) → (i) is obvious. ■

Notice that by (ii), a regular Aronszajn tree is not a Suslin tree. Thus the following theorem is a strong refutation of existence of Suslin trees from $\mathrm{MA}_{\aleph_1}$.

Theorem 54. *If* $\mathrm{MA}_{\aleph_1}$ *holds, then every Aronszajn tree is regular.*

Given an Aronszajn tree T, we shall use $\mathrm{MA}_{\aleph_1}$ to construct a function $f\colon T \to \omega$ that will attest to the regularity of T. First we prove the following lemma.

Lemma 24.2. *If* T *is an Aronszajn tree and* W *is an uncountable collection of finite pairwise disjoint subsets of* T, *then there exist* $S, S' \in W$ *such that any* $x \in S$ *is incomparable with any* $y \in S'$.

Proof. Since uncountably many elements of W have the same size, we may as well assume that there exists a natural number n such that $|S| = n$ for all $S \in W$; furthermore, let us consider a fixed enumeration $\{z_1, \ldots, z_n\}$ of each set $S \in W$. Let D be a uniform ultrafilter over W, i.e., an ultrafilter such that every $X \in D$ is uncountable.

Let us assume that the lemma is false. For each $x \in T$ and each $k = 1, \ldots, n$, let $Y_{x,k}$ be the set of all $S \in W$ such that x is comparable with the kth element of S. Since any S and S' contain comparable elements, we have

$$\bigcup_{x \in S} \bigcup_{k=1}^{k=n} Y_{xk} = W$$

for every $S \in W$. Thus pick, for each $S \in W$, an element $x = x_S$ of S and $k = k_S$ such that $Y_{x,k} \in D$. Now, there is $k \le n$ such that the set $Z = \{S \in W : k_S = k\}$ is uncountable. We shall show that the elements x_S, $S \in Z$, are pairwise comparable; and that will be a contradiction since T has no uncountable branch.

If $S_1, S_2 \in Z$ and $x = x_{S_1}$, $y = x_{S_2}$, then the set $Y = Y_{xk} \cap Y_{yk}$ is in the ultrafilter and thus uncountable. If $S \in Y$, then the kth element of S is comparable with both x and y. Since Y is uncountable, there must exist $S \in Y$ such that the kth element of S is *greater* than both x and y. But then it follows that x and y are comparable. ■

To prove Theorem 54 let T be an Aronszajn tree and let us consider the following notion of forcing $(P, <)$: forcing conditions are functions p such that:

(24.3) (i) $\operatorname{dom}(p)$ is a finite subset of T;
(ii) $\operatorname{ran}(p) \subset \omega$;
(iii) if $x, y \in \operatorname{dom}(p)$ and x and y are comparable, then $p(x) \neq p(y)$.

p is stronger than q just in case p extends q.

Lemma 24.3. *$(P, <)$ satisfies the countable chain condition.*

Proof. Let W be an uncountable subset of P. Note that the set $\{\operatorname{dom}(p) : p \in W\}$ is uncountable (there are only countably many functions from a finite set into ω). By the Δ-lemma (Lemma 22.6), there is an uncountable $W_1 \subset W$, and a finite set $S \subset T$ such that $\operatorname{dom}(p) \cap \operatorname{dom}(q) = S$ for any distinct elements p, $q \in W_1$. Then there is an uncountable $W_2 \subset W_1$ such that $p|S = q|S$ for any p, $q \in W_2$. By Lemma 24.4, there exist p and $q \in W_2$ such that any $x \in \operatorname{dom}(p) - S$ is incomparable with any $y \in \operatorname{dom}(q) - S$. Then $p \cup q$ is clearly a function that satisfies (24.3) and extends both p and q. Thus p and q are compatible elements of W and so $(P, <)$ satisfies the c.c.c. ■

For each $x \in T$, let D_x be the set of all $p \in P$ such that $x \in \operatorname{dom}(p)$; clearly, each D_x is dense in P. Let $\mathscr{D} = \{D_x : x \in T\}$.

It follows from $\mathrm{MA}_{\aleph_1}$, that $(P, <)$ has a $\mathscr{D}$-generic filter G. The elements of G are pairwise compatible and since G is $\mathscr{D}$-generic, every $x \in T$ is in the domain of the function $f = \bigcup G$. The function f maps T into ω and satisfies (24.1). Thus T is a regular Aronszajn tree and the proof of Theorem 54 is complete. ■

A few more facts on Aronszajn trees:

*Exercise 24.3.** The following are equivalent for any Aronszajn tree T: (i) T is special, (ii) the set of all successor nodes of T is the union of countably many antichains, and (iii) there exists a function π from T into the reals such that $x < y$ implies $\pi(x) < \pi(y)$.

In the next exercise, P_T denotes the partially ordered set obtained from $(T, <)$ by reversing the order.

*Exercise 24.4.** The following are equivalent for any normal Aronszajn tree T: (i) there are antichains $\{A_n\}_{n=0}^{\infty}$ such that $\bigcup_{n=0}^{\infty} A_n$ is cofinal in T; (ii) whenever $G \subset P_T$ is $\mathfrak{M}$-generic, then $\aleph_1^{\mathfrak{M}}$ is countable in $\mathfrak{M}[G]$; and (iii) whenever $G \subset P_T$ is $\mathfrak{M}$-generic, then $\mathfrak{M}[G]$ contains a subset ω that is not in $\mathfrak{M}$.

A last word on Aronszajn trees. A simple modification of the construction in Lemma 22.3 gives a homogeneous Aronszajn tree. A construction of a rigid Aronszajn tree is more difficult but possible (a result of Baumgartner).

We shall consider a generalization of Aronszajn trees to higher cardinals later, in the chapter on large cardinal properties.

A tree of height ω_1 whose levels are countable can have at most $2^{\aleph_1}$ uncountable branches. Aronszajn (and Suslin) trees are examples of trees without uncountable branches. While it is easy to construct a tree with countable levels and at most $\aleph_1$ uncountable branches (see Exercise 24.5), existence of such trees with more than $\aleph_1$ branches turns out to be an interesting problem.

Exercise 24.5. Construct a normal tree of height ω_1 which has exactly $\aleph_1$ uncountable branches.

[Let the tree consist of functions $t: \alpha \to \omega$ such that $t(\xi) = 0$ except for finitely many ξ's.]

A tree $(T, <)$ is a *Kurepa tree* if:

(24.4) (i) height $(T) = \omega_1$;
(ii) each level of T is at most countable;
(iii) T has at least $\aleph_2$ uncountable branches.

Theorem 55. *Existence of Kurepa trees is independent of the axioms of set theory:* (a) *There is a model of* ZFC *in which a Kurepa tree exists.* (b) *If there is an inaccessible cardinal, then there is a model of* ZFC *in which there are no Kurepa trees.*

I shall give two examples of generic models; in one, Kurepa trees exist, and in the other, Kurepa trees do not exist. In fact, Kurepa trees exist in the constructible universe, and I shall sketch the construction afterward. The assumption of an inaccessible cardinal in (b) is necessary since one can show that if there is no Kurepa tree, then $\aleph_2$ is inaccessible in L.

First we construct a generic model of ZFC in which there is a Kurepa tree. Given a ground model $\mathfrak{M}$ that satisfies $2^{\aleph_0} = \aleph_1$, we extend it to a model $\mathfrak{M}[G]$ such that $\mathfrak{M}[G]$ has the same cardinals as $\mathfrak{M}$ and contains a Kurepa tree.

To define the notion of forcing $(P, <)$, let us consider the normal α-trees $(\alpha < \omega_1)$, described in (22.6). A condition is a pair $p = (T, f)$, where:

(24.5) (i) T is a normal α-tree, for some $\alpha < \omega_1$;
(ii) f is a one-to-one function whose domain is a countable subset of ω_2 and such that for each $i \in \operatorname{dom}(f)$, $f(i)$ is an α-branch in T.

We let $(T', f') \leq (T, f)$ if:

(24.6) (i) T' is an extension of T, i.e., $\exists\alpha \leq \text{height}(T')$ such that $T = \{t|\alpha : t \in T'\}$;
(ii) $\operatorname{dom}(f') \supseteq \operatorname{dom}(f)$, and for every $i \in \operatorname{dom}(f)$, $f'(i) \supseteq f(i)$.

Let G be an $\mathfrak{M}$-generic set of conditions and let

$$\mathcal{T} = \bigcup \{T : (T, f) \in G \text{ for some } f\}$$

We shall show that in $\mathfrak{M}[G]$ $\mathcal{T}$ is a Kurepa tree.

The same argument that I gave in the generic construction of a Suslin tree shows that $\mathscr{T}$ is a normal tree of height ω_1. Since for every $i \in \omega_2$, the set of conditions (T, f) such that $i \in \operatorname{dom}(f)$ is dense, it follows that from the generic set G one obtains a one-to-one function F such that $\operatorname{dom}(F) = \omega_2$ and each $F(i)$ is an ω_1-branch in $\mathscr{T}$. Thus $\mathscr{T}$ is a Kurepa tree, provided ω_1 and ω_2 are cardinals in $\mathfrak{M}[G]$.

The notion of forcing $(P, <)$ is clearly $\aleph_0$-closed and hence $\aleph_1$ remains a cardinal. In order to show that all cardinals $\geq \aleph_2$ are preserved, we prove that $(P, <)$ satisfies the $\aleph_2$-chain condition. This is proved using a generalization of the Δ-lemma (Lemma 22.6):

Lemma 24.4. *Assume $2^{\aleph_0} = \aleph_1$. Let W be a collection of countable sets, such that $|W| = \aleph_2$. Then there exists $Z \subset W$ such that $|Z| = \aleph_2$, and a set A, $|A| \leq \aleph_0$, such that $X \cap Y = A$ for any distinct elements X, $Y \in Z$.*

Proof. It is not very difficult to give a direct generalization of the proof of Lemma 22.6. Let me give an alternative proof (which can also be used to prove the Δ-lemma).

Since $|\bigcup W| = \aleph_2$, we may as well assume that $X \subset \omega_2$ for each $X \in W$. Let us enumerate W: $W = \{X_\alpha : \alpha < \omega_2\}$. Let $S_0 = \{\alpha < \omega_2 : \operatorname{cf} \alpha = \omega_1\}$. For each $\alpha \in S_0$ there is $\beta < \alpha$ such that $X_\alpha \cap \alpha \subset \beta$; call this $\beta = f(\alpha)$. The function f is regressive on the stationary set S_0, and so there is $\gamma < \omega_2$ and a stationary set $S \subset S_0$ such that $X_\alpha \cap \alpha \subset \gamma$ for all $\alpha \in S$. By the continuum hypothesis, there are only $\aleph_1$ countable subsets of γ, and so there is a set $U \subset S$ of size $\aleph_2$ and $A \subset \gamma$ such that $X_\alpha \cap \alpha = A$ for every $\alpha \in U$. Then one can easily construct (by transfinite induction) a set $Z \subset U$ of size $\aleph_2$ such that $X_\alpha \cap X_\beta = A$ for any distinct $\alpha, \beta \in Z$. ■

Now it is easy to show that the notion of forcing $(P, <)$ defined in (24.5) satisfies the $\aleph_2$-c.c. (provided $2^{\aleph_0} = \aleph_1$ holds in $\mathfrak{M}$): Let W be a subset of P, $|W| = \aleph_2$. The set $\{\operatorname{dom}(f) : (T, f) \in W\}$ has size $\aleph_2$ by the continuum hypothesis; there are only $\aleph_1$ normal trees of height $< \omega_1$, and $\aleph_1$ functions from a countable set into a set of size $\aleph_1$). By Lemma 24.4 there exists $W_1 \subset W$ of size $\aleph_2$, and a countable $A \subset \omega_2$ such that $\operatorname{dom}(f) \cap \operatorname{dom}(f') = A$ for any distinct elements (T, f), $(T', f') \in W_1$. Then there is $W_2 \subset W_1$ of size $\aleph_2$ such that $T = T'$ and $f | A = f' | A$ for any (T, f), $(T', f') \in W_2$. Clearly, any two members of W_2 are compatible. ■

Next we construct a generic model in which there is no Kurepa tree. Let $\mathfrak{M}$ be a model of ZFC and assume that $\mathfrak{M}$ satisfies "there exists an inaccessible cardinal λ." We shall show that in the model $\mathfrak{M}[G]$ obtained by Lévy collapse that makes $\aleph_2^{\mathfrak{M}[G]} = \lambda$, Kurepa trees do not exist.

In the definition (20.7) we let $\kappa = \aleph_1$; thus the notion of forcing $(P, <)$ consists of functions p on subsets of $\lambda \times \omega_1$ such that:

(24.7) (i) $\operatorname{dom}(p)$ is countable;
(ii) $p(\alpha, \xi) < \alpha$ for each $(\alpha, \xi) \in \operatorname{dom}(p)$.

As usual, $p < q$ iff $p \supset q$.

$(P, <)$ is $\aleph_0$-closed, and so $\mathfrak{M}$ and $\mathfrak{M}[G]$ have the same countable sequences of elements of $\mathfrak{M}$. Also, $\aleph_1^{\mathfrak{M}[G]} = \aleph_1$, and $\aleph_2^{\mathfrak{M}[G]} = \lambda$. The following lemma plays the key role in the proof:

Lemma 24.5. *If P is an $\aleph_0$-closed notion of forcing and T is an ω_1-tree in the ground model $\mathfrak{M}$ such that every level of T is countable, then in $\mathfrak{M}[G]$ T has no other branches than those that are in $\mathfrak{M}$.*

Proof. Assume that T has a branch $b \in \mathfrak{M}[G]$ that is not in $\mathfrak{M}$; since $\mathfrak{M}[G]$ has no new countable sets, b has length ω_1. There is a name $\boldsymbol{b}$ for b and a condition $p_0 \in G$ such that

$$p_0 \Vdash \boldsymbol{b} \neq \check{a} \qquad \text{for all} \quad a \in \mathfrak{M} \tag{24.8}$$

We construct, by induction, conditions $p_s < p_0$ and nodes $x_s \in T$ for all finite sequences s of 0's and 1's. Having constructed p_s, we can find (because of (24.8)) two incomparable nodes x_{s0} and x_{s1}, both $> x_s$, and two conditions p_{s0} and p_{s1}, both stronger than p_s such that $p_{s0} \Vdash \check{x}_{s0} \in \boldsymbol{b}$ and $p_{s1} \Vdash \check{x}_{s1} \in \boldsymbol{b}$. Moreover, we can find such x_{s0} and x_{s1} at the same level of T. Let $\alpha < \omega_1$ be such that all x_s lie below level α in T. For each $f\colon \omega \to \{0, 1\}$, let p_f be a condition stronger than all $p_{f|n}$, $n \in \omega$. Since $p_0 \Vdash \boldsymbol{b}$ is uncountable, there exists $q < p_f$ and x_f at the αth level of T such that $q \Vdash \check{x}_f \in \boldsymbol{b}$. Now it is clear that $x_f \neq x_g$ whenever f and g are distinct 0, 1-functions on ω. Thus the αth level of T has at least $2^{\aleph_0}$ elements, contrary to our assumption. ■

It follows immediately from the lemma that in the model $\mathfrak{M}[G]$ (given by (24.7)), no tree $T \in \mathfrak{M}$ whose levels are countable can be a Kurepa tree: since every branch of T in $\mathfrak{M}[G]$ is in $\mathfrak{M}$, T has at most $(2^{\aleph_1})^{\mathfrak{M}}$ branches, but $(2^{\aleph_1})^{\mathfrak{M}} < \lambda = \aleph_2^{\mathfrak{M}[G]}$, and so T has (in $\mathfrak{M}[G]$) less than $\aleph_2$ branches.

A similar argument can be used for any tree in $\mathfrak{M}[G]$, with a slight modification. For each $\alpha < \lambda$, let P_α denote the set of all conditions whose domain is a subset of $\alpha \times \omega_1$; similarly, let $P^\alpha = \{p \in P : \operatorname{dom}(p) \subset (\kappa - \alpha) \times \omega_1\}$. Clearly, P is (isomorphic to) the product $P_\alpha \times P^\alpha$. Let $X \in \mathfrak{M}[G]$ be a subset of ω_1, and let $\boldsymbol{X}$ be a name of X; since P satisfies the λ-c.c., there exists for each $\xi < \omega_1$ a set of conditions $W_\xi \subset P$ of size less than λ such that $\|\check{\xi} \in \boldsymbol{X}\| = \sum \{p : p \in W_\xi\}$. There exists $\alpha < \lambda$ such that $W_\xi \subset P_\alpha$, for all $\xi < \omega_1$. It follows that $X \in \mathfrak{M}[G \cap P_\alpha]$.

Now let $T \in \mathfrak{M}[G]$ be an ω_1-tree with countable levels. Since T can be considered a binary relation on ω_1, there exists $\alpha < \lambda$ such that $T \in \mathfrak{M}[G \cap P_\alpha]$. By the product lemma 20.1, $G \cap P^\alpha$ is P^α-generic over $\mathfrak{M}[G \cap P_\alpha]$ and $\mathfrak{M}[G] = \mathfrak{M}[G \cap P_\alpha][G \cap P^\alpha]$. Since $\mathfrak{M}[G \cap P_\alpha]$ and $\mathfrak{M}$ have the same countable sequences in $\mathfrak{M}$, it follows that P^α is $\aleph_0$-closed not only in $\mathfrak{M}$, but also $\mathfrak{M}[G \cap P_\alpha] \vDash P^\alpha$ is $\aleph_0$-closed. Thus Lemma 24.5 applies and every branch of T in $\mathfrak{M}[G]$ is in $\mathfrak{M}[G \cap P_\alpha]$. However, $(2^{\aleph_1})^{\mathfrak{M}[G \cap P_\alpha]} < \lambda = \aleph_2^{\mathfrak{M}[G]}$, and so T is not a Kurepa tree in $\mathfrak{M}[G]$. This completes the proof of Theorem 55.
■

As I mentioned earlier, $V = L$ implies that Kurepa trees exist, and I shall now sketch the construction in L. But first let me give another formulation of the problem.

Lemma 24.6. *A Kurepa tree exists if and only if there exists a family $\mathscr{F}$ of subsets of ω_1 such that:*

(24.9) (i) $|\mathscr{F}| \geq \aleph_2$;
(ii) for each $\alpha < \omega_1$, $|\{X \cap \alpha : X \in \mathscr{F}\}| \leq \aleph_0$.

Proof. (a) Let $(T, <_T)$ be a Kurepa tree. Since T has size $\aleph_1$, we may assume that $T = \omega_1$, and moreover that $\alpha < \beta$ whenever $\alpha <_T \beta$. If we let $\mathscr{F}$ be the family of all ω_1-branches of T, then $\mathscr{F}$ satisfies (24.9).

(b) Let $\mathscr{F}$ be a family of subsets of ω_1 such that (24.9) holds. For each $X \in \mathscr{F}$, let f_X be the function on ω_1 defined by

$$f_X(\alpha) = X \cap \alpha \qquad (\alpha < \omega_1)$$

For each $\alpha < \omega_1$, let

$$U_\alpha = \{f_X | \alpha : X \in \mathscr{F}\}$$

and let $T = \bigcup_{\alpha<\omega_1} U_\alpha$. Then $(T, \subset)$ is a tree, the U_α are the levels of T and the functions f_X correspond to branches of T. By (24.9)(ii), every U_α is countable, and it follows that T is a Kurepa tree. ■

Now assume $V = L$. I shall construct a family $\mathscr{F}$ of subsets of ω_1 that satisfies (24.9). I present this construction to give the reader a taste of the methods used in the theory of constructible sets. The theory ("the fine structure of L"), a mixture of model theory and recursion theory, is a powerful technique that is used to prove various combinatorial principles in L. (I shall introduce some of them below.) The culmination of the theory is Jensen's covering theorem, which will be discussed in Section 30.

We shall use the facts on models $\langle L_\alpha, \in\rangle$ mentioned in Section 13. In particular, we recall (Exercise 13.14) that if L_α is a model of ZF$^-$, then for every $X \subset L_\alpha$ there is a least elementary submodel $M \prec L_\alpha$ such that $X \subseteq M$. Also (see Exercise 13.16), if $M \prec \langle L_{\omega_2}, \in\rangle$, then $\omega_1 \cap M = \alpha$ for some $\alpha \leq \omega_1$. We also use the fact (from model theory) that elementary submodels of a given model are closed under unions of chains.

For each $\alpha < \omega_1$, there is a least elementary submodel M of $\langle L_{\omega_1}, \in\rangle$ such that $\alpha \in M$. Moreover (see Exercise 13.15), $M = L_\gamma$ for some $\gamma < \omega_1$, and we let $\gamma = f(\alpha)$:

$$f(\alpha) = \text{least } \gamma \text{ such that } \alpha \in L_\gamma \prec \langle L_{\omega_1}, \in\rangle \tag{24.10}$$

Let $\mathscr{F}$ be the following family of subsets of ω_1:

$$\mathscr{F} = \{X \subseteq \omega_1 : X \cap \alpha \in L_{f(\alpha)} \text{ for every } \alpha < \omega_1\} \tag{24.11}$$

It is immediately clear that $\{X \cap \alpha : X \in \mathscr{F}\}$ is countable for each $\alpha < \omega_1$; and hence if we show that $|\mathscr{F}| = \aleph_2$, $\mathscr{F}$ will satisfy (24.9).

Assume that $|\mathscr{F}| \leq \aleph_1$. Then $\mathscr{F}$ has an enumeration

$$C = \langle X_\xi : \xi < \omega_1 \rangle \tag{24.12}$$

and any such enumeration is in L_{ω_2}. If we let C be the $<_L$-least such C in L_{ω_2}, then since the function f is a definable element of L_{ω_2} (by the definition (24.10)) and the X_ξ satisfy (24.11) in $\langle L_{\omega_2}, \in \rangle$, it follows that C is a definable element of $\langle L_{\omega_2}, \in \rangle$.

Now, we construct an elementary chain of submodels of $\langle L_{\omega_2}, \in \rangle$:

$$N_0 \prec N_1 \prec \cdots \prec N_\nu \prec \cdots \prec \langle L_{\omega_2}, \in \rangle \qquad (\nu < \omega_1)$$

as follows: N_0 is the least elementary submodel of L_{ω_2}; $N_{\nu+1}$ is the least $N \prec L_{\omega_2}$ such that $N_\nu \subset N$ and $N_\nu \in N$; if η is a limit ordinal, then $N_\eta = \bigcup_{\nu < \eta} N_\nu$. Note that each N_ν is countable, and $\omega_1 \cap N_\nu = \alpha_\nu$, for some $\alpha_\nu < \omega_1$. Moreover,

$$\langle \alpha_\nu : \nu < \omega_1 \rangle \tag{24.13}$$

is a continuous increasing sequence of countable ordinals.

Now, we let $X = \{\alpha_\nu : \alpha_\nu \notin X_\nu\}$. Obviously, $X \neq X_\xi$ for all $\xi < \omega_1$, and we shall show that X satisfies the condition in (24.11), which will contradict the assumption that (24.12) is an enumeration of *all* elements of $\mathscr{F}$.

We want to show that $X \cap \alpha \in L_{f(\alpha)}$ for all $\alpha < \omega_1$. By induction on α, if α is not a limit point of the sequence (24.13), then let α_ν be the largest $\alpha_\nu < \alpha$. Then either $X \cap \alpha = X \cap \alpha_\nu$ or $X \cap \alpha = (X \cap \alpha_\nu) \cup \{\alpha_\nu\}$;in either case, since $X \cap \alpha_\nu \in L_{f(\alpha_\nu)} \subset L_{f(\alpha)}$ (by the induction hypothesis), we have $X \cap \alpha \in L_{f(\alpha)}$. Thus it suffices to show that $X \cap \alpha_\eta \in L_{f(\alpha_\eta)}$ whenever η is a limit ordinal.

We shall show that

$$\text{(24.14)} \quad \text{(i)} \ \langle \alpha_\nu : \nu < \eta \rangle \in L_{f(\alpha_\eta)}; \qquad \text{(ii)} \ \langle X_\xi \cap \alpha_\eta : \xi < \alpha_\eta \rangle \in L_{f(\alpha_\eta)}.$$

Since $L_{f(\alpha_\eta)}$ is a model of ZF$^-$, the set $X \cap \alpha_\eta$ has then the following definition in $L_{f(\alpha_\eta)}$:

$$X \cap \alpha = \{\alpha_\nu : \nu < \eta \text{ and } \alpha_\nu \notin X_\nu \cap \alpha_\eta\}$$

For each $\nu < \omega_1$, let π_ν be the transitive collapse of N_ν. Each N_ν is isomorphic to some $L_{\delta(\nu)}$, and since $\omega_1 \cap N_\nu = \alpha_\nu$, we have $\pi_\nu(\omega_1) = \alpha_\nu$. Since C is a definable element of L_{ω_2}, we have $C \in N_\nu$ for all ν and one can see that $\pi_\nu(C) = \langle X_\xi \cap \alpha_\nu : \xi < \alpha_\nu \rangle$.

Note that α_η is uncountable in $L_{\delta(\eta)}$, while it is countable in $L_{f(\alpha_\eta)}$. It follows that $\delta(\eta) < f(\alpha_\eta)$, and we have $\pi_\eta(C) \in L_{\delta(\eta)} \subset L_{f(\alpha_\eta)}$, which proves (ii) in (24.14).

To prove (i), let us construct, inside $L_{f(\alpha_\eta)}$ (which is a model of ZF$^-$), an elementary chain N'_ν, $\nu < \eta$ of submodels of $\langle L_{\delta_\eta}, \in \rangle$: N'_0 is the least elementary submodel of L_{δ_η}, $N'_{\nu+1}$ is the least $N \prec L_{\delta_\eta}$ such that $N'_\nu \cup \{N'_\nu\} \subset N$, etc. It is not difficult to show, by induction on $\nu < \eta$, that for each ν, N'_ν is isomorphic to N_ν. Then if π'_ν denotes the transitive collapse of N'_ν, we have $\pi'_\nu[N'_\nu] = L_{\delta(\nu)}$, and so $\langle L_{\delta(\nu)} : \nu < \eta \rangle \in L_{f(\alpha_\eta)}$. It follows that $\langle \alpha_\nu : \nu < \eta \rangle \in L_{f(\alpha_\eta)}$, and we have (i). ■

It should be mentioned that a modification of the above construction proves that if $V = L[A]$ where $A \subseteq \omega_1$, then a Kurepa tree exists. This fact can be used to show that the assumption of an inaccessible cardinal in part (b) of Theorem 55 is necessary:

If ω_2 is not an inaccessible cardinal in L, then there exists $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$ (see Exercise 15.8). There exists a tree $T \in L[A]$ that is a Kurepa tree in $L[A]$. It easily follows that T is a Kurepa tree in the universe (it has height $\aleph_1$, countable levels, and at least $\aleph_2$ ω_1-branches). Thus Kurepa trees exist unless $\aleph_2$ is inaccessible in L.

Almost Disjoint Subsets of ω_1

If A_1, A_2 are subsets of ω_1, then we say that A_1 and A_2 are *almost disjoint* if $|A_1 \cap A_2| \leq \aleph_0$. This definition generalizes the notion of almost disjoint subsets of ω introduced in Section 23. In Lemma 23.9 we showed that there exists an almost disjoint family of infinite subsets of ω, of size $2^{\aleph_0}$. The counterpart for ω_1 is true in case the continuum hypothesis holds:

Exercise 24.6. If $2^{\aleph_0} = \aleph_1$, then there exists a family of $2^{\aleph_1}$ pairwise almost disjoint uncountable subsets of ω_1.

[As in Lemma 23.9, let $S = \bigcup_{\alpha<\omega_1} {}^{\alpha}\{0, 1\}$; $|S| = \aleph_1$.]

If the continuum hypothesis fails, the problem becomes more interesting. To start, we have

Lemma 24.7. *There exists a family of $\aleph_2$ pairwise almost disjoint uncountable subsets of ω_1.*

Proof. Let me prove a little stronger statement: *There exists an almost disjoint family of functions $f: \omega_1 \to \omega_1$, of size $\aleph_2$.* [We recall (cf. Section 8) that f, g are *almost disjoint* if $f(\alpha) \neq g(\alpha)$ for all but countably many α's.] Since functions $f: \omega_1 \to \omega_1$ are uncountable subsets of $\omega_1 \times \omega_1$ and since if f, g are almost disjoint, then $|f \cap g| \leq \aleph_0$, the lemma will follow.

It suffices to show that if given $\aleph_1$ almost disjoint functions $\{f_\nu : \nu < \omega_1\}$, then there exists $f: \omega_1 \to \omega_1$ almost disjoint from all f_ν, $\nu < \omega_1$; this we do by simple diagonalization: Let $f(\alpha) \neq$ all $f_\nu(\alpha)$, $\nu < \alpha$. ■

If T is a Kurepa tree, then the family of all ω_1-branches is an almost disjoint family of uncountable subsets of T. In fact, since the levels of T are countable, we can identify the ω_1-branches with functions from ω_1 into ω and get the following result: *There exists an almost disjoint family of $\aleph_2$ functions $f: \omega_1 \to \omega$.*

Exercise 24.7. Given a family $\mathscr{F}$ of $\aleph_2$ almost disjoint functions $f: \omega_1 \to \omega$, there exists a collection $\mathscr{S}$ of $\aleph_2$ pairwise almost disjoint *stationary* subsets of ω_1.

[Each $f \in \mathscr{F}$ is constant on a stationary set S_f with value n_f. There is $\mathscr{G} \subseteq \mathscr{F}$ of size $\aleph_2$ such that n_f is the same for all $f \in \mathscr{G}$. Let $\mathscr{S} = \{S_f : f \in \mathscr{G}\}$.]

I shall return to the problem of size of almost disjoint families in the section on saturation of ideals (Section 35). One of the results to be proved there is the following: if the ground model $\mathfrak{M}$ satisfies GCH and if $\mathfrak{M}[G]$ is obtained by adjoining any number of Cohen generic reals, then any almost disjoint family of uncountable subsets of ω_1 in $\mathfrak{M}[G]$ has size at most $\aleph_2$.

Now let T be a tree with $\aleph_1$ nodes, of height ω_1; we do not require that T has countable levels; let us ask whether T can have more than $\aleph_1$ ω_1-branches. As in Lemma 24.6, we have:

Exercise 24.8. Let $\lambda \geq \aleph_2$. There is a tree T of size $\aleph_1$ and height ω_1 with λ ω_1-branches if and only if there is a family $\mathscr{F}$ of subsets of ω_1 such that $|\mathscr{F}| = \lambda$ and

$$|\{X \cap \alpha : X \in \mathscr{F} \text{ and } \alpha < \omega_1\}| = \aleph_1$$

Note that if $2^{\aleph_0} = \aleph_1$, then there is such a tree with $2^{\aleph_1}$ ω_1-branches (let $T = \bigcup_{\alpha<\omega_1} {}^{\alpha}\{0, 1\}$). Note also that if there is a tree like that with λ ω_1-branches, then there exists an almost disjoint family of λ functions $f: \omega_1 \to \omega_1$ (and hence an almost disjoint family of λ uncountable subsets of ω_1). However, there exists a generic model (using an inaccessible cardinal) in which $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, and not only is there no Kurepa tree, but *no* tree of size $\aleph_1$ has $\aleph_2$ ω_1-branches.

Finally, if T is an ω_1-tree of size $\aleph_1$, let us ask how many branches, countable or uncountable, T can have. This problem is related to a problem on linearly ordered sets. It can be shown that: There exists a tree of height ω_1 with $\aleph_1$ nodes and at least λ branches if and only if there exists a dense linearly ordered set of size λ with a dense subset of size $\aleph_1$.

Exercise 24.9. There is always a tree of height ω_1 with $\aleph_1$ nodes and $\aleph_2$ branches.

[Distinguish two cases: Either $2^{\aleph_0} = \aleph_1$, or $2^{\aleph_0} \geq \aleph_2$.]

But even in this case T does not necessarily have $2^{\aleph_1}$ branches. Let $\mathfrak{M}[G]$ be the generic model obtained from a model $\mathfrak{M}$ of GCH by adjoining $\aleph_{\omega_1}$ Cohen generic reals. In $\mathfrak{M}[G]$ any tree T of size $\aleph_1$ and height $\aleph_1$ has at most $2^{\aleph_0} = \aleph_{\omega_1}$ countable branches. Moreover, T has at most $\aleph_2$ ω_1-branches since by the result mentioned above, any family of uncountable almost disjoint subsets of ω_1 in $\mathfrak{M}[G]$ has size at most $\aleph_2$. However, $2^{\aleph_1} > \aleph_{\omega_1}$ (because $\operatorname{cf} 2^{\aleph_1} > \aleph_1$), and so T has less than $2^{\aleph_1}$ branches. It follows that in $\mathfrak{M}[G]$ there is no dense linearly ordered set of size $2^{\aleph_1}$ with a dense subset of size $\aleph_1$.

Some More Combinatorial Principles in L

We have so far encountered two combinatorial principles true in L: the diamond and the existence of a Kurepa tree. I shall state below several other principles that have been used in combinatorial set theory, model theory, and algebra and that can be proved in L (although the proofs are in general more complicated than the proofs given above).

Let me start with an easy generalization of $\Diamond$:

(24.15) Given $\aleph_1$ stationary subsets E_ν, $\nu < \omega_1$, of ω_1, there exists a sequence of sets S_α, $\alpha < \omega_1$, such that for every $X \subseteq \omega_1$, and every $\nu < \omega_1$, the set

$$\{\alpha \in E_\nu : X \cap \alpha = S_\alpha\}$$

is stationary.

Exercise 24.10.* If $V = L$, then (24.15) holds.

[Modify the proof of Lemma 22.9. To define S_α, take into account the sets E_ν, $\nu < \alpha$.]

Exercise 24.11. Assume that (24.15) holds. If E_ν, $\nu < \omega_1$, are stationary subsets of ω_1, then there exist pairwise disjoint stationary sets F_ν, $\nu < \omega_1$, such that $F_\nu \subseteq E_\nu$ for all $\nu < \omega_1$.

[Let X_ν, $\nu < \omega_1$, be distinct subsets of ω_1. Let $F'_\nu = \{\alpha \in E_\nu : X_\nu \cap \alpha = S_\alpha\}$; the sets F'_ν are pairwise almost disjoint, so let $F_\nu = F'_\nu - \bigcup_{\eta<\nu} F'_\eta$.]

The following generalization of a Kurepa tree is quite useful:

(24.16) **Silver's Principle** W. There exists a family $\mathcal{F}$ of subsets of ω_1, and a sequence W_α, $\alpha < \omega_1$, of sets such that:

(i) $|\mathcal{F}| \geq \aleph_2$;
(ii) for each $\alpha < \omega_1$, $|\{X \cap \alpha : X \in \mathcal{F}\}| \leq \aleph_0$;
(iii) for each $\alpha < \omega_1$, $|W_\alpha| \leq \aleph_0$;
(iv) for every countable $C \subset \mathcal{F}$, there exists $\alpha_0 < \omega_1$ such that for all $\alpha \geq \alpha_0$, $\{X \cap \alpha : X \in C\} \in W_\alpha$.

Clauses (i) and (ii) just assert that there exists a Kurepa tree (see (24.9)). Each W_α is a countable collection of subsets of the αth level of the Kurepa tree.

The proof of (24.16) in L is quite complicated. It is easier to obtain (24.16) by forcing:

Exercise 24.12.* If $\mathfrak{M}$ has a family $\mathcal{F}$ that satisfies (24.9), then there is a $\aleph_0$-closed notion of forcing such that $\mathfrak{M}[G]$ satisfies Silver's principle W.

[A forcing condition is a pair $(\mathcal{W}, \mathcal{C})$, where $\mathcal{W} = \langle W_\xi : \xi < \alpha \rangle$ and $\mathcal{C}$ is a countable set of countable subsets of $\mathcal{F}$; $(\mathcal{W}', \mathcal{C}')$ is stronger than $(\mathcal{W}, \mathcal{C})$ if $\mathcal{W}'$ extends $\mathcal{W}$, $\mathcal{C}' \supseteq \mathcal{C}$, and for every ξ such that $\alpha \leq \xi < \alpha'$ and every $C \in \mathcal{C}$, $\{X \cap \xi : X \in C\} \in W'_\xi$.]

An example of an application of (24.16) is the following: Using $\Diamond$ and W, one can construct a Suslin algebra of cardinality $\aleph_2$. There are other applications in the partition calculus and in point set topology.

Another theorem of L is the following. It has several applications in algebra. If $\alpha < \omega_2$ is an ordinal of cofinality ω_1, then closed unbounded, stationary, and thin subsets of α are defined in the obvious way.

(24.17) There exists a stationary subset S of ω_2 such that for every $\alpha < \omega_2$ of cofinality ω_1, $S \cap \alpha$ is a thin subset of α.

Exercise 24.13.* There exists a $\aleph_0$-closed, $\aleph_1$-distributive notion of forcing such that $\mathfrak{M}[G]$ satisfies (24.17).

[A forcing condition is a function $p\colon \alpha \to \{0, 1\}$ such that $\alpha < \omega_2$, and that $\{\xi < \gamma : p(\xi) = 1\}$ is thin in γ, for every $\gamma \leq \alpha$ of cofinality ω_1; p' is stronger than p iff $p' \supseteq p$. To verify $\aleph_1$-distributivity, let $\dot{f}$ be a name for a function on ω_1 and let p_0 be a condition. Construct an ω_1-chain of conditions $p_0 \subset p_1 \subset \cdots \subset p_\alpha \subset \cdots, \alpha < \omega_1$, such that each $p_{\alpha+1}$ decides $\dot{f}(\check{\alpha})$ and that for each limit ordinal $\alpha < \omega_1$, if $\gamma_\alpha = \lim_{\xi \to \alpha} (\operatorname{dom} p_\xi)$, then $p_\alpha(\gamma_\alpha) = 0$; then $p = \bigcup_{\alpha < \omega_1} p_\alpha$ is a condition and decides $\dot{f}(\check{\alpha})$ for all $\alpha < \omega_1$.]

Finally, let us consider the following principle (also true in L) which has several applications, notably in model theory:

(24.18) **The Box Principle ($\square$):**

There exists a collection $\{C_\alpha : \alpha < \omega_2, \alpha \text{ a limit ordinal}\}$ such that for every limit $\alpha < \omega_2$,

(i) C_α is a closed unbounded subset of α;
(ii) if $\operatorname{cf} \alpha = \omega$, then $|C_\alpha| \leq \aleph_0$;
(iii) if γ is a limit point of C_α, then $C_\gamma = C_\alpha \cap \gamma$.

(Note that by (ii) and (iii), C_α has order-type ω_1 whenever $\operatorname{cf} \alpha = \omega_1$.)

Exercise 24.14.* There exists an $\aleph_0$-closed, $\aleph_1$-distributive notion of forcing such that $\mathfrak{M}[G]$ satisfies the box principle.

[A forcing condition is a collection $p = \{C_\alpha : \alpha \leq \gamma\}$, where $\gamma < \omega_2$ is a limit ordinal, and the C_α satisfy (24.18). A condition $\{C_\alpha : \alpha \leq \gamma\}$ is stronger than $\{C'_\alpha : \alpha \leq \gamma'\}$ if $\gamma \geq \gamma'$ and $C_\alpha = C'_\alpha$ for all $\alpha < \gamma'$. To verify $\aleph_1$-distributivity, let $\dot{f}$ be a name for a function on ω_1 and let p_0 be a condition. Construct an ω_1-chain of conditions $p_0 \subset p_1 \subset \cdots \subset p_\alpha \subset \cdots, \alpha < \omega_1$, such that each $p_{\alpha+1}$ decides $\dot{f}(\check{\alpha})$ and that for each limit ordinal $\alpha < \omega_1$, if $\gamma_\alpha = \lim_{\xi \to \alpha} (\operatorname{dom} p_\xi)$, then $\gamma_\alpha \in \operatorname{dom} p_\alpha$, and for each limit ordinal $\beta < \alpha$, C_{γ_β} is an initial segment of C_{γ_α}. Then if $\gamma = \lim_{\alpha \to \omega_1} \gamma_\alpha$, let $C_\gamma = \bigcup_{\alpha < \omega_1} C_{\gamma_\alpha}$ and $p = \{C_\xi : \xi \leq \gamma\}$; p is a condition and decides $\dot{f}(\check{\alpha})$ for all $\alpha < \omega_1$.]

Note that the box principle implies (24.17): For each limit ordinal $\xi < \omega_1$, let $S_\xi = \{\alpha < \omega_2 : \operatorname{cf} \alpha = \omega \text{ and } C_\alpha \text{ has order type } \xi\}$. There is ξ such that $S = S_\xi$ is stationary. If $\alpha < \omega_1$ has cofinality ω_1, then $S \cap \alpha$ is not stationary in α because if C'_α denotes the set of all limit points of C_α, then C'_α is closed unbounded in α but $S \cap C'_\alpha$ has at most one element.

Ultrafilters over ω

Since every ultrafilter over ω is a set of subsets of ω, there exist at most $2^{2^{\aleph_0}}$ ultrafilters over ω; more generally, if κ is an infinite cardinal number, then there are at most 2^{2^κ} ultrafilters over κ. The next theorem shows that the number of ultrafilters over κ is exactly 2^{2^κ}. To get a slightly stronger result, recall that an ultrafilter D over κ is *uniform* if $|X| = \kappa$ for all $X \in D$.

Theorem 56 (Pospíšil). *For every infinite cardinal κ, there exist 2^{2^κ} uniform ultrafilters over κ.*

We prove first the following lemma. Let us call a family $\mathscr{C}$ of subsets of κ *uniformly independent* if for any distinct sets $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ in $\mathscr{C}$, the intersection

$$(24.19) \qquad X_1 \cap \cdots \cap X_n \cap (\kappa - Y_1) \cap \cdots \cap (\kappa - Y_m)$$

has cardinality κ.

Lemma 24.8. *There exists a uniformly independent family of subsets of κ such that $|\mathscr{C}| = 2^\kappa$.*

Proof. Let us consider the set P of all pairs $(F, \mathscr{F})$ where F is a finite subset of κ and $\mathscr{F}$ is a finite set of finite subsets of κ. Since $|P| = \kappa$, it suffices to find a uniformly independent family $\mathscr{C}$ of subsets of P, of size 2^κ.

For each $u \subseteq \kappa$, let

$$X_u = \{(F, \mathscr{F}) \in P : F \cap u \in \mathscr{F}\}$$

and let

$$\mathscr{C} = \{X_u : u \subseteq \kappa\}$$

If u and v are distinct subsets of κ, then $X_u \neq X_v$: For example, if $\alpha \in u$ but $\alpha \notin v$, then let $F = \{\alpha\}$, $\mathscr{F} = \{F\}$, and $(F, \mathscr{F}) \in X_u$ while $(F, \mathscr{F}) \notin X_v$. Hence $|\mathscr{C}| = 2^\kappa$.

To show that $\mathscr{C}$ is uniformly independent, let $u_1, \ldots, u_n, v_1, \ldots, v_m$ be distinct subsets of κ. For each $i \leq n$ and each $j \leq m$, let α_{ij} be some element of κ such that either $\alpha_{ij} \in u_i - v_j$ or $\alpha_{ij} \in v_j - u_i$. Now let F be any finite subset of κ such that $F \supseteq \{\alpha_{ij} : i \leq n, j \leq m\}$ (note that there are κ such finite sets). Clearly, we have

$$F \cap u_i \neq F \cap v_j \qquad (\text{any } i \leq n, j \leq m)$$

Thus if we let $\mathscr{F} = \{F \cap u_i : i \leq n\}$, we have

$$(F, \mathscr{F}) \in X_{u_i}, \qquad i \leq n$$

$$(F, \mathscr{F}) \notin X_{v_j}, \qquad j \leq m$$

Consequently, the intersection

$$X_{u_1} \cap \cdots \cap X_{u_n} \cap (\kappa - X_{v_1}) \cap \cdots \cap (\kappa - X_{v_m})$$

has cardinality κ. ■

Proof of Theorem 56. Let $\mathscr{C}$ be a uniformly independent family of subsets of κ. For every function f: $\mathscr{C} \to \{0, 1\}$, consider this family of subsets of κ:

$$(24.20) \quad G_f = \{X : |\kappa - X| < \kappa\} \cup \{X : f(X) = 1\} \cup \{\kappa - X : f(X) = 0\}$$

By (24.19), the family G_f has the finite intersection property, and so there exists an ultrafilter D_f such that $D_f \supseteq G_f$. It follows from (24.20) that D_f is uniform. If $f \neq g$, then for some $X \in C$, $f(X) \neq g(X)$; e.g., $f(X) = 1$ and $g(X) = 0$ and then $X \in D_f$ while $\kappa - X \in D_g$. Thus we obtain 2^{2^κ} distinct uniform ultrafilters over κ. ■

Let D be a nonprincipal ultrafilter over ω. D is called a *p-point* if for every partition $\{A_n : n \in \omega\}$ of ω into $\aleph_0$ pieces such that $A_n \notin D$ for all n, there exists $X \in D$ such that

$$X \cap A_n \text{ is finite, for all } n \in \omega$$

The notion of a p-point comes from point set topology, where the topological properties of p-points in the space βN of all ultrafilters over ω are of some interest.

First we notice that it is easy to find an ultrafilter that is not a p-point: Let $\{A_n : n \in \omega\}$ be any partition of ω into $\aleph_0$ infinite pieces, and let F be the following filter over ω:

(24.21) $X \in F$ iff except for finitely many n, $F \cap A_n$ contains all but finitely many elements of A_n.

If D is any ultrafilter extending F, then D is not a p-point.

Theorem 57 below shows that existence of p-points follows from the continuum hypothesis. A recent result of Shelah states that there exists a model of ZFC in which there are no p-points.

Let me first give a trivial reformulation of the definition of a p-point.

Lemma 24.9. *A nonprincipal ultrafilter D over ω is a p-point if and only if it satisfies the following:*

If $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \cdots$ is a decreasing sequence of elements of D, then there exists $X \in D$ such that for each n,

$$(24.22) \quad X - A_n \textit{ is finite} \quad ■$$

(If (24.22) holds, we may say that "*X is included* in *A_n modulo a finite set*," or:

$$(24.23) \quad X \subseteq A_n \text{ mod.f.})$$

Exercise 24.15. If $(P, <)$ is a countable linearly ordered set and if D is a p-point over P, then there exists $X \in D$ such that the order type of X is either ω or ω^*. (X has order type ω^* iff $X = \{x_n\}_{n=0}^{\infty}$ and $x_0 > x_1 > \cdots > x_n > \cdots$.)

Theorem 57 (a) *If $2^{\aleph_0} = \aleph_1$, then there exists a p-point.* (b) *More generally, if Martin's axiom holds, then there exists a p-point.*

In the proof of Theorem 57 we shall use functions from ω to ω. Let us say that a function $h: \omega \to \omega$ *majorizes* a function $f: \omega \to \omega$ if $h(n) > f(n)$ for all n. A family of functions $\mathscr{H}$ *majorizes* all functions $f: \omega \to \omega$ if for every f there is $h \in \mathscr{H}$ such that h majorizes f.

An easy diagonalization shows that no countable family majorizes all functions $f: \omega \to \omega$. We can use MA to extend this observation:

Lemma 24.10.

(a) *If $\mathscr{H}$ is countable, then $\mathscr{H}$ does not majorize all functions $f: \omega \to \omega$.*
(b) *If Martin's Axiom holds and $|\mathscr{H}| < 2^{\aleph_0}$, then $\mathscr{H}$ does not majorize all functions.*

Proof. (a) If $\mathscr{H} = \{h_n\}_{n=0}^{\infty}$, let f be a function such that $f(n) \geq h_k(n)$ for all $k = 0, \ldots, n$.

(b) Let P be the collection of all finite sequences $p = \langle p(0), \ldots, p(n-1) \rangle$ of natural numbers; $p < q$ iff p extends q. The notion of forcing $(P, <)$ satisfies the c.c.c.

For each n, let $D_n = \{p \in P : n \in \operatorname{dom}(p)\}$; each D_n is dense in P. For each $h \in \mathscr{H}$, let $E_h = \{p \in P : p(n) \geq h(n) \text{ for some } n\}$; again, each E_h is dense in P.

Since $|H| < 2^{\aleph_0}$, there exists a filter G on P that intersects each D_n and each E_h. Then $f = \bigcup G$ is a function from ω to ω that is not majorized by any $h \in \mathscr{H}$. ■

The key element of the proof of Theorem 57 is the following lemma:

Lemma 24.11. *Let $\mathscr{G}$ be a family of subsets of ω with the finite intersection property and let us assume that* (a) *$\mathscr{G}$ is countable, or more generally, that* (b) MA *holds and $|\mathscr{G}| < 2^{\aleph_0}$. Let $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq$ be a decreasing sequence of elements of $\mathscr{G}$. Then there exists $Z \subseteq \omega$ such that:*

(i) *$\mathscr{G} \cup \{Z\}$ has the finite intersection property;*
(ii) *$Z \subseteq A_n$* mod.f., *for all $n \in \omega$.*

Proof. We may assume that if $X, Y \in \mathscr{G}$, then $X \cap Y \in \mathscr{G}$. For each $X \in \mathscr{G}$, let $h_X: \omega \to \omega$ be some function such that $h_X(n) \in X \cap A_n$. By Lemma 24.10, the family $\{h_X : X \in \mathscr{G}\}$ does not majorize all functions; and so there is a function $f: \omega \to \omega$ such that for every $X \in \mathscr{G}$,

$$f(n) \geq h_X(n) \qquad \text{for some } n \tag{24.24}$$

Now we let

$$Z = \bigcup_{n=0}^{\infty} \{k \in A_n : k \leq f(n)\} \tag{24.25}$$

It is readily verified that $Z - A_n$ is finite for each n, and by (24.24), $Z \cap X \neq \emptyset$ for every $X \in \mathscr{G}$. ■

Proof of Theorem 57. Let $\mathscr{A}_\alpha$, $\alpha < 2^{\aleph_0}$, be an enumeration of all decreasing sequences $\{A_n\}_{n=0}^\infty$ of subsets of ω. We construct, by induction on $\alpha < 2^{\aleph_0}$, a chain of families $\mathscr{G}_0 \subseteq \cdots \subseteq \mathscr{G}_\alpha \subseteq \cdots$ of subsets of ω, such that each $\mathscr{G}_\alpha$ is closed under finite intersections and $|\mathscr{G}_\alpha| < 2^{\aleph_0}$ for all α.

We let $\mathscr{G}_0 = \{X \subseteq \omega : \omega - X \text{ is finite}\}$. If α is a limit ordinal, we let $\mathscr{G}_\alpha = \bigcup_{\beta<\alpha} \mathscr{G}_\beta$. Having constructed $\mathscr{G}_\alpha$, we construct $\mathscr{G}_{\alpha+1}$ as follows: Let $\mathscr{A}_\alpha = \{A_n\}_{n=0}^\infty$ be a decreasing sequence of subsets of ω. If some A_n is disjoint from some $X \in \mathscr{G}_\alpha$, then we let $\mathscr{G}_{\alpha+1} = \mathscr{G}$. Otherwise, the family $\mathscr{G} = \mathscr{G}_\alpha \cup \{A_n : n \in \omega\}$ has the finite intersection property and by Lemma 24.11, there exists $Z \subseteq \omega$ such that $Z \subset A_n$ mod.f. for all n, and $\mathscr{G}' = \mathscr{G} \cup \{Z\}$ has the finite intersection property. Then we let $\mathscr{G}_{\alpha+1}$ consist of all finite intersections $X_1 \cap \cdots \cap X_k$ of elements of $\mathscr{G}'$.

Finally, we let $\mathscr{G} = \bigcup \{\mathscr{G}_\alpha : \alpha < 2^{\aleph_0}\}$, and let D be any ultrafilter such that $D \supseteq \mathscr{G}$. We claim that D is a p-point: if $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \cdots$ is any decreasing sequence of elements of D, then $\{A_n\}_{n=0}^\infty = \mathscr{A}_\alpha$ for some $\alpha < 2^{\aleph_0}$ and we have $Z \in \mathscr{G}_{\alpha+1}$ such that $Z \subseteq A_n$ mod.f. for all n. ■

Incidentally, it is easy to obtain a p-point by forcing. Let P be the set of all infinite subsets of ω and let us identify two elements of P if they differ by a finite set. (Strictly speaking, let $P = B - \{0\}$ where B is the Boolean algebra $P(\omega)$/the ideal of finite sets.) Let $X \leq Y$ just in case $X \subseteq Y$ mod.f. Note that if $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$ are infinite subsets of ω, then there exists an infinite $X \subseteq \omega$ such that $X \subseteq X_n$ mod.f. for all n.

Exercise 24.16. $(P, <)$ is an $\aleph_0$-closed notion of forcing and in $\mathfrak{M}[G]$, G is a p-point.

[If $\mathscr{A} = \{A_n\}_{n=0}^\infty$ is a partition of ω, then $\mathscr{A} \in \mathfrak{M}$; consider the dense set $\{X \in P : \text{either } X \subseteq A_n \text{ for some } n, \text{ or } X \cap A_n \text{ is finite for all } n\}$.]

The same argument gives an alternative construction of a p-point if $2^{\aleph_0} = \aleph_1$: Let $\mathscr{A}_\alpha$, $\alpha < \omega_1$, be all partitions of ω; we construct an ω_1-sequence X_α, $\alpha < \omega_1$, of infinite subsets of ω as follows: Given X_α, let $X_{\alpha+1} \subseteq X_\alpha$ be such that either $X_{\alpha+1} \subseteq A$ for some $A \in \mathscr{A}_\alpha$ or that $X_{\alpha+1} \cap A$ is finite for all $A \in \mathscr{A}_\alpha$. If α is a limit, let X_α be such that $X_\alpha \subseteq X_\beta$ mod.f. for all $\beta < \alpha$. Then we let $D = \{X : X \supseteq X_\alpha \text{ for some } \alpha < \omega_1\}$. D is a p-point.

If in the definition of a p-point, one replaces "$X \cap A_n$ is finite" by "$X \cap A_n$ has at most one element," one gets a stronger notion, known as the *Ramsey ultrafilter*. Again, Martin's axiom implies that Ramsey ultrafilters exist. Note that in Exercise 24.16 G is in fact a Ramsey ultrafilter and that the construction of a p-point outlined above can be modified to give a Ramsey ultrafilter (if $2^{\aleph_0} = \aleph_1$). In Theorem 91 in Section 38 we shall construct a model of ZFC in which there are no Ramsey ultrafilters.

Let me conclude this section with two problems that arise naturally in the preceding considerations. First, let us consider the following partial ordering on functions $f\colon \omega \to \omega$:

$$f \prec g \quad \text{iff} \quad \exists n_0\ \forall n \geq n_0\ f(n) < g(n) \tag{24.26}$$

If κ is a cardinal number, we say that $S = \{f_\alpha : \alpha < \kappa\}$ is a *κ-scale* if:

(24.27) (i) $f_\alpha \prec f_\beta$ whenever $\alpha < \beta < \kappa$;
(ii) for each $g\colon \omega \to \omega$ there exists $f_\alpha \in S$ such that $g \prec f_\alpha$.

If $\mathcal{G} \subset {}^\omega\omega$ is countable then we can easily find $f\colon \omega \to \omega$ such that $g \prec f$ for all $g \in \mathcal{G}$. It follows that if $2^{\aleph_0} = \aleph_1$, then there exists an $\aleph_1$-scale:

Let g_α, $\alpha < \omega_1$, be an enumeration of all functions $g\colon \omega \to \omega$. We construct an $\aleph_1$-scale $\{f_\alpha : \alpha < \omega_1\}$ by induction. We let f_α be such that $f_\beta \prec f_\alpha$ for all $\beta < \alpha$, and that $g_\alpha \prec f_\alpha$.

We shall see below that if Martin's axiom holds, then if $\mathcal{G} \subset {}^\omega\omega$ is such that $|\mathcal{G}| < 2^{\aleph_0}$, then there exists f such that $g \prec f$ for all $g \in \mathcal{G}$. Then an induction on $\alpha < 2^{\aleph_0}$ gives us a $2^{\aleph_0}$-scale.

If the continuum hypothesis fails, then an $\aleph_1$-scale may or may not exist. Let P be the notion of forcing consisting of finite sequences $p = \langle s(0), \ldots, s(n-1)\rangle$ of natural numbers. In the generic model $\mathfrak{M}[G]$, let $f = \bigcup G$. Then for all $g \in \mathfrak{M}$ we have $g \not\succ f$ (see the proof of Lemma 24.10b).

Now, let us instead consider the product of $\aleph_2$ copies of P. If $\alpha < \omega_2$, let P_α be the product of the first α copies and P^α the product of the remaining copies of P; let G_α, G^α be the corresponding projections of G. By the product lemma, G^α is P^α-generic over $\mathfrak{M}[G_\alpha]$, and $\mathfrak{M}[G] = \mathfrak{M}[G_\alpha][G^\alpha]$. If $S \subset {}^\omega\omega$ is a set (in $\mathfrak{M}[G]$) of size $\aleph_1$, then (because P satisfies the c.c.c.), there exists $\alpha < \omega_2$ such that $S \in \mathfrak{M}[G_\alpha]$. Then there exists $f \in \mathfrak{M}[G_\alpha][G^\alpha]$ such that $g \not\succ f$ for all $g \in S$. Hence $\mathfrak{M}[G]$ has no $\aleph_1$-scale.

On the other hand, let us consider the following notion of forcing $(P, <)$: Let $\mathcal{H}$ be a given family of functions $h\colon \omega \to \omega$. A forcing condition is a pair $p = (s, E)$, where $s = \langle s(0), \ldots, s(n-1)\rangle$ is a finite sequence of natural numbers and E is a finite subset of $\mathcal{H}$. A condition (s', E') is stronger than (s, E) if:

(24.28) (i) $s \subseteq s'$, and $E \subseteq E'$;
(ii) if $k \in \operatorname{dom}(s') - \operatorname{dom}(s)$, then $s(k) > h(k)$ for all $h \in E$.

If (s_1, E_1) and (s_2, E_2) are conditions and $s_1 = s_2$, then (s_1, E_1) and (s_2, E_2) are compatible. Hence $(P, <)$ satisfies the c.c.c. Let $G \subset P$ be $\mathfrak{M}$-generic; we let $f = \bigcup \{s : (s, E) \in G \text{ for some } E\}$. We claim that $h \prec f$ for all $h \in H$. First, there is a condition $(s, E) \in G$ such that $h \in E$ (by genericity). Secondly, every condition $(s', E') < (s, E)$ satisfies (24.28ii), and so $f(k) > h(k)$ for all $k \notin \operatorname{dom}(s)$. Thus in $\mathfrak{M}[G]$, there is $f\colon \omega \to \omega$ such that $h \prec f$ for all $h \in \mathcal{H}$.

Using iterated forcing, we can get a generic extension $\mathfrak{M}[G]$ of a given ground model $\mathfrak{M}$ such that $\mathfrak{M}[G]$ has an $\aleph_1$-scale: We iterate the above construction $\aleph_1$ times, taking direct limits at limit stages, and letting $\mathcal{H}$ always be the set of all functions $h\colon \omega \to \omega$. We obtain a sequence of functions (in $\mathfrak{M}[G]$) $S = \{f_\alpha : \alpha < \omega_1\}$ that satisfies (24.27i). If $g \in \mathfrak{M}[G]$ is any function $g\colon \omega \to \omega$, then $g \in \mathfrak{M}[G \cap B_\alpha]$ for some $\alpha < \omega_1$ (where B_α is the complete Boolean algebra obtained in the first α stages of the iteration), and we have $g \prec f_\alpha$. Thus

$\mathfrak{M}[G]$ has an $\aleph_1$-scale. If we start with a ground model $\mathfrak{M}$ in which the continuum hypothesis fails, $\mathfrak{M}[G]$ also satisfies $2^{\aleph_0} > \aleph_1$ and still has an $\aleph_1$-scale.

The above notion of forcing can be also used with Martin's axiom and we get:

Lemma 24.12. *Martin's axiom implies that if $\mathscr{H} \subset {}^\omega\omega$ has size $< 2^{\aleph_0}$, then there exists f such that $h \prec f$ for all $h \in \mathscr{H}$. Consequently, there exists an $2^{\aleph_0}$-scale.* ■

Finally, let us consider the relation $\precsim$ on infinite subsets of ω defined as follows:

$$(24.29) \qquad X \precsim Y \quad \text{iff} \quad X - Y \text{ is finite}$$

[$\precsim$ is not a partial ordering since we can have both $X \precsim Y$ and $Y \precsim X$, but it is a partial ordering of the Boolean algebra $P(\omega)$/finite sets.]

Every countable descending sequence $X_0 \succsim X_1 \succsim \cdots \succsim X_n \succsim \cdots$ has a lower bound.

Exercise 24.17. Martin's axiom implies that any descending sequence $X_0 \succsim X_1 \succsim \cdots \succsim X_\alpha \succsim \cdots$, $\alpha < \kappa$, of length $\kappa < 2^{\aleph_0}$ has a lower bound.

[A forcing condition is a pair (s, F) where s is a finite subset of ω and F is a finite subset of κ; $(s', F') < (s, F)$ just in case $s' \supseteq s$, $F' \supseteq F$, and $s' - s \subseteq X_\alpha$ for all $\alpha \in F$. Consider the dense sets $D_n = \{(s, F) : |s| \geq n\}$, $n < \omega$, and $E_\alpha = \{(s, F) : \alpha \in F\}$, $\alpha < \kappa$.]

25. FORCING AND COMPLETE BOOLEAN ALGEBRAS

An important feature of the method of forcing is that properties of the generic extension are determined by properties of the complete Boolean algebra that is used to construct the generic extension. For instance, we have used extensively the lemma which states that if B satisfies the κ-chain condition (in the ground model), then κ remains a cardinal in the generic extension.

In this section, I shall look more closely into this correspondence between properties of B and properties of the B-valued model, and also give examples how, conversely, we may use Boolean-valued models to obtain results on complete Boolean algebras.

The main theme of this section is the correspondence between a complete Boolean algebra B and the B-valued model V^B. Since B-valued arguments are often cumbersome, I intend to use generic extensions whenever possible. For that reason, let $\mathfrak{M}$ be a fixed ground model, a model of ZFC, and let us have an understanding that B, C, D, etc. are complete Boolean algebras in $\mathfrak{M}$, and all properties of these complete Boolean algebras are their properties in the ground model. Also, unless expressly stated otherwise, a statement:

(25.1) "The generic extension $\mathfrak{M}[G]$ has such and such properties."

will mean that for *all* $\mathfrak{M}$-generic $G \subseteq B$, $\mathfrak{M}[G]$ has such properties. In fact, this interpretation of (25.1) will sometimes need the tacit assumption that for every nonzero $u \in B$ there exists G such that $u \in G$ (this is true if $\mathfrak{M}$ is countable; cf. Lemmas 16.1 and 17.5). The correct interpretation of (25.1) is:

$$\|\text{the universe has such and such properties}\|_B = 1$$

Most of the problems considered below deal with the relationship of the extension $\mathfrak{M}[G]$ to the ground model $\mathfrak{M}$. Although in general $\mathfrak{M}$ may not be a class of $\mathfrak{M}[G]$ (i.e., $x \in \mathfrak{M}$ may not be expressible by a formula), we shall treat $\mathfrak{M}$ as if it were a class of $\mathfrak{M}[G]$. The reason is that we can easily compute the Boolean value of $x \in \mathfrak{M}$:

$$\|x \in \mathfrak{M}\| = \sum \{\|x = \check{a}\| : a \in \mathfrak{M}\} \tag{25.2}$$

(A logical purist would perhaps prefer to augment the language with a new unary predicate $\mathfrak{M}$.)

In the first part of this section, I shall present various variations on the main theme. Some of these are collected in Theorem 58 below. First, the relevant definitions.

Let κ and λ be cardinals. A complete Boolean algebra B is (κ, λ)-*distributive* if

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} u_{\alpha\beta} = \sum_{f:\kappa\to\lambda} \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)} \tag{25.3}$$

Note that (25.3) is a special case of (17.27); B is κ-distributive if and only if it is (κ, λ)-distributive for all λ. As in Lemma 17.7, we can reformulate (κ, λ)-distributivity as follows:

Lemma 25.1. *B is (κ, λ)-distributive if and only if every collection of κ partitions of B of size at most λ has a common refinement.* ■

We recall that a complete subalgebra B of a complete Boolean algebra D is (completely) generated by a set $X \subseteq D$ if B is the least complete subalgebra of D such that $X \subseteq B$. Let κ be a cardinal. We say that a complete Boolean algebra B is κ-*generated* if there exists $X \subseteq B$ of size at most κ such that the complete subalgebra of B generated by X is equal to B.

A complete Boolean algebra is *rigid* if there is no automorphism of B except the identity function $\pi a = a$.

For every nonzero $a \in B$, let B_a denote the complete Boolean algebra

$$B_a = \{u \in B : u \leq a\}$$

The properties of complete Boolean algebras defined above are "hereditary," i.e., if B has the property, then each B_a, $a \neq 0$, has the property; more generally, if B_a has the property and $b \leq a$, then B_b has the property. Given such a "hereditary" property, we say that B has this property *locally* if the set of all $a \in B$ such that B_a has the property is dense. Thus, for instance B is *locally rigid* if $\{a \neq 0 : B_a \text{ is rigid}\}$ is dense. We similarly define *locally κ-generated*, etc.

Note that B is locally rigid, (etc.) iff there is a partition W of B such that each B_a, $a \in W$, is rigid (etc.) iff B is the product of rigid algebras (cf. Fig. 17.1).

Theorem 58. *Let B be a complete Boolean algebra (in $\mathfrak{M}$).*

(a) *B is atomless if and only if $\mathfrak{M}[G] \neq \mathfrak{M}$.*
(b) *B is (κ, λ)-distributive if and only if every $f\colon \kappa \to \lambda$ in $\mathfrak{M}[G]$ is in $\mathfrak{M}$.*
(c) *B is locally κ-generated if and only if $\mathfrak{M}[G] = \mathfrak{M}[A]$ for some $A \subseteq \kappa$.*
(d) *B is locally rigid if and only if every set in $\mathfrak{M}[G]$ is $\mathfrak{M}$-definable (in $\mathfrak{M}[G]$).*

The four cases of Theorem 58 show a typical correspondence between complete Boolean algebras and generic extensions. For instance, if there exists a (nontrivial) rigid complete Boolean algebra, then we can obtain a model $\mathfrak{M}[G]$ such that

$$\mathfrak{M} \neq (\mathrm{H}\mathfrak{M}\mathrm{D})^{\mathfrak{M}[G]} = \mathfrak{M}[G] \tag{25.4}$$

On the other hand, if there is a generic model $\mathfrak{M}[G]$ such that (25.4) holds, then there exists a rigid complete Boolean algebra.

Note that clauses (a) and (b) are formulated without the localness assumption; this is because "locally atomless" implies "atomless," and similarly for distributivity.

Proof of Theorem 58a. If B is atomless and G is generic on B, then it is easy to see that $G \notin \mathfrak{M}$ (cf. Exercise 16.6). Conversely, if $a \in B$ is an atom, let $G = \{u \in B : a \leq u\}$. G is clearly a generic ultrafilter on B, and $G \in \mathfrak{M}$; hence $\mathfrak{M}[G] = \mathfrak{M}$. ■

Proof of Theorem 58b. Let B be (κ, λ)-distributive, and let G be generic on B; let us show that every $f\colon \kappa \to \lambda$ in $\mathfrak{M}[G]$ is in $\mathfrak{M}$. (Compare with Lemma 19.6.) Let $f\colon \kappa \to \lambda$ and let f be its name; there is $b \in G$ such that

$$b \Vdash f \text{ is a function from } \check{\kappa} \text{ into } \check{\lambda} \tag{25.5}$$

For each $\alpha < \kappa$ and each $\beta < \lambda$, let $u_{\alpha\beta} = b \cdot \|f(\check{\alpha}) = \check{\beta}\|$. For each $\alpha < \kappa$, the set $W_\alpha = \{u_{\alpha\beta} : \beta < \lambda\}$ is a partition of b, of size at most λ. By distributivity, the partitions W_α, $\alpha < \kappa$, have a common refinement W. There exists $u \in W$ such that $u \in G$; then for each $\alpha < \kappa$ there is a unique $\beta = \beta(\alpha)$ such that $u \leq u_{\alpha\beta}$. Clearly, $f(\alpha) = \beta(\alpha)$ for all $\alpha < \kappa$, and so $f \in \mathfrak{M}$.

Conversely, assume that B is not (κ, λ)-distributive. There exist partitions $W_\alpha = \{u_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$ (some $u_{\alpha\beta}$ may be zero), such that

$$\sum_{g\colon \kappa \to \lambda} \prod_{\alpha < \kappa} u_{\alpha, g(\alpha)} = u < 1 \tag{25.6}$$

Let $b = -u$, and let f be a B-valued name for a function such that

$$\|f(\check{\alpha}) = \check{\beta}\| = b \cdot u_{\alpha\beta} = v_{\alpha\beta} \qquad (\alpha < \kappa, \quad \beta < \lambda)$$

Note that b forces that f is a function from κ into λ. For each $\alpha < \kappa$, the set $\{v_{\alpha\beta} : \beta < \lambda\}$ is a partition of b and

$$\sum_{g:\kappa\to\lambda} \prod_{\alpha<\kappa} v_{\alpha,g(\alpha)} = 0 \tag{25.7}$$

I claim that b forces that $f \notin \mathfrak{M}$. Let $h \in \mathfrak{M}$ be a function from κ into λ, and let $p \leq b$. It follows from (25.7) that there is α such that for some $\beta \neq \gamma$, p is compatible with both $v_{\alpha\beta}$ and $v_{\alpha\gamma}$. Then $p \cdot v_{\alpha\beta} \Vdash f(\check{\alpha}) = \check{\beta}$ and $p \cdot v_{\alpha\gamma} \Vdash f(\check{\alpha}) = \check{\gamma}$. Thus either $p \cdot v_{\alpha\beta}$ or $p \cdot v_{\alpha\gamma}$ forces that $f(\alpha) \neq h(\alpha)$, and the claim follows. ■

***Exercise* 25.1.** If B is $(\kappa, 2)$-distributive, then B is (κ, κ)-distributive.

[Use Theorem 58b and work in $\mathfrak{M}[G]$. If there is a function $f: \kappa \to \kappa$ that is not in $\mathfrak{M}$, then there is a subset of κ that is not in $\mathfrak{M}$.]

***Exercise* 25.2.** Let $\lambda = 2^\kappa$. If B is $(\kappa, 2)$-distributive, then B is (κ, λ)-distributive.

[Work in $\mathfrak{M}[G]$. Assume that every subset of κ is in $\mathfrak{M}$ and show that every $F: \kappa \to \lambda$ is in $\mathfrak{M}$. Let $F: \kappa \to P(\kappa)$. Define $X \subseteq \kappa \times \kappa$ as follows: $(\alpha, \beta) \in X$ iff $\beta \in F(\alpha)$. Since $X \in \mathfrak{M}$, and F is defined from X by $F(\alpha) = \{\beta : (\alpha, \beta) \in X\}$, we also have $F \in \mathfrak{M}$.]

B is called *weakly* (κ, λ)*-distributive*, if

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} u_{\alpha\beta} = \sum_{g:\kappa\to\lambda} \prod_{\alpha<\kappa} \sum_{\beta<g(\alpha)} u_{\alpha\beta} \tag{25.8}$$

Exercise* 25.3. B is weakly (κ, λ)-distributive if and only if every $f: \kappa \to \lambda$ in $\mathfrak{M}[G]$ is majorized by some $g: \kappa \to \lambda$ that is in $\mathfrak{M}$ (i.e., $f(\alpha) < g(\alpha)$ for all $\alpha < \kappa$).

***Exercise* 25.4.** Let P be the notion of forcing that adjoins a single Cohen-generic real. The algebra $B = \text{r.o.}(P)$ is not weakly (ω, ω)-distributive.

[See the proof of Lemma 24.10b.]

***Exercise* 25.5.** B is weakly (ω, ω_1)-distributive iff ω_1 is a cardinal in $\mathfrak{M}[G]$.

Let us turn our attention to part (c) of Theorem 58. To start, the notation $\mathfrak{M}[A]$ needs explanation. If A is a subset of $\mathfrak{M}$, then $\mathfrak{M}[A]$ denotes the least model $\mathfrak{N} \supseteq \mathfrak{M}$ of ZFC (if it exists) that has the same ordinals as $\mathfrak{M}$ and contains A. In general (e.g., if $\mathfrak{M}$ is a set), $\mathfrak{M}[A]$ need not exist. But if $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}$ and $A \in \mathfrak{M}[G]$, then $\mathfrak{M}[A]$ does exist, and in fact (as we prove below) $\mathfrak{M}[A]$ is itself a generic extension of $\mathfrak{M}$.

Let B be a complete Boolean algebra and let D be a complete subalgebra of B. If G is generic on B, then it is easy to see that $G \cap D$ is generic on D, and so $\mathfrak{M}[G \cap D]$ is a model of ZFC, and

$$\mathfrak{M} \subseteq \mathfrak{M}[G \cap D] \subseteq \mathfrak{M}[G] \tag{25.9}$$

We shall prove that every model of ZFC between $\mathfrak{M}$ and $\mathfrak{M}[G]$ is obtained this way, and that for every subset A of $\mathfrak{M}$ in $\mathfrak{M}[G]$ there is a complete subalgebra D of B such that $\mathfrak{M}[G \cap D] = \mathfrak{M}[A]$.

Lemma 25.2. *Let X be a subset of a complete Boolean algebra B such that B is completely generated by X. Then for every generic G on B,*

$$\mathfrak{M}[G] = \mathfrak{M}[X \cap G] \tag{25.10}$$

Proof. We want to show that $\mathfrak{M}[G]$ is the least model such that the set $A = X \cap G$ is in $\mathfrak{M}[G]$. It suffices to show that G can be defined in terms of A (in any model $\mathfrak{N} \supseteq \mathfrak{M}$ containing A).

Since B is generated by X, every element of B can be obtained from elements of X by successive (transfinite) application of the operations $-$ and $\sum$. Thus let X_α be subsets of B defined inductively as follows:

$$\begin{aligned} &X_0 = X \\ &\bar{X}_\alpha = \{-a : a \in X_\alpha\} \\ &X_\alpha = \Big\{a : a = \sum Z \text{ where } Z \subseteq \bigcup_{\beta<\alpha} (X_\beta \cup \bar{X}_\beta)\Big\} \end{aligned} \tag{25.11}$$

Then $B = \bigcup_{\alpha<\theta} X_\alpha$ for some θ (in fact $\theta \leq |B|^+$).

If we denote $G_\alpha = G \cap X_\alpha$, $\bar{G}_\alpha = G \cap \bar{X}_\alpha$, we have

$$\begin{aligned} &G_0 = A \\ &\bar{G}_\alpha = \{-a : a \in X_\alpha - G_\alpha\} \\ &G_\alpha = \{a \in X_\alpha : a = \textstyle\sum Z \text{ where } Z \text{ contains at least one } b \text{ in} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{some } G_\beta \text{ or } \bar{G}_\beta,\ \beta < \alpha\} \end{aligned} \tag{25.12}$$

and $G = \bigcup_{\alpha<\theta} G_\alpha$. Thus given A, we define G_α and $\bar{G}_\alpha$ inductively using (25.12), and let $G = \bigcup_{\alpha<\theta} G_\alpha$. ■

Corollary 1. *If B is κ-generated, then $\mathfrak{M}[G] = \mathfrak{M}[A]$ for some $A \subseteq \kappa$.*

Corollary 2. *If G is generic on B and $A \in \mathfrak{M}[G]$ is a subset of κ, then there exists a κ-generated complete subalgebra D of B such that $\mathfrak{M}[D \cap G] = \mathfrak{M}[A]$.*

Proof. Let $\boldsymbol{A}$ be a name for A. We let $X = \{u_\alpha : \alpha < \kappa\}$, where

$$u_\alpha = \|\check{\alpha} \in \boldsymbol{A}\| \tag{25.13}$$

Now we let D be the complete subalgebra completely generated by X; by the lemma we have $\mathfrak{M}[X \cap G] = \mathfrak{M}[D \cap G]$. It remains to show that $\mathfrak{M}[X \cap G] = \mathfrak{M}[A]$.

On the one hand, we have $A = \{\alpha : u_\alpha \in X \cap G\}$. On the other hand, $X \cap G = \{u_\alpha : \alpha \in A\}$. ■

Lemma 25.3. *Let G be generic on B. If $\mathfrak{N}$ is a model of* ZFC *such that $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{M}[G]$, then there exists a complete subalgebra $D \subseteq B$ such that $\mathfrak{N} = \mathfrak{M}[D \cap G]$.*

Proof. We shall show that $\mathfrak{N} = \mathfrak{M}[A]$, where A is a set of ordinals. Then the lemma follows by Corollary 2 of Lemma 25.1. First we note that since $\mathfrak{N}$ satisfies the axiom of choice, there exists for every $X \in \mathfrak{N}$ a set of ordinals $A_X \in \mathfrak{N}$ such that $X \in \mathfrak{M}[A_X]$. We let $Z = P(B) \cap \mathfrak{N}$, and let $A = A_Z$; we shall show that $\mathfrak{N} = \mathfrak{M}[A]$.

If $X \in \mathfrak{N}$, consider the set of ordinals A_X; by Corollary 2 above, there exists a subalgebra $D_X \subseteq B$ such that $\mathfrak{M}[A_X] = \mathfrak{M}[D_X \cap G]$. Hence $D_X \cap G \in \mathfrak{N}$, and we have $D_X \cap G \in Z$. Since $Z \in \mathfrak{M}[A]$, it follows that $D_X \cap G \in \mathfrak{M}[A]$ and hence $X \in \mathfrak{M}[A]$. Thus $\mathfrak{N} = \mathfrak{M}[A]$. ∎

The following lemma gives a necessary and sufficient condition for a subalgebra to give the same Boolean valued model as the given algebra. Let B be a complete subalgebra of a complete Boolean algebra D. Let us recall the definition of canonical projection h of D onto B:

$$h(x) = \sum \{u \in B : u \geq x\} \tag{25.14}$$

Let a be a nonzero element of D. Even if $a \notin B$, we may define

$$B_a = \{x \cdot a : x \in B\} \tag{25.15}$$

Again, B_a is a complete Boolean algebra. In accordance with the discussion preceding Theorem 58, we say that B is *locally equal* to D, if the set $\{a : B_a = D_a\}$ is dense in D.

Lemma 25.4. *Let B be a complete subalgebra of D. Then B is locally equal to D if and only if $\mathfrak{M}[G] = \mathfrak{M}[G \cap B]$ (for all generic $G \subseteq D$).*

Proof. If B is locally equal to D, and if G is generic on D, then there exists $a \in G$ such that $B_a = D_a$. Since

$$D_a \cap G = B_a \cap G = \{a \cdot x : x \in B \cap G\}$$

it follows that $D_a \cap G \in \mathfrak{M}[B \cap G]$. Clearly, $G \in \mathfrak{M}[D_a \cap G]$, and so $\mathfrak{M}[G] = \mathfrak{M}[G \cap B]$.

Thus let $B \subseteq D$ be such that B is not locally equal to D. There is $d_0 \in D$ such that for all nonzero $a \leq d_0$, there is $v \leq a$ such that $v \notin B_a$. Let G be a generic ultrafilter on D such that $d_0 \in G$. We shall show that $\mathfrak{M}[G \cap B] \neq \mathfrak{M}[G]$.

Let $G_1 = G \cap B$. Let $C \in \mathfrak{M}[G_1]$ be the algebra $C = D/\equiv$, where

$$d_1 \equiv d_2 \quad \text{iff} \quad h(d_1 \triangle d_2) \notin G_1 \tag{25.16}$$

(C is the G_1-interpretation of the B-valued algebra $D:B$ from Exercise 23.6.) Note that if $d_1 \equiv d_2$ and $d_1 \in G$, then $d_2 \in G$ (if $h(d_1 \triangle d_2) \notin G_1$, then $d_1 \triangle d_2 \in G$). Let $[d]$ denote the $\equiv$-equivalence class of d. Let

$$H = \{[d] : d \in G\}$$

By the preceding remark, we have $G = \{d : [d] \in H\}$, and it follows that $\mathfrak{M}[G] = \mathfrak{M}[G_1][H]$.

H is an ultrafilter on C, and an argument similar to the proof of Lemma 23.4b shows that H is $\mathfrak{M}[G_1]$-generic.

We shall show that H is nonprincipal. Then it follows easily that $H \notin \mathfrak{M}[G_1]$ (see Exercise 16.6). Thus let $[u] \in H$; we shall find v such that $[v] \in H$ and $[v] < [u]$. We have $u \in G$, and we may assume that $u \leq d_0$.

Note that if $v \leq a$ is such that $v \notin B_a$, then $h(v) \cdot a > v$; also, $a - v \notin B_a$, and hence $h(a - v) \cdot v \neq 0$. Now for every $a \leq u$, we have $B_a \neq D_a$, and hence the following set is dense below u:

$$\{w \neq 0 : w = v \cdot h(u - v) \text{ for some } v \leq u\}$$

Hence there is $v \leq u$ such that $v \cdot h(u - v) \in G$. Thus $h(v) \in G$, and $h(u - v) \in G$, and we have $v \nVdash 0$ and $u - v \nVdash 0$. In other words, we have found $v \leq u$ such that $[v] \in H$ and $[v] \neq [u]$. ■

Proof of Theorem 58c. Let B be locally κ-generated, and let G be generic on B. There is $a \in G$ such that B_a is κ-generated. By Corollary 1 of Lemma 25.2, $\mathfrak{M}[G \cap B_a] = \mathfrak{M}[A]$ for some $A \subseteq \kappa$. But $\mathfrak{M}[G \cap B_a] = \mathfrak{M}[G]$.

Conversely, assume that $\mathfrak{M}[G] = \mathfrak{M}[A]$ for some $A \subseteq \kappa$, for all generic $G \subseteq B$. Hence there is a name $A \in \mathfrak{M}^B$ such that

$$\|A \subseteq \check{\kappa} \text{ and } \mathfrak{M}[A] = \mathfrak{M}[G]\| = 1$$

Let D be the subalgebra of B generated by $\|\check{\alpha} \in A\|$, $\alpha < \kappa$ (see the proof of Corollary 2 of Lemma 25.2). Then $\mathfrak{M}[G] = \mathfrak{M}[G \cap D]$ for all generic G; and by Lemma 25.4, D is locally equal to B. Since D is κ-generated, it follows that B is locally κ-generated (verify that if $D_a = B_a$, then B_a is κ-generated). ■

Exercise 25.6. If a complete Boolean algebra is κ-generated and λ-saturated, then $|B| \leq \kappa^{<\lambda}$.

Exercise 25.7. Every infinite countably generated c.c.c. complete Boolean algebra has size $2^{\aleph_0}$.

If B and C are complete Boolean algebras and if π is an isomorphism between B and C, then it is clear that for any generic ultrafilter G on B, $\pi[G]$ is a generic ultrafilter on C and $\mathfrak{M}[G] = \mathfrak{M}[\pi[G]]$. We shall now establish the "converse" of this statement.

Lemma 25.5.

(a) *Let B and C be complete Boolean algebras (in $\mathfrak{M}$). Let G be a generic ultrafilter on B and let H be a generic ultrafilter on C such that $\mathfrak{M}[G] = \mathfrak{M}[H]$. Then there exist $b \in G$ and $c \in H$ and an isomorphism π (in $\mathfrak{M}$) between B_b and C_b such that $\pi[G \cap B_b] = H \cap C_b$.*

(b) *Let G and H be generic ultrafilters on B, such that $\mathfrak{M}[G] = \mathfrak{M}[H]$. Then there is an automorphism π of B (in $\mathfrak{M}$) such that $\pi[G] = H$.*

Proof. (a) There is $H \in \mathfrak{M}^B$ such that $i_G(H) = H$ and

$$(25.17) \quad \|H \text{ is a generic ultrafilter on } \check{C} \text{ and } \mathfrak{M}[H] = \mathfrak{M}[G]\|_B = b_0 \in G$$

(where G is the canonical name for a generic ultrafilter on B). Without loss of generality, we shall assume that $b_0 = 1$ (otherwise we replace the algebra B by the algebra B_{b_0}).

Next I claim that there is $c_0 \in H$ such that for every nonzero $a \leq c_0$, $\|\check{a} \in H\| \neq 0$. Let S be the set of all $x \in C$ such that $\|\check{x} \in H\| = 0$, and let $s = \sum S$. Then

$$(25.18) \qquad \|\check{s} \in H\| = \|(\exists x \in \check{S})x \in H\| = \sum_{x \in S} \|\check{x} \in H\| = 0$$

and it follows that $s \notin H$. Thus $c_0 = -s$ satisfies the claim. Again, let us assume that $c_0 = 1$, for otherwise we replace C by C_{c_0}.

Now we define an embedding e of C into B as follows:

$$(25.19) \qquad e(a) = \|\check{a} \in H\|$$

It is easy to verify that e is a complete homomorphism and that for every $a \in C$,

$$(25.20) \qquad a \in H \quad \text{iff} \quad e(a) \in G$$

In fact, we have

$$(25.21) \qquad \|\check{a} \in H\| = \|e(a)^\vee \in G\|$$

I claim that e is one-to-one. It suffices to show that $e(a) \neq 0$ whenever $a \neq 0$. However, this has been guaranteed by the choice of c_0 above.

Let $D = e[C]$; D is a complete subalgebra of B; and for every G generic on B, if $H = i_G(H)$, we have

$$(25.22) \qquad \mathfrak{M}[G \cap D] = \mathfrak{M}[e[H]] = \mathfrak{M}[H] = \mathfrak{M}[G]$$

Thus by Lemma 25.4, D is locally equal to B. Therefore, there exists $b \in G$ such that $D_b = B_b$.

Now, let h be the canonical projection of B onto D, and let ρ be this mapping of $D_{h(b)}$ onto D_b:

$$(25.23) \qquad \rho(d) = d \cdot b \qquad (d \in D_{h(b)})$$

It is easily verified that ρ is an isomorphism between $D_{h(b)}$ and D_b. Thus if we let $c = e^{-1}(h(b))$, the composition of e and ρ gives us an isomorphism π between C_c and B_b. Moreover, it follows that for all $x \in C_c$, $x \in H$ iff $\pi x \in G$.

(b) Let G and H be generic on B such that $\mathfrak{M}[G] = \mathfrak{M}[H]$. If $G = H$, then the assertion holds trivially; if $G \neq H$, then there are disjoint $u, v \in B$ such that $u \in G$ and $v \in H$, and by (a) there exist $a \leq u$ and $b \leq v$ and an isomorphism π between B_a and B_b such that $a \in G$, $b \in H$, and $\pi[G \cap B_a] = \pi[H \cap B_b]$.

We can now extend π to an automorphism ρ of B as follows (see Fig. 25.1):

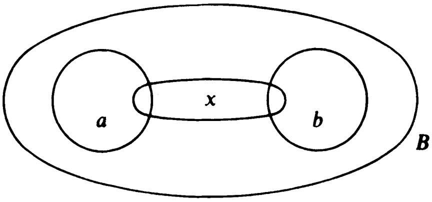

FIGURE 25.1

For $x \le a$, we let $\rho x = \pi x$; for $x \le b$, let $\rho x = \pi^{-1}x$. If x is disjoint from $a + b$, we let $\rho x = x$. In general, each $x \in B$ can be written uniquely as a disjoint sum

$$x = a \cdot x + b \cdot x + y$$

and we define

$$\rho x = \pi(a \cdot x) + \pi^{-1}(b \cdot x) + y$$

It is easily seen that ρ is an automorphism of B, and $\pi[G] = H$.

Lemma 25.5 enables us to give the following characterization of $\mathfrak{M}$-definable sets in the generic extension $\mathfrak{M}[G]$:

Theorem 59 (Vopěnka). *If B is a complete Boolean algebra and G is a generic ultrafilter on B, then*

$$(\mathrm{H}\mathfrak{M}\mathrm{D})^{\mathfrak{M}[G]} = \mathfrak{M}[G \cap B^*]$$

where

$$B^* = \{a \in B : \pi a = a \text{ for every automorphism } \pi \text{ of } B\}$$

Proof. Let us work in $\mathfrak{M}[G]$. First we use Lemma 25.5 to show that $\mathfrak{M}[G \cap B^*] \subseteq \mathrm{H}\mathfrak{M}\mathrm{D}$. If H is a generic ultrafilter on B such that $\mathfrak{M}[H] = \mathfrak{M}[G]$, then $H = \pi[G]$ for some automorphism π of B. Since π is the identity on B^*, we have $H \cap B^* = G \cap B^*$. Thus the set

$$\{H \cap B^* : H \subseteq B \text{ is generic and } \mathfrak{M}[H] \text{ is the universe}\}$$

has exactly one element, $G \cap B^*$. Hence $G \cap B^*$ is definable, and $\mathfrak{M}[G \cap B^*] \subseteq \mathrm{H}\mathfrak{M}\mathrm{D}$ follows.

In order to prove the converse, let us first prove that if X is an $\mathfrak{M}$-definable subset of $\mathfrak{M}$, then $X \in \mathfrak{M}[G \cap B^*]$. Let $X \in \mathfrak{M}^B$ be a name for X; we shall find another name for X, a B^*-valued name $Z \in \mathfrak{M}^{B^*}$. Then $X = i_{G \cap B^*}(Z)$ and hence $X \in \mathfrak{M}[G \cap B^*]$.

Since X is $\mathfrak{M}$-definable, there is a formula $\varphi(X, \ldots)$, with parameters in $\mathfrak{M}$, and some $a \in G$ such that

(25.24) $\quad a \Vdash X$ is the unique X such that $\varphi(X, \ldots)$

Let $A \in \mathfrak{M}$ be such that $a \Vdash X \subseteq \check{A}$. We may assume that for each $x \in A$, $\|\check{x} \in X\| \subseteq a$ (otherwise we replace X by a name X' such that $\|\check{x} \in X'\| = \|\check{x} \in X\| \cdot a$).

Let Z be the name (for a subset of A) defined as follows:

$$\|\check{x} \in Z\| = \sum \{\pi(\|\check{x} \in X\|) : \pi \text{ is an automorphism of } B\} \qquad (x \in A) \tag{25.25}$$

It follows from definition (25.25) that for any $x \in A$ and any automorphism ρ of B, $\rho(\|\check{x} \in Z\|) = \|\check{x} \in Z\|$. Hence Z is B^*-valued. It remains to show that $X = Z$, where $Z = i_G(Z) = i_{G \cap B^*}(Z)$. Since $\|\check{x} \in X\| \leq \|\check{x} \in Z\|$ for every x, we have $X \subseteq Z$. On the other hand, let $x \in Z$.

Since $\|\check{x} \in Z\| \in G$, there exists by (25.25) an automorphism π of B such that $\pi(\|\check{x} \in X\|) \in G$. Let $u = a \cdot \pi(\|\check{x} \in X\|)$ and let $v = \pi^{-1}u$. Note that $u \in G$. We shall show that $u \Vdash \check{x} \in X$ (and hence $x \in X$).

On the one hand, $u \leq \pi(\|\check{x} \in X\|)$ and so

$$u \Vdash \check{x} \in \pi(X) \tag{25.26}$$

On the other hand, we have $v \leq a$ (since $v \leq \|\check{x} \in X\| \leq a$), and so $v \Vdash \varphi(X, \ldots)$, which gives us

$$\pi v = u \Vdash \varphi(\pi X, \ldots) \tag{25.27}$$

But also $u \leq a$, and (25.24), together with (25.27), implies that

$$u \Vdash \pi(X) = X$$

Hence $u \Vdash \check{x} \in X$ and we have $X = Z$.

Now we shall show that the model $\mathfrak{N} = \mathfrak{M}[G \cap B^*]$ includes H$\mathfrak{M}$D. Let $Y \in$ H$\mathfrak{M}$D be of least rank such that $Y \notin \mathfrak{N}$. Then $Y \subseteq \mathfrak{N}$, and there is $A \in \mathfrak{N}$ such that $Y \subseteq A$. There is a one-to-one function $f \in \mathfrak{N}$ such that $f[A] \in \mathfrak{M}$. Then $X = f[Y]$ is an $\mathfrak{M}$-definable subset of $\mathfrak{M}$ and hence $X \in \mathfrak{N}$. Therefore $Y = f^{-1}[X]$ is in $\mathfrak{N}$. ■

Corollary 1. *If B is rigid, then $\mathfrak{M}[G] \Vdash$ every set is $\mathfrak{M}$-definable.*

A complete Boolean algebra is *weakly homogeneous* if for any nonzero u and v there is an automorphism π such that $\pi(u) \cdot v \neq 0$. It is easy to see that then no $u \neq 0, 1$ is fixed under all automorphisms, and we have

Corollary 2. *If B is weakly homogeneous, then* $(\text{H}\mathfrak{M}\text{D})^{\mathfrak{M}[G]} = \mathfrak{M}$.

Exercise 25.8. Let P be the notion of forcing that adjoins a single Cohen generic real, and let $B = \text{r.o.}(P)$. Show that B is weakly homogeneous.

This exercise gives us consistency of the following:

$$L = \text{HOD} \neq V$$

The above Corollary 2 can be proved directly, using:

Exercise 25.9. If B is weakly homogeneous, if $\varphi(x_1, \ldots, x_n)$ is any formula, and if $x_1, \ldots, x_n \in \mathfrak{M}$, then the value

$$\|\varphi(\check{x}_1, \ldots, \check{x}_n)\|$$

is either 0 or 1.

Proof of Theorem 58d. If B is locally rigid and G is generic on B, then there is $a \in G$ such that B_a is rigid, and since $\mathfrak{M}[G] = \mathfrak{M}[G \cap B_a]$, every set in $\mathfrak{M}[G]$ is $\mathfrak{M}$-definable in $\mathfrak{M}[G]$, by Corollary 1 of Theorem 59.

Conversely, assume that B is not locally rigid. Then there is $u \neq 0$ such that for every nonzero $v \leq u$, the algebra B_v has a nontrivial automorphism. Let $G \subseteq B$ be generic such that $u \in G$; we shall show that G is not $\mathfrak{M}$-definable in $\mathfrak{M}[G]$.

Otherwise, there is some $\varphi(X, \ldots)$, with parameters in $\mathfrak{M}$, and $v \leq u$ such that

$$v \Vdash G \text{ is the unique } X \text{ such that } \varphi(X, \ldots) \tag{25.28}$$

Let π be a nontrivial automorphism of B_v; it is easy to find disjoint $z_1, z_2 \leq v$ such that $\pi z_1 = z_2$. Then $z_1 \Vdash \check{z}_1 \in G$ and $z_2 \Vdash \check{z}_1 \notin G$, and $z_2 (= \pi z_1)$ forces $\check{z}_1 \in \pi(G)$, hence

$$z_2 \Vdash G \neq \pi(G)$$

This contradicts (25.28) since $v \Vdash \varphi(G)$ and also $v = \pi v \Vdash \varphi(\pi G)$. ■

By Theorem 58d, there are two possible ways to obtain a rigid complete Boolean algebra. Besides the "direct" algebraic construction, one can also achieve the same result by constructing a notion of forcing such that in the generic extension every set is $\mathfrak{M}$-definable. And indeed, the first construction of a rigid complete Boolean algebra (due to McAloon) was accomplished by the indirect method.

I shall not present the construction here, but let me mention one simple idea, namely how one can obtain a set of integers, not in $\mathfrak{M}$ but definable: Construct a two-step generic extension. In the first step, adjoin a Cohen-generic subset A of ω, and then use Easton's method (Theorem 46) to change the continuum function so that in the resulting model we have $2^{\aleph_n} = \aleph_{n+1}$ if $n \in A$, and $2^{\aleph_n} = \aleph_{n+2}$ otherwise. In the model, the set A is definable by

$$A = \{n \in \omega : 2^{\aleph_n} = \aleph_{n+1}\}$$

The above discussion and Exercise 25.8 show that both $L = \text{HOD} \neq V$ and $L \neq \text{HOD} = V$ are consistent (and so is $L \neq \text{HOD} \neq V$). Moreover, there are models in which $(\text{HOD})^{\text{HOD}} \neq \text{HOD}$: The submodel HOD has fewer definable subsets than the given model. In fact, one can even construct, for any ordinal θ, a transfinite sequence of models $\mathfrak{M}_\alpha$, $\alpha < \theta$, such that

$$\mathfrak{M}_{\alpha+1} = (\text{HOD})^{\mathfrak{M}_\alpha}$$

for every $\alpha < \theta$ (and $\mathfrak{M}_\lambda = \bigcap_{\alpha<\lambda} \mathfrak{M}_\alpha$ if λ is a limit).

Thus the notion of ordinal definability is, unlike constructibility, not absolute.

In the constructible universe, a rigid complete Boolean algebra can be constructed with much less effort, particularly if the reader is already familiar with the construction of a Suslin tree (Section 22). Below, I shall sketch a construction of a rigid complete Boolean algebra using $\diamondsuit$ (or its version (22.20)).

Construction of a Rigid Complete Boolean Algebra in L
(a sketch)

Any normal Suslin tree can be embedded upside down as a dense set in an atomless complete Boolean algebra: Let $(T, \leq_T)$ be a normal Suslin tree, let $\leq \; = \; \geq_T$ and let $B_T = \text{r.o.}(T, \leq)$. Let U_α, for each $\alpha < \omega_1$, denote the αth level of T. Since B satisfies the c.c.c., there is for each $b \in B$ some α and some $u \subseteq U_\alpha$ such that $b = \sum \{t : t \in u\}$; thus elements of B are represented by subsets of levels of T.

In Exercise 22.15 we used $\diamondsuit$ to construct a rigid Suslin tree. Now I shall show that a slight modification yields a tree T such that B_T is a rigid complete Boolean algebra.

If C is a closed unbounded subset of ω_1, let T^C denote the subtree of T consisting of elements at levels $\alpha \in C$:

$$T^C = \bigcup \{U_\alpha : \alpha \in C\} = \{t \in T : \text{o}(t) \in C\}$$

Lemma 25.6. *If π is an automorphism of B_T, then there is a closed unbounded $C \subseteq \omega_1$ such that $\pi \mid T^C$ is an automorphism of T^C.*

Proof. Let C be the set of all α such that for each $t \in U_\alpha$, there is $s \in U_\alpha$ such that $s = \pi(t)$, and there is $z \in U_\alpha$ such that $z = \pi^{-1}(t)$. It is easy to verify that C is closed.

To show that C is unbounded, let α_0 be arbitrary. Let $A \subseteq B$ be the countable set

$$\{\pi t : t \in U_{\alpha_0}\} \cup \{\pi^{-1} t : t \in U_{\alpha_0}\}$$

There exists $\alpha_1 > \alpha_0$ such that all elements of A are represented by subsets of U_{α_1}. In a similar way we find $\alpha_2 > \alpha_1, \alpha_3, \alpha_4, \ldots$, and then let $\alpha = \lim_{n \to \omega} \alpha_n$. I claim that $\alpha \in C$:

If $t \in U_\alpha$, let t_n, $n < \omega$, be the predecessors of t in T at levels α_n. For each n, let $s_n \in U_{\alpha_n}$ be such that $t_{n+1} \in \pi^{-1}(s_n)$. See Fig. 25.2. The s_n form a chain in T and it follows that there is $s \in U_\alpha$ that has the s_n as predecessors; and $s = \pi(t)$. We find $z = \pi^{-1}(t)$ in U_α similarly. ■

By Lemma 25.6, we have reduced the problem of construction of a rigid complete Boolean algebra to a construction of a normal Suslin tree T such that T^C is rigid, for all closed unbounded $C \in \omega_1$.

The key step in the construction is a modification of Exercise 22.6:

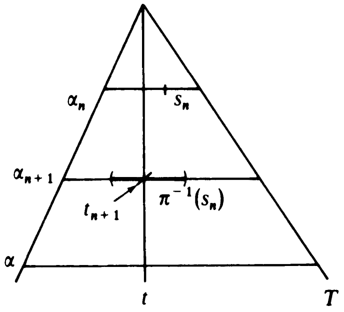

FIGURE 25.2

Lemma 25.7. *If T is a normal α-tree where $\alpha < \omega_1$ is a limit ordinal, if C is a closed unbounded subset of α, and if π is a nontrivial automorphism of T^C, then T has an extension T' of height $\alpha + 1$ such that π cannot be extended to the αth level.* ■

Now we use $\diamondsuit$ and construct a normal Suslin tree as in Lemma 22.8, with the added twist that we also use the version (22.20) of $\diamondsuit$, and whenever the h_α of (22.20) is an automorphism of T_α^C for some closed unbounded $C \subseteq \alpha$, we make sure that h_α does not extend to the αth level. This way we obtain a normal Suslin tree T such that T^C is rigid for all closed unbounded $C \subseteq \omega_1$.

Then B_T is a rigid complete Boolean algebra. ■

Simple Complete Boolean Algebras

Let us consider now the following concept, which is stronger than rigidity: An atomless complete Boolean algebra is *simple* if it has no proper atomless complete subalgebra.

Lemma 25.8. *If B is simple and G is a generic ultrafilter on B, then:*

(i) *every set in $\mathfrak{M}[G]$ is $\mathfrak{M}$-definable in $\mathfrak{M}[G]$;*
(ii) *there is no model $\mathfrak{N}$ of* ZFC *such that $\mathfrak{M} \subsetneq \mathfrak{N} \subsetneq \mathfrak{M}[G]$.*

Proof. The assertion (ii) follows from Lemma 25.3; (i) follows from Corollary 1 of Theorem 59 and the following lemma:

Lemma 25.9. *Every simple complete Boolean algebra is rigid.*

Proof. Let π be a nontrivial automorphism of B; we shall construct a proper atomless subalgebra of B. There exist disjoint a and b such that $\pi a = b$. Let us write each $x \in B$ uniquely as a disjoint sum

$$x = a \cdot x + b \cdot x + y$$

(see Figure 25.1), and let D be the complete subalgebra of B consisting of all those x such that $b \cdot x = \pi(a \cdot x)$. Then $D \neq B$ (since, e.g., $a \notin D$), and is easily seen to be atomless. ■

It can be proved that simple complete Boolean algebras exist in the constructible universe. One possible way to construct such an algebra in L is to modify further the construction of a rigid complete Boolean algebra given above. Given a normal Suslin tree T one can characterize all complete subalgebras of B_T (in fact, they can be identified with certain equivalence relations on T); and using $\diamondsuit$, one can arrange the construction of a Suslin tree T such that B_T is simple.

Thus existence of simple complete Boolean algebras is consistent; however, it is open whether one can prove their existence in ZFC.

The construction of a simple complete Boolean algebra in L mentioned above has another interesting application. The Suslin tree T that yields the simple algebra B_T can be constructed so that the function $\alpha \mapsto T_\alpha$ is Δ_1 over L_{ω_1} (see Exercise 22.18). Then if G is an L-generic ultrafilter on B_T, one can use the fact that B_T is simple and define in $L[G]$ a Σ_1 binary relation which well-orders the universe. Thus we obtain a model of ZFC other than the constructible universe, which has a Σ_1 well-ordering.

Suslin Algebras

In Section 22 we defined a *Suslin algebra* as a complete Boolean algebra that is atomless, $\aleph_0$-distributive, and satisfies the countable chain condition. We have also shown that if T is a normal Suslin tree, then B_T is a Suslin algebra (Lemma 22.2). Conversely, it is shown in Exercise 22.3 that if B is a Suslin algebra, then there is a normal Suslin tree $T \subset B - \{0\}$, such that $\leq_T = \geq_B$. It is not difficult to see that then B_T is a complete subalgebra of B.

If T is a normal Suslin tree, then the Suslin algebra B_T has cardinality $2^{\aleph_0}$. The following theorem shows that every Suslin algebra has cardinality at most $2^{\aleph_1}$. Using a notion of forcing that generalizes the forcing conditions in (22.6), we can obtain consistency of a Suslin algebra of arbitrary prescribed size κ (satisfying $\kappa^{\aleph_0} = \kappa$); also, using combinational principles that hold in L (in fact, $\diamondsuit$ and W, see (24.16)), one can construct in L a Suslin algebra of size $\aleph_2$.

Theorem 60. *If a complete Boolean algebra B is a Suslin algebra, then* $|B| \leq 2^{\aleph_1}$.

Proof. The proof presented here uses Boolean-valued models. It follows closely the proof of Theorem 51 (consistency of Martin's axiom). Let $\kappa = 2^{\aleph_1}$. We shall assume that there is a Suslin algebra of size $> \kappa$ and reach a contradiction.

Let B be a Suslin algebra such that $|B| > \kappa$; without loss of generality we may assume that $|B_u| > \kappa$ for every $u \neq 0$.

We shall construct a transfinite sequence

$$B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots, \qquad \alpha < \kappa \tag{25.29}$$

of complete subalgebras of B, each of size $\leq \kappa$. If $D \subseteq B$ and $|D| \leq \kappa$, then there are $\kappa^{\aleph_1} = \kappa$ D-valued names for binary relations on ω_1; thus for each such

D let R_γ^D, $\gamma < \kappa$, be a fixed enumeration of all such names. Let $\alpha \to (\beta_\alpha, \gamma_\alpha)$ be the canonical mapping of κ onto $\kappa \times \kappa$; we recall that $\beta_\alpha \le \alpha$ for all α.

The sequence (25.29) is constructed as follows: We let $B_0 = \{0, 1\}$; if α is a limit ordinal, then B_α is the complete subalgebra of B generated by $\bigcup_{\nu<\alpha} B_\nu$. If $|B_\nu| \le \kappa$ for each $\nu < \alpha$, then $|B_\alpha| < \kappa$. At successor steps, we construct $B_{\alpha+1}$ as follows: Let $D = B_{\beta_\alpha}$ and let $R = R_{\gamma_\alpha}^D$. If

$$\|(\check{\omega}_1, R) \text{ is a Suslin tree}\|_{B_\alpha} = 1 \tag{25.30}$$

if $C \in V^{B_\alpha}$ is the Suslin algebra (in V^{B_α}) corresponding to the Suslin tree and if $B_\alpha * C$ is (isomorphic to) a complete subalgebra of B, then we let $B_{\alpha+1} = B_\alpha * C$. Otherwise, we let $B_{\alpha+1} = B_\alpha$. In either case, if $|B_\alpha| \le \kappa$, then $|B_{\alpha+1}| \le \kappa$.

Now let B_κ be the complete subalgebra of B generated by $\bigcup_{\alpha<\kappa} B_\alpha$. Clearly, $|B_\kappa| \le \kappa$. Let $A \in V^{B_\kappa}$ be the complete Boolean algebra $B : B_\kappa$ (in V^{B_κ}). Since both B and B_κ satisfy the c.c.c., we have, by Lemma 23.6,

$$\|A \text{ satisfies the c.c.c.}\|_{B_\kappa} = 1$$

Similarly, since both B and B_κ are $\aleph_0$-distributive, we have (using for instance Theorem 58b)

$$\|A \text{ is } \aleph_0\text{-distributive}\|_{B_\kappa} = 1$$

We have assumed that $|B_u| > \kappa$ for all $u \ne 0$, and we also have $|B_\kappa| \le \kappa$. Thus

$$\| \, |A| > \kappa \|_{B_\kappa} = 1$$

and consequently

$$\|A \text{ is not atomic}\|_{B_\kappa} = 1$$

Now we work inside V^{B_κ}: Using Exercise 22.3, we find $T \subseteq A$ such that $(T, \ge_A)$ is Suslin tree; let $B_T \subseteq A$ be the Suslin algebra corresponding to T (B_T is the complete subalgebra of A generated by T). Let R be a binary relation on $\check{\omega}_1$ isomorphic to T.

The name R is B_κ-valued; and since B_κ satisfies the c.c.c., R involves at most $\aleph_1$ elements of B_κ (compare with Lemma 23.8). Since cf $\kappa > \aleph_1$, there exists $\beta < \kappa$ such that $R \in V^{B_\beta}$; furthermore, let $\gamma < \kappa$ be such that R is the γth B_β-valued binary relation on $\check{\omega}_1$, $R = R_\gamma^{B_\beta}$.

Let $\alpha < \kappa$ be such that $\beta = \beta_\alpha$ and $\gamma = \gamma_\alpha$. Since

$$\|(\check{\omega}_1, R) \text{ is a Suslin tree}\|_{B_\kappa} = 1$$

we also have (25.30):

$$\|(\check{\omega}_1, R) \text{ is a Suslin tree}\|_{B_\alpha} = 1$$

If C denotes the corresponding Suslin algebra in V^{B_α}, we have (cf. remark following Lemma 23.3)

$$B_\alpha * C \subseteq B_\kappa * B_T \subseteq B_\kappa * A = B$$

and it follows that $B_{\alpha+1} = B_\alpha * C$. However, as the reader knows, forcing with a Suslin tree destroys its Suslinity, and we have

$$\|(\check{\omega}_1, R) \text{ is not a Suslin tree}\|_{B_{\alpha+1}} = 1$$

Which of course is a contradiction. ■

Collapsing Algebras

In Section 19 we defined a notion of forcing that collapses a cardinal λ onto a cardinal $\kappa \leq \lambda$. As defined in (19.12), the forcing conditions $p \in P$ are functions such that

$$\operatorname{dom}(p) \subseteq \kappa, \qquad \operatorname{ran}(p) \subseteq \lambda, \qquad \text{and } |\operatorname{dom}(p)| < \kappa$$

I shall refer to the corresponding complete Boolean algebra as the (κ, λ)-*collapsing algebra*, and use the notation

(25.31) $$Col(\kappa, \lambda) = \text{r.o.}(P)$$

I shall present below several interesting theorems on the collapsing algebra $Col(\aleph_0, \lambda)$. In particular, it will be shown that every complete Boolean algebra can be embedded in a countably generated complete Boolean algebra.

If λ is an infinite cardinal, we let P_λ denote the set of all finite sequences

(25.32) $$p = \langle p(0), \ldots, p(n-1) \rangle \qquad (n \in \omega)$$

of ordinals less than λ; we have $Col(\aleph_0, \lambda) = \text{r.o. } P_\lambda$. For a fixed λ, we use $u_{n\alpha}$ ($n < \omega$, $\alpha < \lambda$) to denote the following elements of $Col(\aleph_0, \lambda)$:

(25.33) $$u_{n\alpha} = \sum \{p \in P_\lambda : p(n) = \alpha\}.$$

Lemma 25.10. *Every* $(\aleph_0, \lambda)$*-collapsing algebra is* $\aleph_0$*-generated.*

Proof. Let λ be an infinite cardinal and let $B = Col(\aleph_0, \lambda)$. We shall find countably many elements b_{nm} ($n, m < \omega$) of B such that B is the least complete subalgebra of B that contains all b_{nm}.

First we note that the collection $\{u_{n\alpha} : n < \omega, \alpha < \lambda\}$ generates B: Since P_λ is dense in B, it suffices to show that each $p \in P_\lambda$ is so generated. If $p = \langle p(0), \ldots, p(n-1) \rangle$, let

$$u = u_{0,p(0)} \cdot \cdots \cdot u_{n-1,p(n-1)}$$

and let us show that $p = u$. On the one hand, $p \leq u_{k,p(k)}$ for all $k < n$, and so $p \leq u$. On the other hand, let us show that every $q \in P_\lambda$, $q \leq u$, is compatible with p. Otherwise, $q(k) \neq p(k)$ for some $k \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$, and then it follows from the definition (25.33) of $u_{k,p(k)}$ that $q \cdot u_{k,p(k)} = 0$; this contradicts the assumption $q \leq u$.

We define, for every n and $m < \omega$,

(25.34) $$b_{nm} = \sum \{p \in P_\lambda : p(n) \leq p(m)\}$$

and show that each $u_{n\alpha}$ is in the complete subalgebra C of B generated by $\{b_{nm} : n, m < \omega\}$. This we prove by induction on α. Let us assume that all $u_{m\beta}$ ($m < \omega$ and $\beta < \alpha$) are in C and let us show that $u_{n\alpha} \in C$. Since $u_{n\beta} \cdot u_{n\gamma} = 0$ whenever $\beta \neq \gamma$, we have

$$u_{n\alpha} = \sum_{\beta \leq \alpha} u_{n\beta} - \sum_{\beta < \alpha} u_{n\beta}$$

and thus it suffices to show that $\sum_{\beta \leq \alpha} u_{n\beta}$ belongs to C.

We shall show that

$$\sum_{\beta \leq \alpha} u_{n\beta} = \prod_{m=0}^{\infty} \left(b_{nm} + \sum_{\beta < \alpha} u_{m\beta} \right) \tag{25.35}$$

Let u denote the left-hand side of (25.35) and let v denote the right-hand side. Let us first show that $u \leq v$. Given $m < \omega$ and $p \in P_\lambda$, $p \leq u$, it suffices to find $q \leq p$ such that either $q \leq b_{nm}$ or $q \leq \sum_{\beta < \alpha} u_{m\beta}$. So let $p \leq u$; we let $q \leq p$ be arbitrary such that both $n, m \in \mathrm{dom}(q)$. Since $q \leq u$, it is clear that $q(n) \leq \alpha$. Now if $q(m) \geq q(n)$, we have $q \leq b_{nm}$; if $q(m) < q(n)$, we have $q(m) < \alpha$ and so $q \leq \sum_{\beta < \alpha} u_{m\beta}$.

Now let us show that $v \leq u$. We shall show that every $p \leq v$ is compatible with u. Otherwise, there is $p \leq v$ such that $n \in \mathrm{dom}(p)$ and $p(n) > \alpha$. Then we find q stronger than p such that for some m, $q(m) = \alpha$. But then q is incompatible with $\sum_{\beta < \alpha} u_{m\beta}$; and also, since $q(n) > q(m)$, q is incompatible with b_{nm}. This contradicts the assumption that $q \leq p \leq b_{nm} + \sum_{\beta < \alpha} u_{m\beta}$.

Thus each $u_{n\alpha}$ belongs to C and therefore $C = B$. Hence B is $\aleph_0$-generated. ■

Since $|Col(\aleph_0, \lambda)| = |P_\lambda|^{\aleph_0} = \lambda^{\aleph_0}$, we obtain

Theorem 61 (Gaifman, Hales). *There are arbitrarily large $\aleph_0$-generated complete Boolean algebras.* ■

The next theorem will give an even better result.

Lemma 25.11. *Let $(Q, <)$ be a notion of forcing such that $|Q| = \lambda > \aleph_0$ and such that Q collapses λ onto $\aleph_0$, i.e.,*

$$\|\check{\lambda} \text{ is countable}\|_{\mathrm{r.o.}(Q)} = 1$$

Then $\mathrm{r.o.}(Q) = Col(\aleph_0, \lambda)$.

Proof. Without loss of generality we may assume that $(Q, <)$ is a separative partial ordering. We shall find a dense subset of Q isomorphic to P_λ.

Let $B = \mathrm{r.o.}(Q)$, and let G be the canonical name for the generic filter on Q. Let $f \in V^B$ be such that

$$\|f \text{ maps } \check{\omega} \text{ onto } G\|_B = 1$$

For each $p \in P_\lambda$, we shall construct $q(p) \in Q$ such that $D = \{q(p) : p \in P_\lambda\}$ is dense in Q and that $p \mapsto q(p)$ is an isomorphism of P_λ onto D. We construct $q(p)$ by induction on the length of p.

If $p = \langle p(0) \rangle$, we construct $q(p)$ as follows: Since Q collapses λ, there exists an incompatible set $W \subseteq Q$ of size λ. Moreover, we may find such W of size λ with the additional property that each $w \in W$ *decides* $\dot{f}(0)$, i.e., there is $q_w \in Q$ such that $w \Vdash \dot{f}(\check{0}) = \check{q}_w$. Thus let $W_\emptyset$ be a maximal incompatible set of conditions with these properties, $W_\emptyset = \{w_\xi : \xi < \lambda\}$, and for each $p = \langle p(0) \rangle \in P_\lambda$ we let $q(p) = w_\xi$, where $\xi = p(0)$.

Having constructed $q(p)$, where $p = \langle p(0), \ldots, p(n-1) \rangle$, we construct $q(p^\frown \xi)$, $\xi < \lambda$, as follows: We let $W_p = \{w_\xi : \xi < \lambda\}$ be a maximal incompatible set of conditions $w < q(p)$ such that $|W_p| = \lambda$ and that each $w \in W_p$ decides $\dot{f}(n)$. Then we let $q(p^\frown \xi) = w_\xi$, for all $w_\xi \in W_p$.

The set $D = \{q(p) : p \in P_\lambda\}$ is clearly isomorphic to P_λ. Let us show that D is dense in Q. Let $q \in Q$ be arbitrary. Since $q \Vdash \check{q} \in G$, and hence $q \Vdash \check{q} \in \operatorname{ran}(\dot{f})$, there is $r \leq q$ and $n < \omega$ such that $r \Vdash q = \dot{f}(\check{n})$. Now there is $p \in P_\lambda$ of length $n + 1$ such that $q(p)$ is compatible with r; since $q(p)$ decides $\dot{f}(n)$, we necessarily have $q(p) \Vdash \dot{f}(\check{n}) = \check{q}$. Therefore, $q(p) \Vdash \check{q} \in G$. Since Q is separative, it follows that $q(p) \leq q$. This proves that D is dense in Q. ■

Theorem 62 (Kripke). *Every complete Boolean algebra can be embedded into an* $\aleph_0$*-generated complete Boolean algebra; in fact, if* $|B| = \lambda$, *then* B *can be embedded in* $Col(\aleph_0, \lambda)$.

Proof. Let B be a complete Boolean algebra, let $|B| = \lambda$. Then the notion of forcing $Q = (B - \{0\}) \times P_\lambda$ has cardinality λ and collapses λ. By Lemma 25.11, r.o.$(Q) = Col(\aleph_0, \lambda)$. In other words, $B \oplus Col(\aleph_0, \lambda)$ is isomorphic to $Col(\aleph_0, \lambda)$, and so B is isomorphic to a complete subalgebra of $Col(\aleph_0, \lambda)$. Since $Col(\aleph_0, \lambda)$ is $\aleph_0$-generated, the theorem follows. ■

We shall now prove a stronger version of Theorem 62.

Lemma 25.12. *Let* B *be a complete Boolean algebra,* $|B| = \lambda$. *Let* C *be a complete subalgebra of* B *such that* $|C| < \lambda$, *and let* h_0 *be an embedding of* C *in* $Col(\aleph_0, \lambda)$. *Then there exists an embedding* h *of* B *in* $Col(\aleph_0, \lambda)$ *such that* $h(c) = h_0(c)$ *for all* $c \in C$.

Proof. See Fig. 25.3. Let D be the image of C under the embedding h_0. Let Col be an abbreviation for $Col(\aleph_0, \lambda)$; let $\boldsymbol{Col}^C$ and $\boldsymbol{Col}^D$ denote, respectively, the $(\aleph_0, \check{\lambda})$-collapsing algebra in the Boolean valued models V^C and V^D.

First, we find an embedding k of B in $C * \boldsymbol{Col}^C$: Working in V^C, we observe that $\check{\lambda}$ is a cardinal and that $B:C$ is complete Boolean algebra that collapses $\check{\lambda}$ onto $\aleph_0$. Also, since $B:C$ is a quotient of $\check{B}$, $B:C$ has cardinality $\check{\lambda}$. Thus by Theorem 61 (we are still in V^B), there is an embedding of $B:C$ in $\boldsymbol{Col}^C$.

By the remark following Lemma 23.3, there is an embedding k of $C * (B:C)$ into $C * \boldsymbol{Col}^C$ such that $k(c) = c$ for all $c \in C$ (by Lemma 23.3, C is considered a

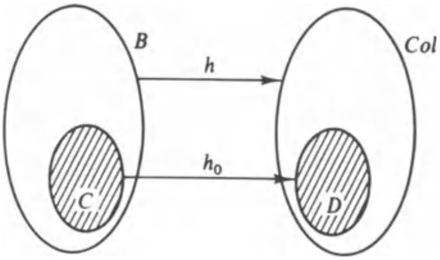

FIGURE 25.3

complete subalgebra of both those algebras). Since $C * (B:C) = C$, we have

$$k\colon B \to C * Col^C$$

such that $k(c) = c$ for all $c \in C$.

Next we find an isomorphism between Col and $D * Col^D$: Working in V^D, we observe that $\check{\lambda}$ is a cardinal, and that $Col:D$ collapses $\check{\lambda}$ onto $\aleph_0$. Also, since the algebra Col^V has a dense subset $\check{P}_\lambda$ of size $\check{\lambda}$, its quotient $Col:D$ has a dense subset Q of size $\check{\lambda}$. Thus by Lemma 25.11 (applied in V^D), there is an isomorphism between $Col:D$ and Col^D.

By the same argument as above, we get an isomorphism π between $Col = D * (Col:D)$ and $D * Col^D$:

$$\pi\colon Col \to D * Col^D$$

such that $\pi(d) = d$ for all $d \in D$.

Since C and D are isomorphic, there exists an isomorphism

$$\sigma\colon C * Col^C \to D * Col^D$$

such that $\sigma(c) = h_0(c)$ for all $c \in C$. Thus we define $h\colon B \to Col$ as follows: $h(b) = \pi^{-1}(\sigma(k(b)))$, for all $b \in B$ (see Fig. 25.4):

$$\begin{tikzcd}
C * Col^C \arrow[r, "\sigma"] & D * Col^D \\
B \arrow[u, "k"] \arrow[r, "h"] & Col \arrow[u, "\pi"']
\end{tikzcd}$$

FIGURE 25.4

Clearly, h is an embedding of B into Col, and $h(c) = h_0(c)$ for all $c \in C$. ■

Exercise 25.10.* Let G be a generic filter on P_λ and let X be a set of ordinals in $\mathfrak{M}[G]$. Then either $\mathfrak{M}[X] = \mathfrak{M}[G]$ or there exists an $\mathfrak{M}[X]$-generic filter H on P_λ such that $\mathfrak{M}[X][H] = \mathfrak{M}[G]$.

[If λ is uncountable in $\mathfrak{M}[X]$, then $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}[X]$ by $Col^{\mathfrak{M}[X]}(\aleph_0, \kappa)$, where $\kappa = |\lambda|^{\mathfrak{M}[X]}$. However, P_κ is isomorphic in $\mathfrak{M}[X]$ to P_λ. If λ is countable in $\mathfrak{M}[X]$ and $\mathfrak{M}[X] \neq \mathfrak{M}[G]$, then $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}[X]$ by a countable atomless notion of forcing Q. There is only one atomless complete Boolean algebra with a countable dense subset (see Exercise 17.6), and so r.o.(Q) is isomorphic (in $\mathfrak{M}[X]$) to r.o.(P_λ).]

The Lévy Algebra

Let λ be an inaccessible cardinal. We shall now consider the notion of forcing P introduced in (20.7), which collapses each $\alpha < \lambda$ onto $\aleph_0$: The conditions are functions p on subsets of $\lambda \times \omega$ such that each dom(p) is finite, and $p(\alpha, n) < \alpha$ whenever $(\alpha, n) \in$ dom(p). Let B denote the corresponding complete Boolean algebra, the *Lévy algebra.*

We have shown in Section 20 that sat$(B) = \lambda$. Since λ is inaccessible, we have $|B| = \lambda$.

For each ν such that $\omega \leq \nu < \lambda$, let P_ν denote the set of all $p \in P$ such that $\alpha < \nu$ whenever $(\alpha, n) \in$ dom(p). Similarly, let P^ν denote the set of all $p \in P$ such that $\alpha \geq \nu$ whenever $(\alpha, n) \in$ dom(p). Clearly, P is isomorphic to $P_\nu \times P^\nu$. Let $B_\nu =$ r.o.(P_ν) and $B^\nu =$ r.o.(P^ν). Since B is isomorphic to $B_\nu \oplus B^\nu$, B_ν is isomorphic to a complete subalgebra of B having P_ν as a dense subset, and so we may identify B_ν with the complete subalgebra of B generated by P_ν. Thus B_ν is a complete subalgebra of B_μ whenever $\nu < \mu < \lambda$. Moreover, if $\gamma > \omega$ is a limit ordinal, B_γ is the completion of $\bigcup_{\nu<\gamma} B_\nu$, and so we have

$$B_\gamma = \lim_{\nu \to \gamma} \text{dir } B_\nu \tag{25.36}$$

If $u \in B$, then because sat$(B) = \lambda$, there exists $X \subset P$ of size less than λ such that $u \in \Sigma X$. Hence $u \in B_\nu$ for some $\nu < \lambda$, and it follows that $B = \bigcup_{\nu<\lambda} B_\nu$. We also note that if κ is an uncountable cardinal, then $|P_{\kappa+1}| = \kappa$ and $P_{\kappa+1}$ collapses κ onto $\aleph_0$, and so by Lemma 25.11, $B_{\kappa+1}$ is (isomorphic to) $Col(\aleph_0, \kappa)$.

If C is a complete Boolean algebra and $|C| < \lambda$, then by Theorem 62 C can be embedded in some B_ν, $\nu < \lambda$, and hence in B. Moreover, by Lemma 25.12 every embedding of a complete subalgebra of C in B can be extended to an embedding of C in B. We use this to prove the following theorem. Also, using Exercise 25.10, we obtain the following *factor lemma* for the Lévy collapse; we shall use this factor lemma in the proof of Theorem 101 in Section 42:

Exercise 25.11.* Let G be a generic filter on the Lévy collapse P, and let X be a countable set of ordinals in $\mathfrak{M}[G]$. Then there exists an $\mathfrak{M}[X]$-generic filter H on P such that $\mathfrak{M}[X][H] = \mathfrak{M}[G]$.

[Let $\nu < \lambda$ be such that $X \in \mathfrak{M}[G \cap B_\nu]$. Let $K \subseteq P_\nu$ be $\mathfrak{M}[X]$-generic such that $\mathfrak{M}[G \cap B_\nu] = \mathfrak{M}[X][K]$. Hence $\mathfrak{M}[G] = \mathfrak{M}[X][K][G \cap B^\nu]$.]

Theorem 63. *The Lévy algebra B is homogeneous, in the following sense: If A and A' are isomorphic complete subalgebras of B such that $|A| = |A'| < |B|$ and if π_0 is an isomorphism between A and A', then there exists an automorphism π of B such that $\pi(a) = \pi_0(a)$ for all $a \in A$.*

Proof. First we construct increasing sequences of complete subalgebras. $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$, $A'_0 \subseteq A'_1 \subseteq \cdots \subseteq A'_n \subseteq \ldots$, as follows:

$$\begin{tikzcd}
A_0 \arrow[d, "\pi_0"] & \subseteq & A_1 \arrow[d, "\pi_1^{-1}"] & \subseteq B_{\nu_2} = & A_2 \arrow[d, "\pi_2"] & \subseteq & A_3 \arrow[d, "\pi_3^{-1} \cdots"] & \subseteq \\
A'_0 & \subseteq B_{\nu_1} = & A'_1 & \subseteq & A'_2 & \subseteq B_{\nu_3} = & A'_3 & \subseteq
\end{tikzcd}$$

FIGURE 25.5

We let $A_0 = A$ and $A'_0 = A'$. There is ν_1 such that $A'_0 \subseteq B_{\nu_1}$; we let $A'_1 = B_{\nu_1}$. The embedding π_0^{-1} of A'_0 in B can be extended to an embedding π_1^{-1} of A'_1 in B, and we let $A_1 = \pi_1^{-1}[A'_1]$. Then there is $\nu_2 > \nu_1$ such that $A_1 \subseteq B_{\nu_2}$; we let $A_2 = B_{\nu_2}$. Then $\pi_1\colon A_1 \to B$ extends to some $\pi_2\colon A_2 \to B$, and we let $A'_2 = \pi_2[A_2]$. We proceed in this manner ad infinitum.

Clearly, $\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} A'_n = \bigcup_{n=1}^{\infty} B_{\nu_n}$, and $\bigcup_{n=0}^{\infty} \pi_n$ is an automorphism of this Boolean algebra. This automorphism extends to a unique automorphism π_ω of the completion, which is B_ν, where $\nu = \lim_{n\to\infty} \nu_n$.

Now $B = B_\nu \oplus B^\nu$, and the automorphism π_ω of B_ν can be extended to an automorphism π of $B_\nu \oplus B^\nu$. (For instance, let us define π on the dense set $(B_\nu - \{0\}) \times (B^\nu - \{0\})$ by $\pi(u, v) = (\pi_\omega u, v)$.) ■

Corollary. *If u and v are elements of the Lévy algebra B such that $u \neq 0, 1$ and $v \neq 0, 1$, then there exists an automorphism π of B such that $\pi(u) = v$.*

It follows (see Exercise 25.9) that for any formula φ and all $x_1, \ldots, x_n \in \mathfrak{M}$, $\|\varphi(\check{x}_1, \ldots, \check{x}_n)\|_B$ is either 1 or 0. I will conclude this section with a stronger property of the Lévy algebra (Lemma 25.14 below).

Lemma 25.13. *Let C be a complete subalgebra of a complete Boolean algebra B and let $u \in B$ be such that $u \notin C$. Then the complete subalgebra D of B generated by $C \cup \{u\}$ has an automorphism π such that $\pi u \neq u$, and $\pi a = a$ for all $a \in C$.*

Proof. First we note that

$$(25.37) \qquad D = \{x \in B : x = a \cdot u + b - u \text{ for some } a, b \in C\}$$

We define an automorphism π of D as follows: Let $v = \sum \{t \in C : t \leq u\}$, and $w = \sum \{t \in C : t \leq -u\}$ and let $z = -(v + w)$. Each $x \in D$ can be written uniquely as a disjoint sum

$$x = a \cdot u + b - u + c$$

where $a, b, c \in C$, and $c \leq v + w$ while $a \leq z$ and $b \leq z$. It suffices to define π separately for $a \cdot u$, $b - u$, and c.

If $c \leq v + w$, we let $\pi c = c$. If $x = a \cdot u$ where $a \leq z$ (see Fig. 25.6) we let

$$\pi x = a - u$$

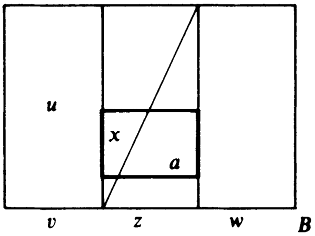

FIGURE 25.6

Similarly, if $x = b - u$ where $b \leq z$, we let

$$\pi x = b \cdot u.$$

It is not difficult to verify that π is an automorphism, and $\pi(z \cdot u) = z - u$. Also, $\pi a = a$ for all $a \in C$. ■

Lemma 25.14. *Let B be the Lévy algebra (in the ground model $\mathfrak{M}$), and let C be a complete subalgebra of B such that $|C| < |B|$. For any formula $\varphi(x)$, if x is a C-valued name,* i.e., *$x \in \mathfrak{M}^C$, then*

$$\|\varphi(x)\|_B \in C$$

Proof. Let $b = \|\varphi(x)\|_B$. If π is an automorphism of B such that $\pi c = c$ for all $c \in C$, then $\pi(x) = x$ and it follows that $\pi(b) = b$. Thus it suffices to show that for any $u \notin C$, there exists an automorphism π of B such that $\pi u \neq u$, and $\pi(c) = c$ for all $c \in C$.

By Lemma 25.13, there exists an automorphism π of a complete subalgebra D of B such that $\pi u \neq u$, and $\pi c = c$ for all $c \in C$; and $|D|$ is still less than $|B|$. By Theorem 63, we can extend such automorphism to an automorphism of B. ■

26. MORE APPLICATIONS OF FORCING

I shall conclude the chapter on forcing with a few more applications and several interesting results on generic models. Further applications of the forcing method will be presented in the chapters on large cardinals and real numbers.

Almost Disjoint Forcing

I shall briefly discuss a useful notion of forcing that has various applications in descriptive set theory. The forcing conditions are constructed with the use of almost disjoint sets; we have already used the method once in Section 23: The forcing conditions were defined in (23.32), in the proof of Theorem 52.

Let $\mathscr{A} = \{A_\alpha : \alpha < \omega_1\}$ be a family of almost disjoint subsets of ω, and let X be a subset of ω_1. We consider the following notion of forcing $P = P_{\mathscr{A},X}$: A condition is a function p from a subset of ω into $\{0, 1\}$ such that:

(26.1) (i) $\operatorname{dom}(p) \cap A_\alpha$ is finite for every $\alpha \in X$;
(ii) $\{n : p(n) = 1\}$ is finite.

The set P is partially ordered by inverse inclusion: $p \leq q$ iff p extends q.

If p and q are incompatible, then $\{n : p(n) = 1\} \neq \{n : q(n) = 1\}$; and it follows that P satisfies the countable chain condition.

If G is a generic filter on P, then $\mathfrak{M}[G] = \mathfrak{M}[A]$, where

$$A = \{n : p(n) = 1 \text{ for some } p \in G\} \tag{26.2}$$

The argument given in Lemma 23.10 shows that

$$X = \{\alpha < \omega_1 : A \cap A_\alpha \text{ is infinite}\} \tag{26.3}$$

A typical application of almost disjoint forcing is the following: Let $\mathfrak{M}[X]$ be a generic extension of the ground model $\mathfrak{M}$ where $X \subseteq \omega_1^{\mathfrak{M}}$; furthermore, assume that $\omega_1^{\mathfrak{M}[X]} = \omega_1^{\mathfrak{M}}$. Let $\mathscr{A} = \{A_\alpha : \alpha < \omega_1\}$ be an almost disjoint family *in* $\mathfrak{M}$, and let us consider the notion of forcing $P = P_{\mathscr{A},X}$ in $\mathfrak{M}[X]$. If $G \subset P$ is generic over $\mathfrak{M}[X]$, then $\mathfrak{M}[X][G] = \mathfrak{M}[X][A]$, where A is defined by (26.2). Note that $\omega_1^{\mathfrak{M}[X][A]} = \omega_1^{\mathfrak{M}}$.

Now in $\mathfrak{M}[X][A]$, the set X satisfies the condition (26.3); and since $\mathscr{A} \in \mathfrak{M}$, it follows that $X \in \mathfrak{M}[A]$, and we have

$$\mathfrak{M}[X][A] = \mathfrak{M}[A]$$

Thus we have found a generic extension $\mathfrak{M}[A]$ of $\mathfrak{M}$ such that $A \subseteq \omega$ and $X \in \mathfrak{M}[A]$.

In other words, under certain conditions one can generically code a set $X \subset \omega_1$ by a set $A \subset \omega$.

Example I. Let $\mathfrak{M}$ be a model of ZFC. There is a generic extension $\mathfrak{M}[A]$ of $\mathfrak{M}$, where $A \subset \omega$, such that $\omega_1^{\mathfrak{M}[A]} = \omega_1^{\mathfrak{M}}$ and $\omega_2^{\mathfrak{M}}$ is collapsed.

Proof. First we add generically (in the usual way) a collapsing map f of ω_1 onto ω_2, such that ω_1 is preserved. There is in $\mathfrak{M}[f]$ a set $X \subset \omega_1$ such that $\mathfrak{M}[X] \models |\omega_2^{\mathfrak{M}}| = \aleph_1$. Then we apply the method described above to find a generic extension $\mathfrak{M}[A]$ of $\mathfrak{M}$ such that $A \subseteq \omega$ and $X \in \mathfrak{M}[A]$, and that $\omega_1^{\mathfrak{M}[A]} = \omega_1^{\mathfrak{M}}$. ■

The method of almost disjoint forcing can be generalized to uncountable cardinals. For instance, let $\mathscr{A} = \{A_\alpha : \alpha < \omega_2\}$ be an almost disjoint family of subsets of ω_1 (i.e., $|A_\alpha \cap A_\beta| \leq \aleph_0$ whenever $\alpha \neq \beta$). If X is a subset of ω_2, we can define $P_{\mathscr{A},X}$ by a straightforward generalization of (26.1), replacing ω by ω_1 and "finite" by "at most countable." $P_{\mathscr{A},X}$ is $\aleph_0$-closed; and assuming $2^{\aleph_0} = \aleph_1$, P satisfies the $\aleph_2$-chain condition and hence preserves cardinals.

This generalization enables us, under certain conditions, to code generically a set $X \subset \omega_2$ by a set $A \subset \omega_1$ (which in turn might be coded by $A' \subset \omega$).

Exercise 26.1. Let $\mathfrak{M}$ be a model of ZFC + GCH. There is a generic extension $\mathfrak{M}[A]$ of $\mathfrak{M}$, where $A \subset \omega$, such that $\omega_1^{\mathfrak{M}[A]} = \omega_1^{\mathfrak{M}}$, $\omega_2^{\mathfrak{M}[A]} = \omega_2^{\mathfrak{M}}$, and $\omega_3^{\mathfrak{M}}$ is collapsed.

Perfect Set Forcing

Several interesting results have been obtained by employing as forcing conditions perfect sets of reals, or a variant of such conditions. I shall present here two such results.

Let $\mathfrak{M}$ be a model of ZFC. We say that a set of ordinals A is *minimal over* $\mathfrak{M}$ if $A \notin \mathfrak{M}$ and if for every set of ordinals $X \in \mathfrak{M}[A]$, we have either $A \in \mathfrak{M}[X]$ or $X \in \mathfrak{M}$.

Theorem 64 (Sacks). *Let $\mathfrak{M}$ be a model of* AFC. *There exists a generic extension $\mathfrak{M}[x]$ of $\mathfrak{M}$ where x is a real, such that x is minimal over $\mathfrak{M}$.*

Exercise 26.2. A Cohen-generic real is not minimal over the ground model.

[Show that P is isomorphic to $P \times P$, and therefore $\mathfrak{M}[x] = \mathfrak{M}[x_1][x_2]$, where x_1 is Cohen-generic over $\mathfrak{M}$ and x_2 is Cohen-generic over $\mathfrak{M}[x_1]$. Consequently, $x_1 \notin \mathfrak{M}$ and $x \notin \mathfrak{M}[x_1]$.]

We note in passing that if B is a simple complete Boolean algebra, then the corresponding generic extension is minimal. In connection with the problem of existence of a simple complete Boolean algebra, it is known that the algebra used in the proof of Theorem 64 is not simple (it is not rigid).

Let me recall that a set of reals is *perfect* if it is nonempty, closed, and has no isolated points. The following lemma is instrumental in the proof of Theorem 64.

Let Seq denote the set $Seq = \bigcup_{n=0}^{\infty} {}^n\{0, 1\}$ of all finite 0, 1-sequences. Let $\{P_s : s \in Seq\}$ be a family of perfect sets such that:

(26.4) (i) $s \subseteq t$ implies $P_s \supseteq P_t$;
(ii) $P_{s0} \cap P_{s1} = \emptyset$, for all $s \in Seq$.

It is easy to see that

$$(26.5)\qquad \bigcap_{n=0}^{\infty} \bigcup_{s \in {}^{n}\{0,1\}} P_s = \bigcup_{f \in {}^{\omega}\{0,1\}} \bigcap_{n=0}^{\infty} P_{f|n}$$

Let us call the set (26.5) the *fusion* of $\{P_s : s \in Seq\}$.

Lemma 26.1 (Fusion Lemma). *Let* $\{P_s : s \in Seq\}$ *be a family of perfect sets satisfying* (26.4). *Furthermore, assume that the diameter of* P_s *tends to* 0 *with increasing length of* s, *i.e.,*

$$(26.6)\qquad \lim_{n \to \infty} (\text{diameter } P_{f|n}) = 0 \qquad (\textit{all } f \in {}^{\omega}\{0, 1\})$$

Then the fusion of $\{P_s : s \in Seq\}$ *is a perfect set.*

Proof. For each $f\colon \omega \to \{0, 1\}$, let $P_f = \bigcap_{n=0}^{\infty} P_{f|n}$; the fusion is the set $P = \bigcup \{P_f : f \in {}^{\omega}\{0, 1\}\}$. By (26.5), P is a closed set. By (26.6), each P_f is nonempty and has exactly one element: $P_f = \{x_f\}$. We shall show that no x_f is an isolated point of P.

Let $\varepsilon > 0$. There exists n such that the diameter of $P_{f|n}$ is $< \varepsilon$. Let $g\colon \omega \to \{0, 1\}$ be such that $g|n = f|n$ and $g(n) \neq f(n)$. Then $x_g \neq x_f$ and $|x_g - x_f| < \varepsilon$. Thus x_f is not isolated and it follows that P is perfect. ■

Let us recall the Cantor–Bendixson analysis of closed sets: Every uncountable closed set is the union of a perfect set and a countable set. One of its consequences is part (a) of the following lemma, which is a very weak instance of a general theorem to be proved in Chapter 7.

Lemma 26.2.

(a) *If* P *and* Q *are perfect sets, then either* $P \cap Q$ *or* $P - Q$ *contains a perfect subset.*

(b) *For every perfect set* P *there exist perfect sets* $Q_1 \subset P$, $Q_2 \subset P$ *such that* $Q_1 \cap Q_2 = \emptyset$.

Proof. (a) Let U be the complement of Q. The open set U is the union $U = \bigcup_{n=0}^{\infty} I_n$ of countably many closed intervals (with rational endpoints). Each $P \cap I_n$ is either perfect or finite, and so $P - Q = P \cap U = \bigcup_{n=0}^{\infty} (P \cap I_n)$ either contains a perfect subset or is at most countable. In the latter case, $P \cap Q$ is uncountable and hence contains a perfect subset.

(b) Easy. ■

Proof of Theorem 64. The forcing conditions are the perfect sets of reals (in $\mathfrak{M}$) and $P \leq Q$ just in case $P \subseteq Q$.

Thus P and Q are incompatible if and only if $P \cap Q$ does not contain a perfect subset. (Then by Lemma 26.2 there exists $Q_1 < Q$ such that $P \cap Q_1 = \emptyset$.)

Let G be an $\mathfrak{M}$-generic set of conditions. (By Lemma 26.2b, G is not in $\mathfrak{M}$.) We shall first show that there is a real $x_G \in \mathfrak{M}[G]$ such that $\mathfrak{M}[G] = \mathfrak{M}[x_G]$.

Every condition P, being a closed set, is the complement of a certain union of open intervals with rational endpoints (in $\mathfrak{M}$). If we carry out the same construction in $\mathfrak{M}[G]$, we obtain a perfect set $P^* = P^{\mathfrak{M}[G]}$; clearly, we have $P = P^* \cap \mathfrak{M}$, and moreover, P^* is the closure of the set P (in $\mathfrak{M}[G]$).

Let us consider (in $\mathfrak{M}[G]$) the collection $\mathscr{F} = \{P^* : P \in G\}$ of perfect sets. $\mathscr{F}$ has the finite intersection property and, by genericity of G, contains sets of arbitrarily small diameter. Hence its intersection contains a single element $x = x_G$:

$$\bigcap_{P \in G} P^* = \{x_G\} \tag{26.7}$$

The following lemma shows that $\mathfrak{M}[G] = \mathfrak{M}[x]$.

Lemma 26.3. $G \in \mathfrak{M}[x]$.

Proof. For each condition P, let $P^x = P^{\mathfrak{M}[x]}$; as before, $P = P^x \cap \mathfrak{M}$, and $P^x = P^* \cap \mathfrak{M}[x]$. We shall show that

$$G = \{P : x \in P^x\} \tag{26.8}$$

thus giving a definition of G in $\mathfrak{M}[x]$.

On the one hand, if $P \in G$, then $x \in P^*$ and so $x \in P^x$. On the other hand, let $P \notin G$. By the genericity of G, if a condition is compatible with every element of G, then it is itself in G. Thus there is $Q_1 \in G$ such that P and Q_1 are incompatible. Now it follows by Lemma 26.2a (and by genericity again) that there is $Q \in G$ such that $Q \subset Q_1$ and $P \cap Q = \emptyset$.

We may also assume that Q is bounded, and then P and Q have positive distance; i.e., there is $\varepsilon > 0$ such that any $p \in P$ and $q \in Q$ have distance $\geq \varepsilon$. Thus in $\mathfrak{M}[G]$, P^* and Q^* have distance $\geq \varepsilon$. This means that we have found $Q \in G$ such that $P^* \cap Q^* = \emptyset$.

Hence $x \in Q^*$ and therefore $x \notin P^*$. Thus we have $x \notin P^x$, and (26.7) is proved. ■

We shall show that x is minimal over $\mathfrak{M}$. Let X be a set of ordinals in $\mathfrak{M}[G]$; we wish to show that either $X \in \mathfrak{M}$ or $x \in \mathfrak{M}[X]$.

Let $\underset{\sim}{X}$ be a name for X. Let D_1 and D_2 be the following sets of conditions:

$$D_1 = \{P : P \text{ decides } \check{\alpha} \in \underset{\sim}{X} \text{ for all } \alpha\}$$

$$D_2 = \{P : (\forall Q \leq P)\exists\alpha \text{ such that } Q \text{ does not decide } \check{\alpha} \in \underset{\sim}{X}\}$$

Clearly $D = D_1 \cup D_2$ is a dense set of conditions and so $G \cap D \neq \emptyset$. If there is $P_1 \in G \cap D_1$, then $X \in \mathfrak{M}$ because

$$X = \{\alpha : P_1 \Vdash \check{\alpha} \in \underset{\sim}{X}\}$$

So let us assume that $G \cap D_1$ is empty, let $P_2 \in G \cap D$ and let us show that $x_G \in \mathfrak{M}[X]$.

For every $P \subseteq P_2$, we construct a family $\{P_s : s \in Seq\}$ as follows:
For $s = \emptyset$, we let $P_\emptyset = P$.
Having chosen P_s for all $s \in Seq$ of length n, we pick P_{s0} and P_{s1} so that:

(26.9) (i) $P_{s0} \subset P_s$ and $P_{s1} \subset P_s$;
(ii) $P_{s0} \cap P_{s1} = \emptyset$;
(iii) the diameter of both P_{s0} and P_{s1} is at most $1/n$;
(iv) there exists $\alpha = \alpha_s$ such that

$$P_{s0} \Vdash \check{\alpha}_s \in X \qquad \text{and} \qquad P_{s1} \Vdash \check{\alpha}_s \notin X$$

Condition (iv) can be satisfied because $P_s \subseteq P_2$ and $P_2 \in D_2$.

The family $\{P_s : s \in Seq\}$ satisfies (26.4) and (26.6), and therefore its fusion Q_P is a perfect set. Since the set of conditions $\{Q_P : P \subseteq P_2\}$ is dense below P_2 and $P_2 \in G$, there exists $P \subseteq P_2$ such that $Q_P \in G$. Let us consider such P, and the corresponding family $\{P_s : s \in Seq\}$.

For every $n \in \omega$, we have

$$Q_P \subseteq \bigcup \{P_s : s \in {}^n\{0, 1\}\}$$

and so there is a unique s of length n such that $P_s \in G$. Moreover, if $P_s \in G$, then by (26.9(iv))

$$P_{s0} \in G \quad \text{iff} \quad \alpha_s \in X$$

and

$$P_{s1} \in G \quad \text{iff} \quad \alpha_s \notin X$$

Thus if we define in $\mathfrak{M}[X]$ a function $f\colon \omega \to \{0, 1\}$ as follows (by induction):

$$f(n) = \begin{cases} 0 & \text{if } \alpha_{f|n} \in X \\ 1 & \text{if } \alpha_{f|n} \notin X \end{cases}$$

we have for each $s \in Seq$,

$$P_s \in G \quad \text{iff} \quad s \subset f \tag{26.10}$$

We shall complete the proof by showing that $x_G \in \mathfrak{M}[X]$: Let P^X denote, for each condition P, the perfect set $P^{\mathfrak{M}[X]}$ in $\mathfrak{M}[X]$, and let us consider the intersection (in $\mathfrak{M}[X]$)

$$\bigcap_{n=0}^{\infty} P^X_{f|n} \tag{26.11}$$

By (26.9)(iii), the set in (26.11) has exactly one element y. Similarly, the intersection (in $\mathfrak{M}[G]$)

$$\bigcap_{n=0}^{\infty} P^*_{f|n} \tag{26.12}$$

(where $P^* = P^{\mathfrak{M}[G]}$) has also exactly one element; and since $x_G \in P^*_{f|n}$ for all n (because $P_{f|n} \in G$), it follows that $y = x_G$. This shows that $x_G \in \mathfrak{M}[X]$, and so x_G is minimal over $\mathfrak{M}$. ■

An argument employing the fusion lemma is used below to show that forcing with perfect sets does not collapse $\aleph_1$. If, in addition, one assumes that $2^{\aleph_0} = \aleph_1$ holds in $\mathfrak{M}$, then there are only $\aleph_1$ forcing conditions, and so all cardinals above (and including) $\aleph_2$ are preserved.

Lemma 26.4. *If $f \in \mathfrak{M}[G]$ is a function from ω into $\omega_1^{\mathfrak{M}}$, then the range of f is bounded by some $\alpha < \omega_1^{\mathfrak{M}}$.*

Proof. Let f be a name for a function from ω into ω_1. It suffices to show that for every condition P there exists a stronger condition Q and an ordinal $\alpha < \omega_1$ such that

$$Q \Vdash \operatorname{ran}(f) \subseteq \check{\alpha} \tag{26.13}$$

Let P be a condition. We construct a family $\{P_s : s \in Seq\}$ as follows:
For $s = \emptyset$, we let $P_\emptyset = P$.
Having chosen P_s for all $s \in Seq$ of length n, we pick P_{s0} and P_{s1} so that:

(26.14)
(i) $P_{s0} \subset P_s$ and $P_{s1} \subset P_s$;
(ii) $P_{s0} \cap P_{s1} = \emptyset$;
(iii) the diameter of both P_{s0} and P_{s1} is at most $1/n$;
(iv) there exist countable ordinals α_{s0} and α_{s1} such that

$$P_{s0} \Vdash f(\check{n}) = \check{\alpha}_{s0} \qquad \text{and} \qquad P_{s1} \Vdash f(\check{n}) = \check{\alpha}_{s1}$$

Then we let Q be the fusion of $\{P_s : s \in Seq\}$, and let α be a countable ordinal greater than all α_s, $s \in Seq$.

Let us show that Q and α satisfy (26.13). Otherwise, there is $Q_1 \subseteq Q$, and there is $n \in \omega$ such that

$$Q_1 \Vdash f(\check{n}) \geq \check{\alpha} \tag{26.15}$$

However, since $Q \subseteq \bigcup \{P_s : s \in {}^n\{0, 1\}\}$, there exists $s \in Seq$ of length n such that $Q_1 \cap P_s$ contains a perfect subset Q_2. Then by (26.14)(iv), Q_2 forces $f(\check{n}) = \check{\alpha}_s$, contrary to assumption (26.15). Hence Q forces that the range of f is bounded by α, and the lemma follows. ∎

Exercise 26.3. Every function f: $\omega \to \omega$ in $\mathfrak{M}[G]$ is majorized by some g: $\omega \to \omega$, $g \in \mathfrak{M}$.
[Use fusion: For each $s \in {}^n\{0, 1\}$ consider m_s such that $P_s \Vdash f(\check{n}) = \check{m}_s$; then let $g(n) = \max\{m_s : s \in {}^n\{0, 1\}\}$.]

In the proof of Theorem 64 we could have used perfect sets in the Baire space, i.e., perfect trees $T \subseteq Seq$; cf. Section 4 (with the same results since ${}^N N$ is homeomorphic to the set of all irrationals). In that case, if G is a generic set of perfect trees, one has $\mathfrak{M}[G] = \mathfrak{M}[f]$, where f is the unique infinite path through all $T \in G$.

Exercise 26.4. Formulate and prove a fusion lemma for perfect trees $T \subseteq Seq$.

The following example (due to Namba) uses a generalization of perfect trees.

Example II. Let $\mathfrak{M}$ be a model of ZFC and $2^{\aleph_0} = \aleph_1$. There is a generic extension $\mathfrak{M}[G]$ of $\mathfrak{M}$ such that $\omega_1^{\mathfrak{M}[G]} = \omega_1^{\mathfrak{M}}$ and $\mathrm{cf}^{\mathfrak{M}[G]}(\omega_2^{\mathfrak{M}}) = \omega$.

What makes this example interesting is that ω_2 is collapsed, but its cofinality is changed not to ω_1 (as is the case when the "ordinary" collapse is performed) but to ω. It follows from Jensen's work on singular cardinals that in the absence of large cardinals, this example is the only possible nontrivial change of cofinality. For more, see Jensen's covering theorem in Section 30.

Let S be the set of all finite sequences of ordinals less than ω_2:

$$S = \bigcup_{n=0}^{\infty} {}^{n}\omega_2$$

A *tree* is a set $T \subseteq S$ such that

(26.16) if $t \in T$ and $s = t \upharpoonright n$ for some n, then $s \in T$.

A nonempty tree $T \subseteq S$ is *perfect* if every $t \in T$ has $\aleph_2$ extensions $s \supset t$ in T. (Note that then every $t \in T$ has $\aleph_2$ *incompatible* extensions in T.)

In analogy with perfect sets in the Baire space, we have a Cantor–Bendixson analysis of trees $T \subseteq S$: Let

$$T' = \{t \in T : t \text{ has } \aleph_2 \text{ extensions in } T\}$$

and let $T_0 = T$, $T_{\alpha+1} = T'_\alpha$, $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$ if α is a limit. Let $\theta < \omega_3$ be the least θ such that $T'_\theta = T_\theta$. Then T_θ is either empty or perfect.

If T has no perfect $\bar{T} \subseteq T$, then the above procedure leads to $T_\theta = \emptyset$, and we can associate with each $t \in T$ an ordinal number

$$h_T(t) = \text{the least } \alpha \text{ such that } t \notin T_{\alpha+1} \tag{26.17}$$

If $s \subseteq t$, then $h_T(s) \geq h_T(t)$, and for every $t \in T$,

$$|\{s \in T : t \subseteq s \text{ and } h_T(s) = h_T(t)\}| < \aleph_2 \tag{26.18}$$

Now let me describe the notion of forcing that is used in this example.

Let P be (in the ground model $\mathfrak{M}$) the set of all perfect trees $T \subseteq S$, partially ordered by inclusion. We shall show that in the generic extension, ω_2 has cofinality ω and ω_1 is preserved.

If G is a generic set of conditions, we define in $\mathfrak{M}[G]$ a function f: $\omega \to \omega_2^{\mathfrak{M}}$ as follows:

$$f(n) = \alpha \quad \text{iff} \quad \exists T \in G \ \exists s \in T \text{ such that } s(n) = \alpha$$

An easy argument using genericity of G shows that $f(n)$ is uniquely defined for each n and that the function f maps ω cofinally into $\omega_2^{\mathfrak{M}}$.

Let me prove now that ω_1 is preserved in the extension by showing that every f: $\omega \to \{0, 1\}$ in $\mathfrak{M}[G]$ is in the ground model. Thus let T be a condition, let $f \in \mathfrak{M}^B$, and let

$$T \Vdash f \text{ is a function from } \check{\omega} \text{ into } \{0, 1\}$$

We shall find a stronger condition that decides each $f(\check{n})$; that is, we shall find a function $g\colon \omega \to \{0, 1\}$ such that some condition stronger than T forces $f(\check{n}) = g(n)$, for all n.

We proceed as follows. By induction on length of s, we construct, for each $s \in S$, conditions T_s and numbers $\alpha_s \in \{0, 1\}$ such that:

(26.19) (i) if $s_1 \subseteq s_2$, then $T_{s_1} \supseteq T_{s_2}$

(ii) if length$(s) = n$, then $T_s \Vdash f(\check{n}) = \check{\alpha}_s$

(iii) for every n, the conditions T_s, $s \in {}^n\omega_2$, are mutually incompatible, and moreover, there are mutually incompatible sequences $t_s \in S$, $s \in {}^n\omega_2$ such that for each $s \in {}^n\omega_2$

$$t_s \in T_s \qquad \text{and} \qquad \text{for all } t \in T_s, \quad \text{either } t \subseteq t_s \text{ or } t_s \subseteq t$$

The "moreover" clause in (iii) is stronger than incompatibility of the conditions and implies that any condition stronger than $\bigcup_{s \in {}^n\omega_2} T_s$ is compatible with some T_s, $s \in {}^n\omega_2$.

The construction of conditions satisfying (26.19) is straightforward: We let $t_\emptyset = \emptyset$ and $T_\emptyset \subseteq T$ be any condition that decides $f(0)$: $T_\emptyset \Vdash f(\check{0}) = \alpha_\emptyset$. Having defined T_s, t_s, and α_s for $s \in {}^n\omega_2$ we first pick $\aleph_2$ incompatible extensions t_{si}, $i < \omega_2$, of t_s in T_s, and then find $T_{si} \subseteq T_s$ and α_{si}, $i < \omega_2$, such that $T_{si} \Vdash f(n + 1) = \alpha_{si}$ and that each $t \in T_{si}$ is compatible with t_{si}. Note that if $s_1 \subseteq s_2$, then $t_{s_1} \subseteq t_{s_2}$.

For any function $g\colon \omega \to \{0, 1\}$, we define a tree $T(g) \subseteq S$ (not necessarily a perfect tree) as follows: If $\bar{\beta} = \langle \beta_0, \ldots, \beta_n \rangle$ is a finite sequence of zeros and ones, we let

$$(26.20) \quad T(\bar{\beta}) = \bigcup \{T_s : s \in {}^n\omega_2 \text{ and } \langle \beta_0, \ldots, \beta_n \rangle = \langle \alpha_\emptyset, \ldots, \alpha_{s|k}, \ldots, \alpha_s \rangle\}$$

and

$$(26.21) \qquad T(g) = \bigcap_{n=1}^{\infty} T(g|n)$$

Each $T(\bar{\beta})$ is a condition (a perfect tree) and by the remark following (26.19), we have

$$T(\bar{\beta}) \Vdash f(\check{k}) = \check{\beta}_k \qquad (k = 0, \ldots, n)$$

Thus, if we show that there is at least one $g\colon \omega \to \{0, 1\}$ such that the tree $T(g)$ contains a perfect subtree, our proof will be complete.

Lemma 26.5. *There exists $g\colon \omega \to \{0, 1\}$ such that $T(g)$ contains a perfect subtree.*

Proof. Let us assume that no $T(g)$ has a perfect subtree. Then by (26.18) there exists, for each $g\colon \omega \to \{0, 1\}$, a function $h_g\colon T(g) \to \omega_3$ such that $h_g(s) \geq h_g(t)$ whenever $s \subseteq t$, and that for each $t \in T(g)$, there are at most $\aleph_1$ elements $s \supseteq t$ in $T(g)$ such that $h_g(s) = h_g(t)$.

By induction, we construct a sequence $s_0 \subset s_1 \subset \cdots \subset s_n \subset \cdots$ such that for each n, $s_n \in {}^n\omega_2$. Having constructed s_n, we consider the node t_{s_n} of T_{s_n}. Since there are only $\aleph_1$ functions $g: \omega \to \{0, 1\}$, there exists $i < \omega_2$ such that $h_g(t_{s_n i}) < h_g(t_{s_n})$ for all g for which $h_g(t_{s_n i})$ is defined. We let $s_{n+1} = s_n{}^\frown i$.

Given the sequence s_n, $n = 0, 1, \ldots$, we consider the function $g: \omega \to \{0, 1\}$ defined by

$$g(n) = \alpha_{s_n}, \qquad n = 0, 1, \ldots$$

Now, by (26.20) and (26.21), each t_{s_n} belongs to $T(g)$, and so $h_g(t_{s_n})$ is defined for all n. However, this yields a descending sequence of ordinals $h_g(t_{s_0}) > h_g(t_{s_1}) > \cdots$, a contradiction. ∎

Degrees of Constructibility

The relation between two sets of ordinals

(26.22) x is constructible from y, i.e., $x \in L[y]$

is a transitive relation (and reflexive). Thus one may be tempted to investigate the partial ordering of the corresponding equivalence classes, the *degrees of constructibility*. One might even expect a certain similarity between degrees of constructibility and degrees of recursive unsolvability (Turing degrees) investigated in recursion theory.

Unlike the case for Turing degrees, there has been no systematic study of degrees of constructibility. There are a few facts known and a few results that I should mention.

Theorem 64 provides an example of a minimal degree. Using a similar technique, models have been obtained where the degrees of constructibility form various finite patterns. Not much is known about, however, how, for instance, the initial segments of degrees look in general. (One result worth mentioning is a model in which the nonconstructible degrees have order-type ω^*, i.e., the order-type of negative integers.)

Any two degrees have a least upper bound. (This is easy: If A and B are sets of ordinals, let $C = \Gamma[A \times B]$ where Γ is the canonical mapping of Ord^2 onto Ord; the degree of C is the l.u.b. of the degrees of A and B.) The following theorem of Hájek, stated here without proof, shows that the same is not true for lower bounds:

It is consistent that there are two reals X, Y, Cohen-generic over L, such that the constructibility degrees of X and of Y have no greatest lower bound.

It follows that $L[X] \cap L[Y]$ is not a model of ZFC (and it is unclear whether it is even a model of ZF).

The situation is similar for suprema and infima of countable sequences of degrees. For instance, if $d_0, d_1, \ldots, d_n, \ldots$ is an increasing (or decreasing) sequence of L-degrees, then its supremum (infimum) may or may not exist.

A related problem is the structure of transitive models of ZFC (containing all ordinals). Above, I mentioned an example of two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of ZFC such that $\mathfrak{M}_1 \cap \mathfrak{M}_2$ is not a model of ZFC. Similarly, there is an example of a decreasing sequence $\mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \cdots \supset \mathfrak{M}_n \supset \cdots$ of models such that $\bigcap_{n=0}^{\infty} \mathfrak{M}_n$ is not a model. Or consider this:

Exercise 26.5. If $\mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_n \subset \cdots$ is a strictly increasing sequence of models (containing all ordinals), then $\mathbf{U} = \bigcup_{n=0}^{\infty} \mathfrak{M}_n$ is not a model of ZF.

[If $\mathbf{U} \cap V_\alpha \in \mathbf{U}$ for all α, then there is n such that $\mathbf{U} \cap V_\alpha \in \mathfrak{M}_n$ for arbitrarily large α; hence $\mathbf{U} \subseteq \mathfrak{M}_n$.]

A somewhat better picture can be obtained when generic extensions are used. For instance:

Exercise 26.6. If $B_0 \supseteq B_1 \supseteq \cdots \supseteq B_n \supseteq \cdots$ is a sequence in $\mathfrak{M}$ of complete Boolean algebras and if $G \subset B$ is generic over $\mathfrak{M}$, then

$$\mathfrak{N} = \bigcap_{n=0}^{\infty} \mathfrak{M}[G \cap B_n]$$

is a model of ZF.

[$\mathfrak{N}$ is transitive and closed under Gödel operations. To show that for each α, $V_\alpha \cap \mathfrak{N} \in \mathfrak{N}$, note that for every n, $V_\alpha \cap \mathfrak{N} \in \mathfrak{M}[G \cap B_n]$:

$$V_\alpha \cap \mathfrak{N} = \bigcap_{k=n}^{\infty} (V_\alpha \cap \mathfrak{M}[G_n \cap B_k])$$

where $G_n = G \cap B_n$.]

Even when we have a sequence of models $\mathfrak{M}[G \cap B_n]$ as in Exercise 26.6, the intersection need not satisfy the axiom of choice. I shall conclude this discussion with the following lemma, which shows that in some cases the intersection of a decreasing sequence of models of ZFC is a model of ZFC:

Lemma 26.6. *Let κ be a cardinal and let B be a κ-distributive complete Boolean algebra. Let $B_0 \supseteq B_1 \supseteq \cdots \supseteq B_\alpha \supseteq \cdots$, $\alpha < \kappa$, be a descending κ-sequence of complete subalgebras of B, and let $B_\kappa = \bigcap_{\alpha<\kappa} B_\alpha$. If G is an $\mathfrak{M}$-generic ultrafilter on B, and $G_\alpha = G \cap B_\alpha$ for all $\alpha \leq \kappa$, then*

$$\bigcap_{\alpha<\kappa} \mathfrak{M}[G_\alpha] = \mathfrak{M}[G_\kappa]$$

Proof. Clearly, $G_\kappa \in \mathfrak{M}[G_\alpha]$ for all $\alpha < \kappa$, and so $\mathfrak{M}[G_\kappa]$ is included in the intersection. To prove the opposite inclusion, it suffices to show that every set of ordinals A that is in each $\mathfrak{M}[G_\alpha]$, $\alpha < \kappa$, belongs to $\mathfrak{M}[G_\kappa]$.

Let A be a set of ordinals, $A \in \mathfrak{M}[G_\alpha]$ for all $\alpha < \kappa$. Working in $\mathfrak{M}[G]$, we assign to each $\alpha < \kappa$ a B_α-valued name A_α for A. Since B is κ-distributive, it follows that the sequence $\{A_\alpha : \alpha < \kappa\}$ is in $\mathfrak{M}$.

We wish to define a B_κ-valued name A_κ for A. For each ordinal ξ, we shall define $w_\xi = \|\xi \in A_\kappa\| \in B_\kappa$, and then show that A is the G_κ-interpretation of A_κ; i.e., that $w_\xi \in G$ if and only if $\xi \in A$.

Let ξ be an ordinal; I shall describe the construction of $w = w_\xi$. For each $\alpha < \kappa$, let $u_\alpha = \|\xi \in A_\alpha\| \in B_\alpha$. By κ-distributivity, there exists a partition $\{p_i : i \in I\}$ of B such that for each $\alpha < \kappa$ and each $i \in I$, either $p_i \leq u_\alpha$ or $p_i \cdot u_\alpha = 0$. Hence for each $\alpha < \kappa$ there exists $I_\alpha \subseteq I$ such that $u_\alpha = \sum \{p_i : i \in I_\alpha\}$. We let

$$w = \sum \{p_i : \exists\alpha_0\ \forall\alpha \geq \alpha_0\ p_i \leq u_\alpha\} \tag{26.23}$$

We shall show that $w \in B_\kappa$, i.e., $w \in B_\alpha$ for all $\alpha < \kappa$. Let $\alpha < \kappa$. It suffices to show that for each $p_i \leq w$ there is $v \in B_\alpha$ such that $p_i \leq v \leq w$. Thus let $p_i \leq w$. Then there is $\alpha_0 \geq \alpha$ such that $p_i \leq u_\beta$ for all $\beta \geq \alpha_0$. Let $v = \prod \{u_\beta : \beta \geq \alpha_0\}$. Clearly $v \in B_\alpha$; and also $p_i \leq v$. By the choice of the p_i, there is $J \subseteq I$ such that $v = \sum \{p_i : j \in J\}$. Now if $p_j \leq v$, then by (26.23), $p_j \leq w$; hence $v \leq w$, and we have $p_i \leq v \leq w$. It follows that $w \in B_\alpha$ for all $\alpha < \kappa$.

It remains to show that for each ξ, the element $w = w_\xi$ is in G if and only if $\xi \in A$. Let p_i be the unique p_i that is in G. Then $w \in G$ just in case $p_i \leq w$. If $\xi \in A$, then $u_\alpha \in G$ and $p_i \leq u_\alpha$ for each α, and hence $p_i \leq w$. Conversely, if $p_i \leq w$, then $p_i \leq u_\alpha$ for eventually all α; hence $u_\alpha \in G$ and so $\xi \in A$. ■

Genericity of Extensions

At the conclusion of Part II, I shall very briefly touch the question of genericity of extensions of models. If $\mathfrak{M}$ and $\mathfrak{N}$ are models of ZFC (with the same ordinals), the question is, When is the model $\mathfrak{N}$ a generic extension of the model $\mathfrak{M}$? Or more specifically, Which transitive models of ZFC containing all ordinals are generic extensions of the constructible universe?

It is known that in the presence of large cardinals, the universe is not a generic extension of L. There are also several results that characterize generic extensions. The following theorem shows that the universe is a generic extension of the class of all hereditarily ordinal definable sets (at least if $V = L[A]$ for some set A).

Theorem 65 (Vopěnka). *Let $V = L[A]$ where A is a set of ordinals. Then V is a generic extension of the model $\mathfrak{M} =$ HOD. There is a Boolean algebra $B \in \mathfrak{M}$, complete in $\mathfrak{M}$, and there is an ultrafilter $G \subset B$, generic over $\mathfrak{M}$ such that $V = \mathfrak{M}[G]$.*

Proof. Let κ be such that $A \subseteq \kappa$. We let

$$C = \text{OD} \cap P(P(\kappa))$$

be the family of all ordinal definable sets of subsets of κ. Let us consider the partial ordering $(C, \subseteq)$.

First I claim that there is a hereditarily ordinal definable partially ordered set $(B, \leq)$ and an ordinal definable isomorphism π between $(C, \subseteq)$ and $(B, \leq)$: There is a definable one-to-one mapping $\mathbf{F}$ of OD into the ordinals. The set C is an ordinal definable set of ordinal definable sets and so $\mathbf{F} | C$ is an

OD one-to-one mapping of C onto $\mathbf{F}[C]$. We let $B = \mathbf{F}[C]$, and define the partial ordering of B so that $(B, \leq)$ is isomorphic to $(C, \subseteq)$. Since $\subseteq \cap C^2$ is an OD relation, we have $(B, \leq) \in \text{HOD}$.

Now $(C, \subseteq)$ is clearly a Boolean algebra. Moreover, if $X \subseteq C$ is ordinal definable, then $\bigcup X$ is ordinal definable and so $\bigcup X = \sum^C X$. Hence the algebra C is OD-complete; and using the OD isomorphism π, we conclude that $(B, \leq)$ is a complete Boolean algebra in $\mathfrak{M} = \text{HOD}$.

Now we let

$$H = \{u \in C : A \in u\}$$

Clearly, H is an ultrafilter on C, and if $X \subset H$ is OD, then $\bigcap X \in H$. Hence $G = \pi[H]$ is an $\mathfrak{M}$-generic ultrafilter on B.

It remains to show that $V = \mathfrak{M}[G]$. Let $f\colon \kappa \to B$ be the function defined by $f(\alpha) = \pi(\{Z \subseteq \kappa : \alpha \in Z\})$. Clearly f is OD, and so $f \in \mathfrak{M} = \text{HOD}$. Now we note that for every $\alpha < \kappa$,

$$\alpha \in A \quad \text{iff} \quad f(\alpha) \in G$$

and therefore $A \in \mathfrak{M}[G]$. Now it follows that $V = L[A] = \mathfrak{M}[G]$. ■

PART III

Large Sets

CHAPTER 5

MEASURABLE CARDINALS

27. THE MEASURE PROBLEM

Let S be an infinite set. A *measure* on S (a *nontrivial σ-additive* measure, to be more precise) is a real-valued function μ on $P(S)$ such that:

(27.1) (i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$;
(ii) if $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$;
(iii) $\mu(\{a\}) = 0$ for all $a \in S$ (nontriviality);
(iv) if X_n, $n = 0, 1, 2, \ldots$, are pairwise disjoint, then

$$\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = \sum_{n=0}^{\infty} \mu(X_n) \qquad (\sigma\text{-additivity})$$

It follows from (ii) that $\mu(X)$, the *measure of* X, is nonnegative for every $X \subseteq S$; in a special case of (iv) we get $\mu(X \cup Y) = \mu(X) + \mu(Y)$ whenever $X \cap Y = \emptyset$ (finite additivity).

More generally, let $\mathscr{F}$ be a σ-complete field of sets. A *measure* on $\mathscr{F}$ is a real-valued function μ on $\mathscr{F}$ satisfying (i)–(iv). Thus a measure on S is in fact a measure on $P(S)$.

An example of a measure on a σ-complete field of sets is the Lebesgue measure on the field of all Lebesgue measurable subsets of the interval $[0, 1] = \{x : 0 \leq x \leq 1\}$. The Lebesgue measure has, in addition to (i)–(iv), the following property:

(27.2) if X is congruent by translation to a measurable set Y, then X is measurable and $\mu(X) = \mu(Y)$.

It is well known that there exist sets of reals that are not Lebesgue measurable, and in fact that there is no measure on $[0, 1]$ with the property (27.2) (*translation invariant* measure):

***Exercise* 27.1.** Let M be a maximal (under $\subseteq$) subset of $[0, 1]$ with the property that $x - y$ is not a rational number, for any pair of distinct $x, y \in M$. Show that M is not Lebesgue measurable.

[Consider the sets $M_q = \{x + q : x \in M\}$ where q is rational. They are pairwise disjoint and $[0, 1] \subseteq \bigcup \{M_q : q \in Q \cap [-1, 1]\} \subseteq [-1, 2]$.]

The natural question to ask is whether the Lebesgue measure can be extended to some measure (not translation invariant) such that all subsets of $[0, 1]$ are measurable, or whether there exists any measure on $[0, 1]$. Or, whether there exists a measure on some set S.

The investigation of this problem has lead to important discoveries in set theory, opening up a new field, the theory of large cardinal numbers, which has far-reaching consequences both in "abstract" set theory and in descriptive set theory.

A measure μ on S is *two-valued* if $\mu(X)$ is either 0 or 1 for all $X \subseteq S$. If μ is a two-valued measure on S, let

$$U = \{X \subseteq S : \mu(X) = 1\} \tag{27.3}$$

It is easy to verify that U is an ultrafilter over S. For instance, let me show that if $X \in U$ and $Y \in U$, then $X \cap Y \in U$. If $\mu(X) = \mu(Y) = 1$, then $X = (X - Y) \cup (X \cap Y)$ and $Y = (Y - X) \cup (X \cap Y)$. If $\mu(X \cap Y)$ were not 1, then $\mu(X - Y) = \mu(Y - X) = 1$, and we would have $\mu(X \cup Y) = 2$.

Next we note that the ultrafilter U is σ-complete. This is so because μ is σ-additive, and an ultrafilter U over S is σ-complete if and only if there is no partition of S into countably many disjoint parts $S = \bigcup_{n=0}^{\infty} X_n$ such that $X_n \notin U$, for all n. In fact:

***Exercise* 27.2.** An ultrafilter over S is κ-complete if and only if there is no partition of S into less than κ pieces, $S = \bigcup_{\alpha<\gamma} X_\alpha$, $\gamma < \kappa$, such that $X_\alpha \notin U$ for all $\alpha < \gamma$.

Thus if μ is a two-valued measure on S, U is a σ-complete ultrafilter over S. Conversely, if U is an σ-complete ultrafilter over S, then the following function is a two-valued measure on S:

$$\mu(X) = \begin{cases} 1 & \text{if } X \in U \\ 0 & \text{if } X \notin U \end{cases} \tag{27.4}$$

Let μ be a measure on S. A set $A \subseteq S$ is an *atom* of μ if $\mu(A) > 0$ and if for every $X \subseteq A$, we have either $\mu(X) = 0$ or $\mu(X) = \mu(A)$.

If μ has an atom A, then

$$U = \{X \subseteq S : \mu(X \cap A) = \mu(A)\} \tag{27.5}$$

is again a σ-complete ultrafilter over S.

A measure μ on S is *atomless* if it has no atoms. Then every set $X \subseteq S$ of positive measure can be split into two disjoint sets of positive measure: $X = Y \cup Z$, and $\mu(Y) > 0$, $\mu(Z) > 0$.

***Exercise* 27.3.** If μ is an atomless measure on S, there exists $Z \subset S$ such that $\mu(Z) = \frac{1}{2}$. More generally, given $Z_0 \subseteq S$, there exists $Z \subset Z_0$ such that $\mu(Z) = \frac{1}{2}\mu(Z_0)$.

[Construct a sequence $S = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_\alpha \supseteq \cdots$, $\alpha < \omega_1$, such that $\mu(S_\alpha) \geq \frac{1}{2}$, and if $\mu(S_\alpha) > \frac{1}{2}$, then $\frac{1}{2} \leq \mu(S_{\alpha+1}) < \mu(S_\alpha)$; if α is a limit ordinal, let $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$. There exists $\alpha < \omega_1$ such that $\mu(S_\alpha) = \frac{1}{2}$.]

We shall eventually prove various strong consequences of existence of a nontrivial σ-additive measure and establish the relationship between the measure problem and large cardinals. Our starting point is the following theorem which shows that if a measure exists, then there exists at least a weakly inaccessible cardinal.

Theorem 66 (Ulam): *If there is a σ-additive nontrivial measure on S, then either there exists a two-valued measure on S and $|S| \geq$ the least (strongly) inaccessible cardinal, or there exists an atomless measure on $2^{\aleph_0}$ and $2^{\aleph_0} \geq$ the least weakly inaccessible cardinal.*

Theorem 66 will be proved in a sequence of lemmas, which will also provide additional information on the measure problem and introduce basic notions and methods of the theory of large cardinals. First we make the following observation. Let κ be the least cardinal that carries a nontrivial σ-additive two-valued measure. Clearly, κ is uncountable and is also the least cardinal that has a nonprincipal countably complete ultrafilter. And we observe that such an ultrafilter is in fact κ-complete:

Lemma 27.1. *Let κ be the least cardinal with the property that there is a nonprincipal σ-complete ultrafilter over κ, and let U be such an ultrafilter. Then U is κ-complete.*

Proof. Let U be a σ-complete ultrafilter over κ, and let us assume that U is not κ-complete. Then there exists a partition $\{X_\alpha : \alpha < \gamma\}$ of κ such that $\gamma < \kappa$ and $X_\alpha \notin U$ for all $\alpha < \gamma$. We shall now use this partition to construct a nonprincipal σ-complete ultrafilter over γ, thus contradicting the choice of κ as the least cardinal that carries such an ultrafilter.

Let f be the mapping of κ onto γ defined as follows:

$$f(x) = \alpha \quad \text{iff} \quad x \in X_\alpha \qquad (x \in \kappa)$$

Now, any mapping of κ onto γ induces a σ-complete ultrafilter over γ; we define $D \subseteq P(\gamma)$ thus:

$$Z \in D \quad \text{iff} \quad f_{-1}(Z) \in U \tag{27.6}$$

(We leave it to the reader to verify this.) Moreover, our mapping f is such that D is nonprincipal: Assume that $\{\alpha\} \in D$ for some $\alpha < \gamma$. Then $X_\alpha \in U$, contrary to our assumption on X_α. Thus γ carries a σ-complete nonprincipal ultrafilter. ■

We are now ready to define the central notion of this chapter. An uncountable cardinal κ is *measurable* if there exists a κ-complete nonprincipal ultrafilter U over κ.

By Lemma 27.1, the least cardinal that carries a nontrivial two-valued σ-additive measure is measurable. Note that if U is a κ-complete nonprincipal ultrafilter over κ, then every set $X \in U$ has cardinality κ because every set of smaller size is the union of less than κ singletons. For similar reasons, κ is a regular cardinal because if κ is singular, then it is the union of less than κ small sets. The next lemma gives a first link of the measure problem with large cardinals.

Lemma 27.2. *Every measurable cardinal is inaccessible.*

Proof. We have just given an argument why a measurable cardinal is regular. Let me show that measurable cardinals are strong limit cardinals. Let κ be measurable, and let us assume that there exists $\lambda < \kappa$ such that $2^\lambda \geq \kappa$; we shall reach a contradiction.

Let S be a set of functions $f\colon \lambda \to \{0, 1\}$ such that $|S| = \kappa$, and let U be a κ-complete nonprincipal ultrafilter over S. For each $\alpha < \lambda$, let X_α be that one of the two sets $\{f \in S : f(\alpha) = 0\}$, $\{f \in S : f(\alpha) = 1\}$ which is in U, and let ε_α be 0 or 1 accordingly. Since U is κ-complete, the set $X = \bigcap_{\alpha<\lambda} X_\alpha$ is in U. However, X has at most one element, viz. the function f that has the values $f(\alpha) = \varepsilon_\alpha$. A contradiction. ■

Let us now turn our attention to measures that are not necessarily two-valued. Let μ be a nontrivial σ-additive measure on a set S. In analogy with (27.3) we consider the ideal of all *null sets*:

$$I_\mu = \{X \subseteq S : \mu(X) = 0\} \tag{27.7}$$

I_μ is a nonprincipal σ-complete ideal over S. Moreover, it has these properties:

(27.8) (i) $\{x\} \in I$ for every $x \in S$;

(ii) every family of pairwise disjoint sets $X \subseteq S$ that are not in I is at most countable.

To see that (ii) holds, note that if W is a disjoint family of set of positive measure, then for each integer $n > 0$, there are only finitely many sets $X \in W$ of measure $\geq 1/n$.

A σ-complete nonprincipal ideal I over S is called *σ-saturated* if it satisfies (27.8). More generally, a κ-complete nonprincipal ideal I over S is *κ-saturated* if it satisfies (27.8)(i) and if every disjoint family $W \subset P(S)$ of sets not in I has size $< \kappa$.

Let me make a small digression: Let I be a κ-complete nonprincipal ideal over S, and let B be the quotient Boolean algebra $B = P(S)/I$. Note that B is κ-saturated (cf. (17.15)) just in case there is no $W \subset P(S)$ of size κ such that $X \notin I$ for each $X \in W$ and $X \cap Y \in I$ whenever X and Y are distinct elements of W.

Exercise 27.4. Let S, I, and B be as above. Show that B is κ-saturated if and only if I is κ-saturated.

[If $W \subset P(S) - I$ has size κ, and $X \cap Y \in I$ whenever $X \neq Y \in W$, let $W = \{X_\alpha : \alpha < \kappa\}$, and let $Z_\alpha = X_\alpha - \bigcup_{\beta < \alpha} X_\beta$; the family $\{Z_\alpha : \alpha < \kappa\}$ is disjoint, and $Z_\alpha \notin I$ for each α.]

The following lemma is an analog of Lemma 27.1:

Lemma 27.3.

(a) *Let κ be the least cardinal that carries a nontrivial σ-additive measure and let μ be such a measure on κ. Then the ideal I_μ of null sets is κ-complete.*

(b) *Let κ be the least cardinal with the property that there is a σ-complete σ-saturated ideal over κ, and let I be such an ideal. Then I is κ-complete.*

Proof. (a) Let us assume that I_μ is not κ-complete. There exists a collection of null sets $\{X_\alpha : \alpha < \gamma\}$ such that $\gamma < \kappa$ and that their union X has positive measure. We may assume without loss of generality that the sets X_α, $\alpha < \gamma$, are pairwise disjoint; let $m = \mu(X)$.

Let f be the following mapping of X onto γ:

$$f(x) = \alpha \quad \text{iff} \quad x \in X_\alpha \qquad (x \in X)$$

The mapping f induces a measure ν on γ:

$$\nu(Z) = \frac{1}{m} \cdot \mu(f_{-1}(Z)) \tag{27.9}$$

The measure ν is σ-additive and is nontrivial since $\nu\{\alpha\} = \mu(X_\alpha) = 0$ for each $\alpha \in \gamma$. This contradicts the choice of κ as the least cardinal that carries a measure.

(b) The proof is similar. We define an ideal J over γ by: $Z \in J$ iff $f_{-1}(Z) \in I$. The induced ideal J is σ-complete and σ-saturated. ∎

Let $\{r_i : i \in I\}$ be a collection of nonnegative real numbers. We define

$$\sum_{i \in I} r_i = \sup \{ \sum_{i \in E} r_i : E \text{ is a finite subset of } I\} \tag{27.10}$$

Note that if the sum (27.10) is finite (i.e., is not ∞), then at most countably many terms r_i are nonzero.

Let κ be an uncountable cardinal. A measure μ on S is called *κ-additive* if for every $\gamma < \kappa$ and for every disjoint collection X_α, $\alpha < \gamma$, of subsets of S, we have

$$\mu\left(\bigcup_{\alpha < \gamma} X_\alpha\right) = \sum_{\alpha < \gamma} \mu(X_\alpha) \tag{27.11}$$

Exercise 27.5. Let μ be a two-valued measure, and let U be the ultrafilter of all sets of measure one. The measure μ is κ-additive if and only if U is κ-complete.

If μ is a κ-additive measure, then the ideal I_μ of null sets is κ-complete. The converse is also true, and we get a better analog of Lemma 27.1 for real-valued measures:

Lemma 27.4. *Let μ be a measure on S, and let I_μ be the ideal of null sets. If I_μ is κ-complete, then μ is κ-additive.*

Proof. Let $\gamma < \kappa$, and let X_α, $\alpha < \gamma$, be disjoint subsets of S. Since the X_α are disjoint, at most countably many of them have positive measure. Thus let us write

$$\{X_\alpha : \alpha < \gamma\} = \{Y_n : n = 0, 1, 2, \ldots\} \cup \{Z_\alpha : \alpha < \gamma\}$$

where each Z_α has measure 0. Then we have

$$\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \mu\left(\bigcup_{n=0}^{\infty} Y_n\right) + \mu\left(\bigcup_{\alpha<\gamma} Z_\alpha\right)$$

Now first μ is σ-additive, and we have

$$\mu\left(\bigcup_{n=0}^{\infty} Y_n\right) = \sum_{n=0}^{\infty} \mu(Y_n)$$

and secondly I_μ is κ-complete and

$$\mu\left(\bigcup_{\alpha<\gamma} Z_\alpha\right) = 0 = \sum_{\alpha<\gamma} \mu(Z_\alpha)$$

Thus $\mu(\bigcup_\alpha X_\alpha) = \sum_\alpha \mu(X_\alpha)$. ∎

Corollary. *Let κ be the least cardinal that carries a nontrivial σ-additive measure and let μ be such a measure. Then μ is κ-additive.*

An uncountable cardinal κ is *real-valued measurable* if there exists a nontrivial κ-additive measure μ on κ. By the corollary above, the least cardinal that carries a nontrivial σ-additive measure is real-valued measurable. We shall show that if a real-valued measurable cardinal κ is not measurable, then $\kappa \leq 2^{\aleph_0}$. Note that if μ is a nontrivial κ-additive measure on κ, then every set of size $< \kappa$ has measure 0, and moreover κ cannot be the union of less than κ sets of size $< \kappa$. Thus a real-valued measurable cardinal is regular. We shall show that it is weakly inaccessible.

We shall first prove the first claim made in the preceding paragraph.

Lemma 27.5.

(a) *If there exists an atomless nontrivial σ-additive measure, then there exists a nontrivial σ-additive measure on some $\kappa \leq 2^{\aleph_0}$.*

(b) *If I is a σ-complete σ-saturated ideal over S, then either there exists $Z \subseteq S$ such that $I | Z = \{X \subset Z : X \in I\}$ is a prime ideal, or there exists a σ-complete σ-saturated ideal over some $\kappa \leq 2^{\aleph_0}$.*

Proof. (a) Let μ be a measure on S. We construct a tree T of subsets of S, partially ordered by reverse inclusion. The 0th level of T is $\{S\}$. Each level of T

consists of pairwise disjoint subsets of S of positive measure. Each $X \in T$ has two immediate successors: We choose two sets Y, Z of positive measure such that $Y \cup Z = X$ and $Y \cap Z = \emptyset$. If α is a limit ordinal, then the αth level consists of all intersections $X = \bigcap_{\xi<\alpha} X_\xi$ such that each X_ξ is on the ξth level of T and such that X has positive measure.

We observe that every branch of T has countable length: If $\{X_\xi : \xi < \alpha\}$ is a branch in T, then the set $\{Y_\xi : \xi < \alpha\}$, where $Y_\xi = X_\xi - X_{\xi+1}$, is a disjoint collection of sets of positive measure. Consequently, T has height at most ω_1. Similarly, each level of T is at most countable, and it follows that T has at most $2^{\aleph_0}$ branches.

Let $\{b_\alpha : \alpha < \kappa\}$, $\kappa \le 2^{\aleph_0}$, be an enumeration of all branches $b = \{X_\xi : \xi < \gamma\}$ such that $\bigcap_{\xi<\gamma} X_\xi$ is nonempty; for each $\alpha < \kappa$, let $Z_\alpha = \bigcap \{X : X \in b_\alpha\}$. The collection $\{Z_\alpha : \alpha < \kappa\}$ is a partition of S into κ sets of measure 0.

Using what is by now a standard argument, we induce a measure ν on κ as follows: Let f be the mapping of S onto κ defined by

$$f(x) = \alpha \qquad \text{iff} \qquad x \in Z_\alpha \qquad (x \in S)$$

and let

$$\nu(Z) = \mu(f_{-1}(Z))$$

for all $Z \subseteq \kappa$. It follows that ν is a nontrivial σ-additive measure on κ.

(b) The proof is similar. We define a tree T as above and then induce an ideal J over κ by letting $Z \in J$ iff $f_{-1}(Z) \in I$. ■

The proof of Lemma 27.5 shows that if μ is atomless, then there is a partition of S into at most $2^{\aleph_0}$ null sets; in other words, μ is not $(2^{\aleph_0})^+$-additive. Hence if κ carries an atomless κ-additive measure, then $\kappa \le 2^{\aleph_0}$ and we have:

Corollary. *If κ is a real-valued measurable cardinal, then either κ is measurable or $\kappa \le 2^{\aleph_0}$.*

Similarly, if κ carries a κ-complete σ-saturated ideal, then either κ is measurable or $\kappa \le 2^{\aleph_0}$.

The measure ν obtained in (a) above is atomless; this follows from the fact that $\kappa \le 2^{\aleph_0}$ and Lemma 27.2.

Exercise 27.6.* Prove directly that the measure ν defined in the proof of Lemma 27.5a is atomless.

[Assume that Z is an atom of ν, and let $Y = f_{-1}(Z)$. If $X \in T$ is such that $\mu(Y \cap X) \ne 0$ and if X_1, X_2 are the two immediate successors of X, then either $\mu(Y \cap X_1) = 0$ or $\mu(Y \cap X_2) = 0$. Prove by induction that on each level of T there is a unique X such that $\mu(Y \cap X) \ne 0$, and that these X's constitute a branch in T of length ω_1; a contradiction.]

By Lemma 27.5a, if there exists an atomless σ-additive measure, then there is one on some $\kappa \le 2^{\aleph_0}$. Clearly, such a measure can be extended to a measure on $2^{\aleph_0}$: For $X \subseteq 2^{\aleph_0}$, we let $\mu(X) = \mu(X \cap \kappa)$. Thus we conclude that there exists an atomless σ-additive measure on the set $\boldsymbol{R}$ of all reals. It turns out that using the same assumption, we can obtain a σ-additive measure on $\boldsymbol{R}$ that extends Lebesgue measure. This can be done by a slight modification of the proof of Lemma 27.5:

Using Exercise 27.3, we construct for each finite 0–1 sequence s, a set $X_s \subseteq S$ such that $X_\emptyset = S$, and for every $s \in Seq$, $X_{s0} \cup X_{s1} = X_s$, $X_{s0} \cap X_{s1} = \emptyset$, and $\mu(X_{s0}) = \mu(X_{s1}) = \frac{1}{2}\cdot \mu(X_s)$. Then we define a measure ν_1 on ${}^\omega 2$ by

$$\nu_1(Z) = \mu(\bigcup \{X_f : f \in Z\})$$

where $X_f = \bigcap_{n=0}^{\infty} X_{f|n}$ for each $f \in {}^\omega 2$. Using the mapping $F: {}^\omega 2 \to [0, 1]$ defined by

$$F(f) = \sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}}$$

we obtain a nontrivial σ-additive measure ν on $[0, 1]$. This measure agrees with Lebesgue measure on all intervals $[k/2^n, (k + 1)/2^n]$, and hence on all Borel sets. Every set of Lebesgue measure 0 is included in a Borel (in fact, G_δ) set of Lebesgue measure 0 and hence has ν-measure 0. Every Lebesgue measurable set X can be written as $X = (B - N_1) \cup N_2$, where N_1 and N_2 have Lebesgue measure 0, and hence the Lebesgue measure of X is equal to $\nu(X)$. Thus ν agrees with Lebesgue measure on all Lebesgue measurable subsets of $[0, 1]$.

We shall now show that a real-valued measurable cardinal is weakly inaccessible. The proof is by a combinatorial argument, using matrices of sets.

An *Ulam matrix* (more precisely, an Ulam $(\aleph_1, \aleph_0)$-matrix) is a collection $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ of subsets of ω_1 such that:

(27.12) (i) if $\alpha \neq \beta$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$, for every $n < \omega$;
(ii) for each α, the set $\omega_1 - \bigcup_{n=0}^{\infty} A_{\alpha,n}$ is at most countable.

See Fig. 27.1.

$$\begin{array}{ccccc} A_{00} & A_{01} & \cdots & A_{0n} & \cdots \\ A_{10} & A_{11} & \cdots & A_{1n} & \cdots \\ \vdots & \vdots & & \vdots & \\ A_{\alpha 0} & A_{\alpha 1} & \cdots & A_{\alpha n} & \cdots \\ \vdots & \vdots & & \vdots & \end{array}$$

FIGURE 27.1. An $(\aleph_1, \aleph_0)$-matrix.

An Ulam matrix has $\aleph_1$ rows and $\aleph_0$ columns. Each column consists of pairwise disjoint sets, and the union of each row contains all but countably many elements of ω_1.

Lemma 27.6. *An Ulam matrix exists.*

Proof. For each $\xi < \omega_1$, let f_ξ be a function on ω such that $\xi \subseteq \operatorname{ran}(f_\xi)$. Let us define $A_{\alpha n}$ for $\alpha < \omega_1$ and $n < \omega$ by

$$\xi \in A_{\alpha n} \quad \text{iff} \quad f_\xi(n) = \alpha \tag{27.13}$$

If $n < \omega$, then for each $\xi \in \omega_1$ there is only one α such that $\xi \in A_{\alpha n}$, namely $\alpha = f_\xi(n)$; and we have property (i) of (27.12). If $\alpha < \omega_1$, then for each $\xi > \alpha$ there is an n such that $f_\xi(n) = \alpha$ and hence $(\omega_1 - \bigcup_{n=0}^\infty A_{\alpha n}) \subseteq \alpha$; that verifies property (ii). ■

Using an Ulam matrix, we can show that there is no measure on ω_1:

Lemma 27.7. *There is no nontrivial σ-additive measure on ω_1. More generally, there is no σ-complete σ-saturated ideal over ω_1.*

Proof. Let $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ be an Ulam matrix. Assuming that we have a measure on ω_1, there is for each α some $n = n_\alpha$ such that $A_{\alpha n}$ has positive measure (because of (27.12)ii). Hence there is an uncountable set $W \subseteq \omega_1$ and some $n < \omega$ such that $n_\alpha = n$ for all $\alpha \in W$. Then $\{A_{\alpha n} : \alpha \in W\}$ is an uncountable, pairwise disjoint (by (27.12)i) family of sets of positive measure; a contradiction. ■

A straightforward generalization of Lemmas 27.6 and 27.7 gives the result mentioned above:

Lemma 27.8. *If $\kappa = \lambda^+$, then there is no κ-complete σ-saturated ideal over κ (not even a κ-saturated ideal).*

Corollary. *Every real-valued measurable cardinal is weakly inaccessible.*

Proof. For each $\xi < \lambda^+$, we let f_ξ be a function on λ such that $\xi \subseteq \operatorname{ran}(f_\xi)$, and let

$$\xi \in A_{\alpha\eta} \quad \text{iff} \quad f_\xi(\eta) = \alpha$$

Then $\{A_{\alpha,\eta} : \alpha < \lambda^+, \eta < \lambda\}$ is an Ulam (λ^+, λ)-matrix, that is a collection of subsets of λ such that:

$$\begin{aligned} &\text{(i)} \quad A_{\alpha\eta} \cap A_{\beta\eta} = \emptyset \text{ whenever } \alpha \neq \beta < \lambda^+, \text{ and } \eta < \lambda; \\ &\text{(ii)} \quad \left|\lambda^+ - \textstyle\bigcup_{\eta<\lambda} A_{\alpha\eta}\right| \leq \lambda \text{ for each } \alpha < \lambda^+. \end{aligned} \tag{27.14}$$

The proof of Lemma 27.7 easily generalizes to show that no κ-saturated κ-complete ideal exists over κ. ■

Lemma 27.8 completes the proof of Theorem 66: If there is a σ-additive nontrivial measure on S, then either the measure has an atom A and we can construct a two-valued measure on S via a σ-complete nonprincipal ultrafilter over A, and then $|S| \geq$ the least measurable cardinal, which is inaccessible; or the measure on S is atomless and we construct, as in Lemma 27.5, an atomless measure on $2^{\aleph_0}$, and then $2^{\aleph_0} \geq$ the least real-valued measurable cardinal, which is weakly inaccessible. ■

We conclude this section with an application of the concept of a scale that we introduced in Section 24. Historically this result is interesting since it preceded Theorem 66 by showing that the assumption of existence of a nontrivial σ-additive measure violates the continuum hypothesis.

We recall (cf. (24.27)) that a κ-scale is a κ-sequence of functions f_α: $\omega \to \omega$, $\alpha < \kappa$, such that $f_0 \prec f_1 \prec \cdots \prec f_\alpha \prec \cdots$ and that for every $g \in \omega \to \omega$ there is $\alpha < \kappa$ with $g \prec f_\alpha$.

Lemma 27.9. *If there exists a κ-scale, then κ is not a real-valued measurable cardinal.*

Proof. Let f_α, $\alpha < \kappa$, be a κ-scale. We define an $(\aleph_0, \aleph_0)$-matrix of subsets of κ as follows: For $n, k < \omega$, let

$$\text{(27.15)} \qquad \alpha \in A_{nk} \quad \text{iff} \quad f_\alpha(n) = k \qquad (\alpha \in \kappa)$$

Since for each n and each α there is k such that $\alpha \in A_{nk}$, we have

$$\bigcup_{k=0}^{\infty} A_{nk} = \kappa$$

for every $n = 0, 1, 2, \ldots$.

Now assume that μ is a nontrivial κ-additive measure on κ. For each n, let k_n be such that

$$\mu(A_{n0} \cup A_{n1} \cup \cdots \cup A_{nk_n}) \geq 1 - (1/2^{n+2})$$

and let $B_n = A_{n0} \cup \cdots \cup A_{nk_n}$. If we let $B = \bigcap_{n=0}^{\infty} B_n$, then we clearly have $\mu(B) \geq \frac{1}{2}$.

Let g: $\omega \to \omega$ be the function $g(n) = k_n$. If $\alpha \in B$, then by the definition of B and by (27.15), we have

$$f_\alpha(n) \leq g(n)$$

for all $n = 0, 1, 2, \ldots$; hence $g \not\prec f_\alpha$. However, since B has positive measure, B has size κ, and therefore we have $g \not\prec f_\alpha$ for cofinally many $\alpha < \kappa$. This contradicts the assumption that the f_α form a scale. ■

Corollary 1. (which also follows from Lemma 27.7). *If there is a measure on $2^{\aleph_0}$, then $2^{\aleph_0} > \aleph_1$.*

(Because an $\aleph_1$-scale exists.)

Corollary 2. *If Martin's axiom holds, then $2^{\aleph_0}$ is not a real-valued measurable cardinal.*

28. ULTRAPOWERS AND ELEMENTARY EMBEDDINGS

In the preceding section we introduced measurable cardinals and showed that measurable cardinals are inaccessible. That result was obtained by a more or less straightforward application of the definition of measurability. To obtain much stronger results, we shall borrow from model theory a powerful method which is well suited for measurable cardinals—the method of *ultrapowers*.

In Section 7 (cf. (7.7)) we defined the *reduced product* of a collection of sets modulo a filter (over the set of indexes). Reduced products, not just of sets but of models, are a useful tool in model theory, and become particularly helpful when the filter is an ultrafilter.

Let S be a set and let U be an ultrafilter over S. For each $x \in S$, let $\mathfrak{A}_x$ be a model of some language $\mathcal{L}$, the same language for eeach $x \in S$. For clarity of exposition, let me assume that $\mathcal{L}$ is the language of set theory, $\mathcal{L} = \{\in\}$, and that all formulas of $\mathcal{L}$ have only the binary predicates $=$ and $\in$. (The subsequent discussion of ultraproducts is however valid for arbitrary languages as well.)

Thus for each $x \in S$, we have a model

$$\mathfrak{A}_x = \langle A_x, E_x \rangle$$

of the language of set theory. Let us consider the Cartesian product

$$P = \prod_{x \in S} A_x$$

and let us consider the following equivalence relation $=^*$ over P:

$$\text{(28.1)} \qquad f =^* g \quad \text{iff} \quad \{x \in S : f(x) = g(x)\} \in U \qquad (f, g \in P)$$

As we have seen in Section 7, the relation $=^*$ is indeed an equivalence relation, due solely to the fact that U is a filter.

For each $f \in P$, let $[f]$ be the equivalence class mod $=^*$, and let A^* be the collection of all equivalence classes of all $f \in P$:

$$A^* = (\prod_{x \in S} A_x)/=^*$$

For $f, g \in P$, we define

$$\text{(28.2)} \qquad f \in^* g \quad \text{iff} \quad \{x \in S : f(x) E_x g(x)\} \in U$$

and we note that $=^*$ is a congruence with respect to $\in^*$, i.e., if $f_1 =^* f_2$, $g_1 =^* g_2$, and $f_1 \in^* g_1$, then $f_2 \in^* g_2$. Thus we can define a binary relation over A^* (and by abuse of notation denote it $\in^*$ as well) as follows:

$$\text{(28.3)} \qquad [f] \in^* [g] \quad \text{iff} \quad f \in^* g$$

Now, consider the model $\mathfrak{A}^* = \langle A^*, \in^* \rangle$, which is called the *ultraproduct* of $\mathfrak{A}_x$, $x \in S$ (mod U):

$$\mathfrak{A}^* = Ult_U\{\mathfrak{A}_x : x \in S\} \tag{28.4}$$

The importance of ultraproducts is due mainly to the following basic property of ultraproducts:

Lemma 28.1 (Fundamental Theorem of Ultraproducts). *Let U be an ultrafilter over S and let $\mathfrak{A}^*$ be the ultraproduct of $\mathfrak{A}_x$, $x \in S$ (mod U).*

(a) *If φ is a formula, then for any $f_1, \ldots, f_n \in \prod_{x \in S} A_x$,*

$$\mathfrak{A}^* \models \varphi[[f_1], \ldots, [f_n]] \quad \text{iff} \quad \{x \in S : \mathfrak{A}_x \models \varphi[f_1(x), \ldots, f_n(x)]\} \in U \tag{28.5}$$

(b) *If σ is a sentence, then*

$$\mathfrak{A}^* \models \sigma \quad \text{iff} \quad \{x \in S : \mathfrak{A}_x \models \sigma\} \in U$$

Part (b) is a consequence of (a). Note that by the lemma, the satisfaction of φ at $[f_1], \ldots, [f_n]$ does not depend on the choice of representatives $f_1, \ldots, f_n$ for the equivalence classes $[f_1], \ldots, [f_n]$. Thus we may further abuse the notation and write

$$\mathfrak{A}^* \models \varphi[f_1, \ldots, f_n]$$

It will also be convenient to adopt a measure theoretic terminology. If

$$\{x \in S : \mathfrak{A}_x \models \varphi[f_1(x), \ldots, f_n(x)]\} \in U$$

we say that $\mathfrak{A}_x$ satisfies $\varphi(f_1, \ldots, f_n)$ for *almost all* x, or that $\mathfrak{A}_x \models \varphi(f_1, \ldots, f_n)$ holds *almost everywhere*. In this terminology, the fundamental theorem states that $\varphi(f_1, \ldots, f_n)$ holds in the ultraproduct just in case for almost all x, $\varphi(f_1, \ldots, f_n)$ holds in $\mathfrak{A}_x$.

Proof of Lemma 28.1. We shall prove (a) by induction on complexity of formulas. We shall prove that (28.5) holds for atomic formulas, and then prove the induction step for $\neg$, $\wedge$, and $\exists$; then it follows that (28.5) is true for all formulas.

Atomic formulas: First we consider the formula $u = v$, and we have

$$\begin{aligned}
\mathfrak{A}^* \models [f] = [g] \quad & \text{iff} \quad [f] = [g] \\
& \text{iff} \quad f =^* g \\
& \text{iff} \quad \{x : f(x) = g(x)\} \in U \\
& \text{iff} \quad \{x : \mathfrak{A}_x \models f(x) = g(x)\} \in U
\end{aligned}$$

For $u \in v$, we have

$$\begin{aligned}
\mathfrak{A}^* \models [f] \in [g] \quad & \text{iff} \quad [f] \in^* [g] \\
& \text{iff} \quad \{x : f(x)\ E_x\ g(x)\} \in U \\
& \text{iff} \quad \{x : \mathfrak{A}_x \models f(x) \in g(x)\} \in U
\end{aligned}$$

Logical connectives: First we assume that (28.5) holds for φ and show that it also holds for $\neg\varphi$ (here we use that U is an ultrafilter):

$$\begin{aligned} \mathfrak{A}^* \models \neg\varphi[f] \quad & \text{iff} \quad \text{not } \mathfrak{A}^* \models \varphi[f] \\ & \text{iff} \quad \{x : \mathfrak{A}_x \models \varphi[f(x)]\} \notin U \\ & \text{iff} \quad \{x : \mathfrak{A}_x \not\models \varphi[f(x)]\} \in U \\ & \text{iff} \quad \{x : \mathfrak{A}_x \models \neg\varphi[f(x)\} \in U \end{aligned}$$

Similarly, if 28.5 is true for φ and ψ, we have

$$\begin{aligned} \mathfrak{A}^* \models \varphi \wedge \psi \quad & \text{iff} \quad \mathfrak{A}^* \models \varphi \text{ and } \mathfrak{A}^* \models \psi \\ & \text{iff} \quad \{x : \mathfrak{A}_x \models \varphi\} \in U \text{ and } \{x : \mathfrak{A}_x \models \psi\} \in U \\ & \text{iff} \quad \{x : \mathfrak{A}_x \models \varphi \wedge \psi\} \in U \end{aligned}$$

(The last equivalence uses this fact: $X \in U$ and $Y \in U$ iff $X \cap Y \in U$.)

Existential quantifier. We assume that (28.5) is true for $\varphi(u, v_1, \ldots, v_n)$ and show that it remains true for the formula $\exists u\, \varphi$. Let us assume first that

$$\mathfrak{A}^* \models \exists u\, \varphi[f_1, \ldots, f_n] \tag{28.6}$$

Then there is $g \in \prod_{x \in S} A_x$ such that $\mathfrak{A}^* \models \varphi[g, f_1, \ldots, f_n]$, and therefore

$$\{x : \mathfrak{A}_x \models \varphi[g(x), f_1(x), \ldots, f_n(x)]\} \in U, \tag{28.7}$$

and it clearly follows that

$$\{x : \mathfrak{A}_x \models \exists u\, \varphi[u, f_1(x), \ldots, f_n(x)]\} \in U \tag{28.8}$$

Now let us assume that (28.8) holds. For each $x \in S$, let $u_x \in A_x$ be such that $\mathfrak{A}_x \models \varphi[u_x, f_1(x), \ldots, f_n(x)]$ if such as u_x exists, and arbitrary otherwise. If we define $g \in \prod_{x \in S} A_x$ by $g(x) = u_x$, then we have (28.7) and therefore

$$\mathfrak{A}^* \models \varphi[g, f_1, \ldots, f_n]$$

Now (28.6) follows. ■

Exercise 28.1. Let U be a principal ultrafilter over S, such that $\{x\} \in U$. Show that the ultraproduct $\mathfrak{A}^*$ is isomorphic to $\mathfrak{A}_x$.

Let us consider now the special case of ultraproducts, when each $\mathfrak{A}_x$ is the same model $\mathfrak{A}$. Then the ultraproduct is called an *ultrapower* of $\mathfrak{A}$:

$$\mathfrak{A}^* = Ult_U(\mathfrak{A}) \tag{28.9}$$

An important corollary of the fundamental theorem is

Lemma 28.2. *An ultrapower of a model $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{A}$, i.e., for any sentence σ we have*

$$\mathfrak{A}^* \models \sigma \quad \text{iff} \quad \mathfrak{A} \models \sigma$$

Proof. By part (b) of Lemma 28.1 we have $\mathfrak{A}^* \vDash \sigma$ iff $\{x : \mathfrak{A} \vDash \sigma\} \in U$, but $\{x : \mathfrak{A} \vDash \sigma\}$ is either S or empty, according to whether $\mathfrak{A} \vDash \sigma$ or not. ■

We shall now show that a model $\mathfrak{A}$ is elementarily embeddable in its ultrapower. If U is an ultrafilter over S and $\mathfrak{A}^* = Ult_U(\mathfrak{A})$, we define the *natural embedding* j: $\mathfrak{A} \to \mathfrak{A}^*$ as follows: For each $a \in A$, let c_a be the *constant function* with value a:

(28.10) $$c_a(x) = a \qquad \text{(for every } x \in S)$$

and let

(28.11) $$j(a) = [c_a]$$

Lemma 28.3. *The natural embedding* j: $\mathfrak{A} \to \mathfrak{A}^*$ *is an elementary embedding, i.e.,*

(28.12) $$\mathfrak{A} \vDash \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad \mathfrak{A}^*[j(a_1), \ldots, j(a_n)]$$

for any formula φ.

Proof. By the fundamental theorem: $\mathfrak{A}^* \vDash \varphi[j(a)]$ iff $\mathfrak{A}^* \vDash \varphi[c_a]$ iff $\mathfrak{A} \vDash \varphi[a]$ for almost all x iff $\mathfrak{A} \vDash \varphi[a]$. ■

Exercise 28.2. If U is a principal ultrafilter, then j is an isomorphism between $\mathfrak{A}$ and $Ult(\mathfrak{A})$.

In Chapter 2 we introduced models of set theory as (possibly proper) classes endowed with a binary relation, the interpretation of the membership predicate $\in$. As is the case with many model-theoretical concepts and constructions, the method of ultraproducts can be used in the context of these models of set theory. For instance, let us consider the ultrapower of the universe $(V, \in)$ modulo a given ultrafilter U over S.

Let us consider the class of all functions f with domain S; the equivalence relation $=^*$ and the relation $\in^*$ are defined as before:

(28.13) $$f =^* g \quad \text{iff} \quad \{x \in S : f(x) = g(x)\} \in U$$

(28.14) $$f \in^* g \quad \text{iff} \quad \{x \in S : f(x) \in g(x)\} \in U$$

We encounter a slight difficulty when dealing with the equivalence classes $[f]$ since each equivalence class is a proper class. However, we can avoid all complications by using the trick introduced in (9.3): We simply define

(28.15) $$[f] = [g : f =^* g \text{ and } \forall h\ (h =^* f \rightarrow \text{rank } h \geq \text{rank } g)\}$$

Now we let

$$Ult = Ult_U(V)$$

be the class of all $[f]$ where f is a function on S, and consider the model $Ult = (Ult, \in^*)$ (here we make the same abuse of notation as in (28.3) and furthermore use the symbol Ult to denote both the class and the model).

A quick glance at the proof of the fundamental theorem of ultraproducts should convince the reader that the theorem is true also in the present context. That is, if $\varphi(x_1, \ldots, x_n)$ is a formula of set theory, then

$$(28.16) \quad Ult \vDash \varphi([f_1], \ldots, [f_n]) \quad \text{iff} \quad \{x \in S : \varphi(f_1(x), \ldots, f_n(x)\} \in U$$

If σ is a sentence, then

$$(28.17) \qquad Ult \vDash \sigma \quad \text{iff} \quad \sigma$$

and so we may say that the ultrapower is *elementarily equivalent* to the universe. Moreover, using the *constants* c_a (defined in (28.10)), we obtain a function (which is a proper class definable from U)

$$j_U \colon V \to Ult$$

defined by

$$(28.11) \qquad j_U(a) = [c_a]$$

and the function j_U, the *natural embedding* of V in Ult, is an elementary embedding of the universe in the model Ult: We have

$$(28.18) \qquad \varphi(a_1, \ldots, a_n) \quad \text{iff} \quad Ult \vDash \varphi(ja_1, \ldots, ja_n)$$

whenever $\varphi(x_1, \ldots, x_n)$ is a formula of set theory.

The most important applications of ultrapowers in set theory are those in which the $(Ult, \in^*)$ is well founded. As we shall show presently, well-founded ultrapowers are very closely related to measurable cardinals.

The model $Ult_U(V)$ is well-founded (cf. (9.8)) just in case (i) every nonempty set $X \subset Ult$ has a $\in^*$-minimal element, and (ii) $\text{ext}(f)$ is a set for every f, where

$$\text{ext}(f) = \{[g] : g \in^* f\}$$

The second condition is clearly satisfied for any ultrafilter U: for every $g \in^* f$ there is $h =^* g$ such that rank $h \le$ rank f. As for condition (i), this is satisfied if and only if there exists no infinite descending $\in^*$ sequence

$$f_0 \ni^* f_1 \ni^* \cdots \ni^* f_k \ni^* \cdots \qquad (k \in \omega)$$

of elements of the ultrapower.

Exercise 28.3. Let U be a nonprincipal ultrafilter over ω. Then $Ult_U(V)$ is not well founded.

[For each $k \in \omega$, let f_k be a function on ω such that $f_k(n) = n - k$ for all $n \ge k$. Then $f_0 \ni^* f_1 \ni^* f_2 \ni^* \cdots$ is a descending $\in^*$-sequence in Ult.]

Lemma 28.4. *If U is a σ-complete ultrafilter, then $(Ult, \in^*)$ is a well-founded model.*

Proof. We shall show that there is no infinite descending $\in^*$-sequence in Ult if U is a σ-complete ultrafilter over S. Let us assume that $f_0, f_1, \ldots, f_n, \ldots$ is such a descending sequence. Thus for each n, the set

$$X_n = \{x \in S : f_{n+1}(x) \in f_n(x)\}$$

is in the ultrafilter. Since U is σ-complete, the intersection $X = \bigcap_{n=0}^{\infty} X_n$ is also in U and hence nonempty; let x be an arbitrary element of X. Then we have

$$f_0(x) \ni f_1(x) \ni f_2(x) \ni \cdots$$

an infinite descending $\in$-sequence, which is a contradiction. ∎

In fact, σ-completeness of U is a necessary and sufficient condition for well-foundedness of the ultrapower:

Exercise 28.4. If U is not σ-complete, then $Ult_U(V)$ is not well-founded.

[There exists a countable partition $\{X_n : n = 0, 1, 2, \ldots\}$ of S such that $X_n \notin U$ for all n. For each k, let f_k be a function on S such that $f_k(x) = n - k$ for all $x \in X_n$.]

By the Mostowski collapsing theorem (Theorem 28), every well-founded model is isomorphic to a transitive model (and the collapsing isomorphism is unique). Thus if U is σ-complete, there exists a one-to-one mapping π of Ult onto a transitive class such that $f \in^* g$ if and only if $\pi([f]) \in \pi([g])$. In order to simplify notation, we shall identify each $[f]$ with its image $\pi([f])$. Thus if U is σ-complete, the symbol Ult denotes the transitive collapse of the ultrapower, and for each function f on S, $[f]$ is an element of the transitive class Ult; we say the function f *represents* $[f] \in Ult$.

Exercise 28.5. Every ordinal number α is represented by a function $f \colon S \to Ord$.

If U is principal and $x_0 \in S$ is such that $U = \{X : x_0 \in X\}$, then $Ult = V$ and $[f] = f(x_0)$ for each f. The natural embedding j_U is the identity mapping. The situation is more interesting when U is a nonprincipal σ-complete ultrafilter. Let us consider (without loss of generality) a nonprincipal σ-complete ultrafilter U over a cardinal number κ. Let us denote the ultrapower by M, $M = Ult_U(V)$, and let $j = j_U$ be the natural embedding:

$$j \colon V \to M \tag{28.19}$$

Since j is elementary, it is clear that if α is an ordinal, then $j(\alpha)$ is an ordinal; moreover, $\alpha < \beta$ implies $j(\alpha) < j(\beta)$. Thus we have $\alpha \le j(\alpha)$ for every ordinal number α. Note that $j(\alpha + 1) = j(\alpha) + 1$, and $j(n) = n$ for all natural numbers n. It is also easy to see that $j(\omega) = \omega$: If $[f] < \omega$, then $f(\alpha) < \omega$ for almost all $\alpha < \kappa$, and by σ-completeness, there exists $n < \omega$ such that $f(\alpha) = n$ for almost all α. By the same argument, if U is λ-complete, then $j(\gamma) = \gamma$ for all $\gamma < \lambda$.

We have proved (Lemma 27.1) that if there exists a nonprincipal σ-complete ultrafilter, then there exists a cardinal κ that carries a nonprincipal κ-complete ultrafilter. Thus let us assume further that U is a nonprincipal κ-complete ultrafilter over κ. (And hence $j(\gamma) = \gamma$ for all $\gamma < \kappa$.) We shall show that $j(\kappa) > \kappa$. Let d (the *diagonal function*) be the function on κ defined by

$$d(\alpha) = \alpha \qquad (\alpha < \kappa) \tag{28.20}$$

Since U is κ-complete every bounded subset of κ has measure 0 and so for every $\gamma < \kappa$, we have $d(\alpha) > \gamma$ for almost all α. Hence $[d] > \gamma$ for all $\gamma < \kappa$ and thus $[d] \geq \kappa$. However, we clearly have $[d] < j(\kappa)$ and it follows that $j(\kappa) > \kappa$.

Exercise 28.6. Let U be a nonprincipal σ-complete ultrafilter over S and let λ be the largest cardinal such that U is λ-complete. Then $j(\lambda) > \lambda$.

[Let $\{X_\alpha : \alpha < \lambda\}$ be a partition of S into sets of measure 0; let f be a function on S such that $f(x) = \alpha$ if $x \in X_\alpha$. Then $[f] \geq \lambda$.]

We have thus proved that if there is a measurable cardinal, then there is an elementary embedding j of the universe in a transitive model M such that j is not the identity mapping; let us call j a *nontrivial elementary embedding of the universe*. We shall use this to prove the following theorem:

Theorem 67 (Scott). *If $V = L$, then there are no measurable cardinals.*

Proof. Let us assume $V = L$ and that measurable cardinals exist; let κ be the least measurable cardinal. Let U be a nonprincipal κ-complete ultrafilter over κ and let j: $V \to M$ be the corresponding natural embedding. As we have shown, $j(\kappa) > \kappa$.

Since $V = L$, the only transitive model containing all ordinals is the universe itself: $V = M = L$. Since j is an elementary embedding and κ is the least measurable cardinal, we have

$$M \vDash j(\kappa) \text{ is the least measurable cardinal;}$$

and hence, $j(\kappa)$ is the least measurable cardinal. This is impossible since $j(\kappa) > \kappa$. ■

If there exists a measurable cardinal, then we obtain a nontrivial elementary embedding of the universe. Let me show that conversely, if j: $V \to M$ is a nontrivial elementary embedding then there exists a measurable cardinal. (A word of caution: "There exists a nontrivial elementary embedding of V" or even "j is an elementary embedding of V" is not a statement expressible in the language of ZF.)

Lemma 28.5. *If j is a nontrivial elementary embedding of the universe, then there exists a measurable cardinal.*

Proof. Let j: $V \to M$ be a nontrivial elementary embedding. Notice that there exists an ordinal α such that $j(\alpha) \neq \alpha$; otherwise, we would have $\text{rank}(jx) = \text{rank}(x)$ for all x, and then we could prove by induction on rank that $j(x) = x$ for all x.

Thus let κ be the least ordinal number such that $j(\kappa) \neq \kappa$ (and hence $j(\kappa) > \kappa$). It is clear that $j(n) = n$ for all n and $j(\omega) = \omega$ since 0, $n + 1$, and ω are absolute notions and j is elementary. Hence $\kappa > \omega$. We shall show that κ is a measurable cardinal.

Let D be the collection of subsets of κ defined as follows:

$$X \in D \quad \text{iff} \quad \kappa \in j(X) \qquad (X \subseteq \kappa) \tag{28.21}$$

Since $\kappa < j(\kappa)$, i.e., $\kappa \in j(\kappa)$, we have $\kappa \in D$; also $\emptyset \notin D$ because $j(\emptyset) = \emptyset$. Using the fact that $j(X \cap Y) = j(X) \cap j(Y)$ and that $j(X) \subseteq j(Y)$ whenever $X \subseteq Y$, we see that D is a filter: If $\kappa \in j(X)$ and $\kappa \in j(Y)$, then $\kappa \in j(X \cap Y)$; if $X \subseteq Y$ and $\kappa \in j(X)$, then $\kappa \in j(Y)$. Similarly, $j(\kappa - X) = j(\kappa) - j(X)$ and thus D is an ultrafilter.

D is a nonprincipal ultrafilter: For every $\alpha < \kappa$, we have $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$, and so $\kappa \notin j(\{\alpha\})$ and we have $\{\alpha\} \notin D$. We shall now show that D is κ-complete. Let $\gamma < \kappa$ and let $\mathscr{X} = \langle X_\alpha : \alpha < \gamma \rangle$ be a sequence of subsets of κ such that $\kappa \in j(X_\alpha)$ for each $\alpha < \gamma$. We shall show that $\bigcap_{\alpha<\gamma} X_\alpha \in D$. In M (and thus in V), $j(\mathscr{X})$ is a sequence of length $j(\gamma)$ of subsets of $j(\kappa)$; for each $\alpha < \gamma$, the $j(\alpha)$th term of $j(\mathscr{X})$ is $j(X_\alpha)$. Since $j(\alpha) = \alpha$ for all $\alpha < \gamma$ and $j(\gamma) = \gamma$, it follows that $j(\mathscr{X}) = \langle j(X_\alpha) : \alpha < \gamma \rangle$. Hence if $X = \bigcap_{\alpha<\gamma} X_\alpha$, we have $j(X) = \bigcap_{\alpha<\gamma} j(X_\alpha)$. Now it is clear that $\kappa \in j(X)$ and hence $X \in D$. ■

The construction of a κ-complete ultrafilter from an elementary embedding described in (28.21) gives rise to a neat commutative diagram of elementary embeddings:

Lemma 28.6. *Let j: $V \to M$ be a nontrivial elementary embedding, let κ be the least ordinal moved, and let D be the ultrafilter over κ defined in* (28.21). *Let j_D: $V \to Ult$ be the natural embedding of V in the ultrapower $Ult_D(V)$. Then there is an elementary embedding k of Ult in M such that $k(j_D(a)) = j(a)$ for all a:*

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "j_D"'] & M \\
Ult \arrow[ur, "k"'] &
\end{tikzcd}$$

FIGURE 28.1

Proof. For each $[f] \in Ult$, let

$$k([f]) = (j(f))(\kappa) \tag{28.22}$$

(Here f is a function on κ and $j(f)$ is a function on $j(\kappa)$.)

We shall first show that definition (28.22) does not depend on the choice of f representing $[f]$. If $f =_D g$, then the set $X = \{\alpha : f(\alpha) = g(\alpha)\}$ is in D and hence κ

is in the set

$$j(X) = \{\alpha < j(\kappa) : (jf)(\alpha) = (jg)(\alpha)\}$$

Therefore $(jf)(\kappa) = (jg)(\kappa)$.

Next we show that k is elementary. Let $\varphi(x)$ be a formula and let $Ult \models \varphi([f])$; we shall show that $M \models \varphi(k([f]))$. The set $X = \{\alpha : \varphi(f(\alpha))\}$ is in D and hence κ belongs to

$$j(X) = \{\alpha < j(\kappa) : M \models \varphi((jf)(\alpha))\}$$

Since $(jf)(\kappa) = k([f])$, we have $M \models \varphi(k([f]))$.

Finally, we show that $k(j_D(a)) = j(a)$ for all a. Since $j_D(a) = [c_a]$, where c_a is the constant function on κ with value a, we have $k(j_D(a)) = (j(c_a))(\kappa)$. Now $j(c_a)$ is the constant function on $j(\kappa)$ with value $j(a)$ and hence $(j(c_a))(\kappa) = j(a)$. ■

Before we continue our investigation of elementary embeddings, notice that if $j\colon V \to M$, then the model M is determined by the function j:

Exercise 28.7. If j is an elementary embedding of the universe in a transitive model M, then $M = \bigcup_{\alpha \in Ord} j(V_\alpha)$.

If X is a class defined by a formula φ, then $j(X)$ is the class of the model M, defined in M by the same formula φ. Note that $j(X) = \bigcup_{\alpha \in Ord} j(X \cap V_\alpha)$. In particular, $M = j(V)$.

We proved in Lemma 27.2 that every measurable cardinal is inaccessible. We shall now obtain some stronger results, as a consequence of the following simple lemma.

Lemma 28.7. *Let j be an elementary embedding of the universe and let κ be the least ordinal moved (i.e., $j(\kappa) > \kappa$). If C is a closed unbounded subset of κ, then $\kappa \in j(C)$.*

Proof. Since $j(\alpha) = \alpha$ for all $\alpha < \kappa$, we have $j(C) \cap \kappa = C$. Thus $j(C) \cap \kappa$ is unbounded in κ; and because $j(C)$ is closed (in $j(V)$ and hence in the universe), we have $\kappa \in j(C)$. ■

Corollary. *Every measurable cardinal κ is a Mahlo cardinal, and hence κ is the* κth *inaccessible cardinal.*

Proof. Let κ be a measurable cardinal; we shall show that the set X of all regular cardinals $\alpha < \kappa$ is stationary in κ. Let $j\colon V \to M$ be an elementary embedding such that κ is the least ordinal moved (e.g., let $j = j_U$ where U is a nonprincipal κ-complete ultrafilter). Since κ is a regular cardinal in M, we have $\kappa \in j(X)$. Now if C is any closed unbounded subset of κ, we have $j(X \cap C) \neq \emptyset$ because $\kappa \in j(X) \cap j(C)$ and hence $X \cap C \neq \emptyset$. Thus X is stationary. ■

The above argument can be pushed further: Let X denote the set of all Mahlo cardinals below κ. Since κ is Mahlo, it is a Mahlo cardinal in M and so

$\kappa \in j(X)$. The same argument as above shows that X is stationary and thus κ is the κth Mahlo cardinal.

One can formulate an abstract version of the preceding argument. We define the *Mahlo operation* on classes of ordinals: If X is a class, let

$$\text{(28.23)} \qquad \mathrm{M}(X) = \{\alpha : X \cap \alpha \text{ is stationary in } \alpha\}$$

If X is the class of all inaccessible cardinals, then $\mathrm{M}(X)$ is the class of all Mahlo cardinals.)

Exercise 28.8. Let j be an elementary embedding of the universe and let κ be the least ordinal moved. If X is a class of ordinals such that $\kappa \in j(X)$, then $\kappa \in \mathrm{M}(X)$.

While measurable cardinals induce elementary embeddings of the universe in a transitive model, the following theorem shows that no nontrivial elementary embedding exists of V into itself.

Theorem 68 (Kunen). *If j: $V \to M$ is a nontrivial elementary embedding, then $M \neq V$.*

The proof below makes use of the following combinatorial lemma:

Lemma 28.8. *Let λ be an infinite cardinal such that $2^\lambda = \lambda^{\aleph_0}$. There exists a function F: ${}^\omega\lambda \to \lambda$ such that whenever A is a subset of λ of size λ and $\gamma < \lambda$, there exists $s \in {}^\omega A$ such that $F(s) = \gamma$.*

(The lemma is in fact true for every infinite λ.)

Proof. Let $\{(A_\alpha, \gamma_\alpha) : \alpha < 2^\lambda\}$ be an enumeration of all pairs (A, γ) where $\gamma < \lambda$ and A is a subset of λ of size λ. We define, by induction on α, a sequence s_α, $\alpha < 2^\lambda$, of elements of ${}^\omega\lambda$ as follows: If $\alpha < \lambda$, then since $\lambda^{\aleph_0} = 2^\lambda > |\alpha|$, there exists $s_\alpha \in {}^\omega A_\alpha$ such that $s_\alpha \neq s_\beta$ for all $\beta < \alpha$.

For each $\alpha < 2^\lambda$, we define $F(s_\alpha) = \gamma_\alpha$ (and let $F(s)$ be arbitrary if s is not one of the s_α). The function F has the required property: If $A \subseteq \lambda$ has size λ and $\gamma < \lambda$, then $(A, \gamma) = (A_\alpha, \gamma_\alpha)$ for some α, and then $\gamma_\alpha = F(s_\alpha)$. ■

Proof of Theorem 68. Let us assume that j is a nontrivial elementary embedding of V in V. Let $\kappa = \kappa_0$ be the least ordinal moved; κ_0 is measurable, and so are $\kappa_1 = j(\kappa_0)$, $\kappa_2 = j(\kappa_1)$, and every κ_n, where $\kappa_{n+1} = j(\kappa_n)$. Let $\lambda = \lim_{n\to\infty} \kappa_n$. Since $j(\langle\kappa_n : n < \omega\rangle) = \langle j(\kappa_n) : n < \omega\rangle = \langle\kappa_{n+1} : n < \omega\rangle$, we have $j(\lambda) = \lim_{n\to\infty} j(\kappa_n) = \lambda$. Let $G = j[\lambda] = \{j(\alpha) : \alpha < \lambda\}$; we shall use the set G and Lemma 28.8 to obtain a contradiction.

The cardinal λ is the limit of a sequence of measurable cardinals and hence is a strong limit cardinal. Since $\operatorname{cf} \lambda = \omega$, we have $2^\lambda = \lambda^{\aleph_0}$. By Lemma 28.8 there is a function F: ${}^\omega\lambda \to \lambda$ such that $F[{}^\omega A] = \lambda$ for all $A \subseteq \lambda$ of size λ. Since j is elementary, and $j\omega = \omega$ and $j\lambda = \lambda$, the function $j(F)$ has the same property. Thus, considering the set $A = G$, there exists $s \in {}^\omega G$ such that $(jF)(s) = \kappa$.

Now, s is a function, s: $\omega \to j[\lambda]$, and hence there is t: $\omega \to \lambda$ such that

$s(n) = j(t(n))$ for all $n < \omega$. It follows that $s = j(t)$. Thus we have $\kappa = (jF)(jt) = j(F(t))$; in other words, $\kappa = j(\alpha)$ where $\alpha = F(t)$. However, this is impossible since $j(\alpha) = \alpha$ for all $\alpha < \kappa$, and $j(\kappa) > \kappa$. ∎

Exercise 28.9. If $j\colon V \to M$ is a nontrivial elementary embedding, if κ is the least ordinal moved, and if $\lambda = \lim\{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$, then there exists $A \subset \lambda$ such that $A \notin M$.

[Assuming that M contains all bounded subsets of λ, the above proof that $G \notin M$ goes through.]

Let me now consider ultrapowers and the corresponding natural embeddings $j_U\colon V \to Ult$. To introduce the following lemma, let us observe that if $j\colon V \to M$ and if κ is the least ordinal moved, then $j(x) = x$ for every $x \in V_\kappa$, and $j(X) \cap V_\kappa = X$ for every $X \subseteq V_\kappa$. Hence $V^M_{\kappa+1} = V_{\kappa+1}$ (and $P^M(\kappa) = P(\kappa)$).

Lemma 28.9. *Let U be a nonprincipal κ-complete ultrafilter over κ, let $M = Ult_U(V)$ and let $j = j_U$ be the natural embedding of V in M.*

(a) ${}^\kappa M \subset M$, *i.e., every κ-sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of M is itself a member of M.*

(b) $U \notin M$.

(c) $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.

(d) *If λ is a limit ordinal and if* $\operatorname{cf}\lambda = \kappa$, *then* $j(\lambda) > \lim_{\alpha\to\lambda} j(\alpha)$; *if* $\operatorname{cf}\lambda \neq \kappa$, *then* $j(\lambda) = \lim_{\alpha\to\lambda} j(\alpha)$.

(e) *If $\lambda > \kappa$ is a strong limit cardinal and* $\operatorname{cf}\lambda \neq \kappa$, *then* $j(\lambda) = \lambda$.

Proof. (a) Let $\langle a_\xi : \xi < \kappa\rangle$ be a κ-sequence of elements of M. For each $\xi < \kappa$, let g_ξ be a function that represents a_ξ, and let h be a function that represents κ:

$$[g_\xi] = a_\xi, \qquad [h] = \kappa$$

We shall construct a function F such that $[F] = \langle a_\xi : \xi < \kappa\rangle$. We let, for each $\alpha < \kappa$,

$$F(\alpha) = \langle g_\xi(\alpha) : \xi < h(\alpha)\rangle$$

Since for each α, $F(\alpha)$ is an $h(\alpha)$-sequence, $[F]$ is a κ-sequence. Let $\xi < \kappa$; we want to show that the ξth term of $[F]$ is a_ξ. Since $[h] > \xi$, we have $\xi < h(\alpha)$ for almost all α; and for each α such that $\xi < h(\alpha)$, the ξth term of $F(\alpha)$ is $g_\xi(\alpha)$. But $[c_\xi] = \xi$ and $[g_\xi] = a_\xi$, and we are done.

(b) Assume that $U \in M$, and let us consider the mapping e of ${}^\kappa\kappa$ onto $j(\kappa)$ defined by $e(f) = [f]$. Since ${}^\kappa\kappa \in M$ and $U \in M$, the mapping e is in M. It follows that $M \models |j(\kappa)| \leq 2^\kappa$. This is a contradiction since $\kappa < j(\kappa)$ and $j(\kappa)$ is inaccessible in M.

(c) $2^\kappa \leq (2^\kappa)^M$ holds because $P^M(\kappa) = P(\kappa)$ and $M \subset V$; $(2^\kappa)^M$ is less than $j(\kappa)$ since $j(\kappa)$ is inaccessible in M; finally, we have $|j(\kappa)| = 2^\kappa$ and hence $j(\kappa) < (2^\kappa)^+$.

(d) If $\operatorname{cf}\lambda = \kappa$, let $\lambda = \lim_{\alpha\to\kappa} \lambda_\alpha$ and let $f(\alpha) = \lambda_\alpha$ for all $\alpha < \kappa$. Then $[f] >$

$j(\lambda_\alpha)$ for all $\alpha < \kappa$ and $[f] < j(\lambda)$. If cf $\lambda > \kappa$, then for every $f\colon \kappa \to \lambda$ there exists $\alpha < \lambda$ such that $[f] < j(\alpha)$. If cf $\lambda = \gamma < \kappa$, let $\lambda = \lim_{\nu \to \gamma} \lambda_\nu$; for every $f\colon \kappa \to \lambda$ there exists (by κ-completeness) $\nu < \gamma$ such that $[f] < j(\lambda_\nu)$.

(e) For every $\alpha < \lambda$, the ordinals below α are represented by functions $f\colon \kappa \to \alpha$; hence $|j(\alpha)| \leq |{}^\kappa\alpha| < \lambda$; by (d) we have $j(\lambda) = \lim_{\alpha \to \lambda} j(\alpha) = \lambda$. ■

Remark. Note that in (e) it suffices to assume that cf $\lambda \neq \kappa$ and $\alpha^\kappa < \lambda$ for all cardinals $\alpha < \lambda$.

Normal Measures

We defined the notion of a *normal filter* in (7.18); a filter is normal if it is closed under diagonal intersections. If D is a nonprincipal κ-complete ultrafilter over κ and if D is normal (as a filter), then we call D a *normal (two-valued) measure* on κ.

Lemma 28.10. *Let D be a nonprincipal κ-complete ultrafilter over κ. Then the following are equivalent:*

(i) *D is normal.*
(ii) *If f is such that $f(\alpha) < \alpha$ for almost all α, then there is $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for almost all α.*
(iii) *In the ultrapower $Ult_D(V)$,*

$$\kappa = [d]$$

where d is the diagonal function.

(iv) *For every $X \subseteq \kappa$, $X \in D$ if and only if $\kappa \in j_D(X)$.*

Thus D is normal just in case the diagonal function is the least nonconstant function (i.e., the least function that is not constant almost everywhere).

Proof. (i) → (ii): Let us recall that the diagonal intersection of $\{X_\gamma : \gamma < \kappa\}$ is the set

$$\mathop{\triangle}_{\gamma<\kappa} X_\gamma = \{\alpha < \kappa : \alpha \in \bigcap_{\gamma<\alpha} X_\gamma\}$$

Let D be closed under diagonal intersections and let f be a regressive function on a set $Y \in D$. If we assume that f is not constant almost everywhere, then for each $\gamma < \kappa$, the set $X_\gamma = \{\alpha \in Y : f(\alpha) \neq \gamma\}$ is in D. Let $X = \triangle_{\gamma<\kappa} X_\gamma$. For each $\alpha \in X$, we have $f(\alpha) \geq \alpha$ because $f(\alpha) = \gamma$ for all $\gamma \subset \alpha$. A contradiction.

(ii) → (iii): Obvious.

(iii) → (iv): If $X \subseteq \kappa$, then $X \in D$ if and only if $d(\alpha) \in X$ for almost all α, that is if and only if $[d] \in j_D(X)$. If $[d] = \kappa$, we get $X \in D$ iff $\kappa \in j(X)$.

(iv) → (i): Let us assume that $X \in D$ iff $\kappa \in j_D(X)$ and let us show that D is closed under diagonal intersections. Let X_γ, $\gamma < \kappa$, be such that $\kappa \in j(X_\gamma)$ for all $\gamma < \kappa$, let $X = \triangle_{\gamma<\kappa} X_\gamma$ and let us show that $\kappa \in j(X)$. Now $j(X)$ is (in Ult) the diagonal intersection of $j(\{X_\gamma : \gamma < \kappa\})$, and $\kappa \in j(X)$ just in case $\kappa \in j(X_\gamma)$ for all $\gamma < \kappa$. Thus we have $\kappa \in j(X)$ and so $X \in D$. ■

The following lemma shows that normal measures on measurable cardinals exist and in fact can be obtained from a given measure in a canonical way:

Lemma 28.11. *Every measurable cardinal carries a normal measure. If U is a nonprincipal κ-complete ultrafilter over κ, then there exists a function f: $\kappa \to \kappa$ such that*

$$D = \{X \subseteq \kappa : f_{-1}(D) \in U\} \tag{28.24}$$

is a normal measure.

We denote the measure D obtained from U as in (28.24) by $D = f_*(U)$, and call D a *projection* of U (by f).

Proof. Let f: $\kappa \to \kappa$ be a function such that $[f] = \kappa$ in the ultrapower $Ult_U(V)$. It is easy to see that for any function f: $\kappa \to \kappa$, $D = f_*(U)$ is a κ-complete ultrafilter (see also (27.6)). Since $[f] > \gamma$ for every $\gamma < \kappa$, we have $f_{-1}(\{\gamma\}) \notin U$, and so D is nonprincipal.

In order to show that D is normal, let us consider a function h: $\kappa \to \kappa$ such that $h(\xi) < \xi$ on a set $X \in D$, and let us show that h is constant on some set in D. Let g be the function on κ defined by

$$g(\alpha) = h(f(\alpha))$$

We have $g(\alpha) < f(\alpha)$ for all $\alpha \in f_{-1}(X)$; and since f is the least nonconstant function in Ult_U, there is $Y \in U$ such that g is constant on Y. Then h is constant on $f[Y]$ and $f[Y] \in D$. ■

Exercise 28.10. Let j be an elementary embedding of the universe, let κ be the least ordinal moved, and let $D = \{X \subseteq \kappa : \kappa \in j(X)\}$ (cf. 28.21). Then D is a normal measure on κ.

[Let f be a function regressive on some $X \in D$. Then $(jf)(\kappa) < \kappa$ and show that if $\gamma = (jf)(\kappa)$, then $f(\alpha) = \gamma$ almost everywhere.]

If D is a normal measure on κ, then every closed unbounded subset of κ has measure 1. (In other words, every $X \in D$ is stationary.) This follows either from the characterization of normality in Lemma 28.10(iv) or using the fact that if a set $X \subseteq \kappa$ is disjoint from a closed unbounded set C, then there exists a regressive function f on X that is unbounded and nondecreasing:

$$f(\alpha) = \sup(C \cap \alpha) \qquad (\alpha \in X) \tag{28.25}$$

This and the fact that κ is represented by the diagonal function are used in numerous applications of normal measures. For instance, we have:

Lemma 28.12. *If D is a normal measure on κ, then the set of all inaccessible cardinals below κ has measure one. Similarly, almost all cardinals below κ are Mahlo.*

Proof. Simply because $Ult_D \vDash [d]$ is inaccessible, and $Ult_D \vDash [d]$ is Mahlo (compare with the corollary of Lemma 28.7). ■

One important reason the above proof works is that since κ is inaccessible (in the universe), κ is also inaccessible in the ultrapower; similarly, since κ is Mahlo, κ is Mahlo in *Ult*, etc. We shall use arguments like this for other properties of κ; however, measurability of κ need not be preserved (and is not preserved if κ is the least measurable cardinal).

Exercise 28.11. If κ is measurable, then there exists a normal measure D on κ such that $Ult_D(V) \vDash \kappa$ is not measurable.

[Let D be a normal measure such that $j_D(\kappa)$ is the least possible ordinal; let $M = Ult_D(V)$. If κ is measurable in M, then there is a normal measure U on κ such that $U \in M$. Since $P(\kappa) \subset M$, the ultrapower Ult_U is a class in M. By Lemma 28.9c (applied in M), we have $j_U(\kappa) < ((2^\kappa)^+)^M$. Since $((2^\kappa)^+)^M < j_D(\kappa)$, we get a contradiction.]

Note that in the exercise above almost all (mod D) cardinals below κ are nonmeasurable.

If D is a normal measure on κ, then D extends the closed unbounded filter on κ. We shall see later that this property does not necessarily imply normality. Let f be a function on κ; let us say that f is *monotone* if $f(\alpha) \le f(\beta)$ whenever $\alpha < \beta$.

Exercise 28.12. Let U be a nonprincipal κ-complete ultrafilter over κ. Then U extends the closed unbounded filter if and only if the diagonal function is the least nonconstant monotone function in Ult_U.

[If U extends the closed unbounded filter and if f is monotone and regressive on some $X \in U$, then since X is stationary, f is constant on an unbounded set and hence constant almost everywhere. If U does not extend the closed unbounded filter, then use (28.25) to get a nonconstant monotone function regressive on $X \in U$.]

Let us consider the relation $D \le U$ (the *Rudin–Keisler order*) among nonprincipal κ-complete ultrafilters over κ, defined as follows:

$$(28.26) \qquad D \le U \quad \text{iff} \quad D = h_*(U) \quad \text{for some} \quad h\colon \kappa \to \kappa$$

The relation $\le$ is transitive and induces a partial ordering among the equivalence classes, where $D \equiv U$ means $D \le U$ and $U \le D$. Let $D < U$ denote $D \le U$ and $U \not\le D$.

Exercise 28.13. If $D = h_*(U)$, then the function $k\colon Ult_D(V) \to Ult_U(V)$ defined by $k([f]_D) = [f \circ h]_U$ is an elementary embedding.

It follows that if f is an ordinal function, then $[f]_D \le [f \circ h]_U$, and in particular $[d]_D \le [h]_U$. Another consequence is that if $D \le U$, then $j_D(\kappa) \le j_U(\kappa)$.

If the function h in (28.26) is one-to-one, or at least one-to-one on some set $X \in U$, then $D \equiv U$ because $U = g_*(D)$ where $g = h^{-1}$. Moreover, $[f]_D = [f \circ h]_U$ for every f, and $Ult_D(V) = Ult_U(V)$. Also, $[d]_D = [h]_U$.

If U is a nonprincipal κ-complete ultrafilter over κ, let $\delta(U) = [f]_U$ where f is the least ordinal function that is one-to-one on a set $X \in U$.

Exercise 28.14. If $D < U$, then $\delta(D) < \delta(U)$. Consequently, the Rudin–Keisler order is well-founded.

[Let $D < U$. In view of the discussion above, we may assume that the diagonal function is the least one-to-one function in both Ult_D and Ult_U. Let $D = h_*(U)$. Claim: $h(\alpha) \leq \alpha$ almost everywhere. (Otherwise let $f(\xi)$ be the least ξ such that $h(\alpha) = \xi$; f is one-to-one since $\xi = h(f(\xi))$, and $f(\xi) < \xi$ almost everywhere mod D.) Since $D < U$ we have $[h]_U < [d]_U$, and so $\delta(D) = [d]_D \leq [h]_U < [d]_U = \delta(U)$.]

Measurable Cardinals and Cardinal Arithmetic

One of the remarkable consequences of existence of measurable cardinals is that if the generalized continuum hypothesis holds below a measurable cardinal κ, then $2^\kappa = \kappa^+$. In fact, as we shall see in the next chapter, the failure of the GCH at a measurable cardinal has rather strong consequences. For now, we shall just use the ultrapower to show that 2^κ depends on the behavior of 2^α below κ.

Lemma 28.13. *Let κ be a measurable cardinal. If $2^\kappa > \kappa^+$, then the set $\{\alpha < \kappa : 2^\alpha > \alpha^+\}$ has measure one, for every normal measure on κ.*

Consequently, if $2^\alpha = \alpha^+$ for all cardinals $\alpha < \kappa$, then $2^\kappa = \kappa^+$.

(As for the notation, note that $\{\alpha < \kappa : \alpha \text{ is a cardinal}\}$ is closed unbounded and hence has a normal measure one.)

Proof. Let D be a normal measure on κ, and let $M = Ult_D(V)$. If $2^\alpha = \alpha^+$ for almost all α, then, since $[d]_D = \kappa$, we have

$$M \models 2^\kappa = \kappa^+$$

In other words, there is a one-to-one mapping in M between $P^M(\kappa)$ and $(\kappa^+)^M$. However, $P^M(\kappa) = P(\kappa)$ and $(\kappa^+)^M \leq \kappa^+$ (in fact, $(\kappa^+)^M = \kappa^+$ because $P^M(\kappa) = P(\kappa)$), and so $2^\kappa = \kappa^+$. ■

Similarly, we have:

Exercise 28.15. If D is a normal measure on κ and $\{\alpha : 2^\alpha \leq \alpha^{++}\} \in D$, then $2^\kappa \leq \kappa^{++}$. More generally, if $\beta < \kappa$ and $\{\aleph_\alpha : 2^{\aleph_\alpha} \leq \aleph_{\alpha+\beta}\} \in D$, then $2^{\aleph_\kappa} \leq \aleph_{\kappa+\beta}$.

[If f is such that $f(\aleph_\alpha) = \aleph_{\alpha+\beta}$ for all $\alpha \subset \kappa$, then $[f]_D = (\aleph_{\kappa+j(\beta)})^M \leq \aleph_{\kappa+\beta}$.]

Exercise 28.16. If D is a normal measure on κ and $\{\alpha : 2^{\aleph_\alpha} < \aleph_{\alpha+\alpha}\} \in D$, then $2^{\aleph_\kappa} < \aleph_{\kappa+\kappa}$.

[If $f(\alpha) = \aleph_{\alpha+\alpha}$, then $[f] = (\aleph_{\kappa+\kappa})^M$.]

A similar method can be used for singular cardinals of measurable cofinality. The following results (which have been partly superseded by the more general theorems of Silver and Galvin–Hajnal; see Section 8) illustrate the method.

Lemma 28.14. *Let κ be a measurable cardinal, let D be a normal measure on κ and let $j\colon V \to M$ be the corresponding elementary embedding. Let $\lambda > \kappa$ be a strong limit cardinal of cofinality κ. Then $2^\lambda < j(\lambda)$.*

Proof. We recall that by Lemma 28.9(d), since cf $\lambda = \kappa$, we have $j(\lambda) > \lambda$. We shall show that

$$2^\lambda = \lambda^\kappa \le (\lambda^\kappa)^M \le (\lambda^{j(\kappa)})^M < j(\lambda) \tag{28.27}$$

The first equality in (28.27) holds because λ is a strong limit. We have $\lambda^\kappa \le (\lambda^\kappa)^M$ because $({}^\kappa\lambda)^M = {}^\kappa\lambda$. As for the last inequality, we have

$$M \vDash j(\lambda) \text{ is a strong limit cardinal}$$

and since $\lambda < j(\lambda)$ and $j(\kappa) < j(\lambda)$, we have $M \vDash \lambda^{j(\kappa)} < j(\lambda)$. ■

For applications of the lemma, see Exercises 28.18–28.20 below.

Exercise 28.17. Let λ be a singular cardinal such that $\kappa = \text{cf } \lambda$ is measurable. If $2^\alpha = \alpha^+$ for all $\alpha < \lambda$, then $2^\lambda = \lambda^+$.

[Let D be a normal measure on κ; let $\lambda = [e]_D$. For almost all α, $e(\alpha)^\alpha \le (e(\alpha))^+$, and so $Ult \vDash \lambda^\kappa \le \lambda^+$. Thus $2^\lambda = \lambda^\kappa \le (\lambda^\kappa)^M \le (\lambda^+)^M \le \lambda^+$.]

The result of this exercise is a special case of Theorem 23. In fact, Silver's original proof was motivated by measurable cardinals and is similar to the present method. I shall present the ultrapower proof in Chapter 6.

Exercise 28.18. Let κ be measurable and let $\lambda = \aleph_{\kappa+\kappa}$ be a strong limit. Then $2^\lambda < \aleph_{(2^\kappa)^+}$.

[Use Lemma 28.14: $j(\lambda) = (\aleph_{j(\kappa+\kappa)})^M \le \aleph_{j(\kappa)+j(\kappa)}$; $j(\kappa) + j(\kappa) < (2^\kappa)^+$.]

Exercise 28.19. Let κ be measurable, let λ be a strong limit, cf $\lambda = \kappa$, such that $\lambda < \aleph_\lambda$. Then $2^\lambda < \aleph_\lambda$. (Compare with Theorem 24.)

[Use Lemma 28.14: If $\lambda = \aleph_\alpha$, then $j(\lambda) = (\aleph_{j(\alpha)})^M \le \aleph_{j(\alpha)}$, and $j(\alpha) < (\alpha^\kappa)^+ < \lambda$.]

Exercise 28.20. Let $\Phi(\alpha)$ denote the αth fixed point of $\aleph$, i.e., the αth ordinal ξ such that $\aleph_\xi = \xi$. Let κ be measurable and let $\lambda = \Phi(\kappa + \kappa)$ be a strong limit. Then $2^\lambda < \Phi((2^\kappa)^+)$.

[Use Lemma 28.14 as in Exercise 28.18, and use the fact that $(\Phi(\alpha))^M \le \Phi(\alpha)$ for all α.]

The above applications of the ultrapower modulo a normal measure on κ can be expressed in a general abstract form (see also Lemma 32.2):

Lemma 28.15. *Let κ be a measurable cardinal and let D be a normal measure on κ. Then the model $\langle V_{\kappa+1}, \in\rangle$ is isomorphic to the ultraproduct $Ult_D\{\langle V_{\alpha+1}, \in\rangle : \alpha < \kappa\}$. The function*

$$\pi(x) = [\langle x \cap V_\alpha : \alpha < \kappa\rangle] \qquad (x \in V_{\kappa+1})$$

is an isomorphism.

Proof. Exercise. ■

Corollary. *Let φ be a formula. Under the assumptions of the lemma, if $x \in V_\kappa$ and $S \subseteq V_\kappa$, we have*

$$(28.28) \qquad V_{\kappa+1} \vDash \varphi(x, S) \quad \text{iff} \quad \{\alpha : V_{\alpha+1} \vDash \varphi(x, S \cap V_\alpha)\} \in D$$

29. INFINITARY COMBINATORICS

In this section, I shall introduce a new method, which in conjunction with the method of ultrapowers will enable us to obtain various consequences of large cardinal axioms. The method, which uses combinatorial arguments, has wide applications in model theory, and its most remarkable application in set theory (Section 30) is a mixture of combinatorial and model-theoretic techniques.

Partition Properties

Let us consider the following argument (*the pigeonhole principle*): If seven pigeons occupy three pigeonholes, then at least one pigeonhole is occupied by three pigeons. More generally: If an infinite set is partitioned into finitely many pieces, then at least one piece is infinite.

Recall that a *partition* of a set S is a pairwise disjoint family $P = \{X_i : i \in I\}$ such that $\bigcup_{i \in I} X_i = S$. With the partition P we can associate a function $F: S \to I$ such that $F(x) = F(y)$ if and only if x and y are in the same $X \in P$. Conversely, any function $F: S \to I$ determines a partition of S. (I shall sometimes say that F *is* a partition of S.)

For any set A and any natural number $n > 0$,

$$(29.1) \qquad [A]^n = \{X \subseteq A : |X| = n\}$$

is the set of all subsets of A that have exactly n elements. It is sometimes convenient, when A is a set of ordinals, to identify $[A]^n$ with the set of all sequences $\langle \alpha_1, \ldots, \alpha_n \rangle$ in A such that $\alpha_1 < \cdots < \alpha_n$. We shall consider partitions of sets $[A]^n$ for various infinite sets A and natural numbers n. Our starting point is the theorem of Ramsey dealing with finite partitions of $[\omega]^n$.

If $\{X_i : i \in I\}$ is a partition of $[A]^n$, then a set $H \subseteq A$ is *homogeneous* for the partition if for some i, $[H]^n$ is included in X_i; that is, if all the n-element subsets of H are in the same piece of the partition.

Lemma 29.1 (Ramsey's Theorem). *Let n and k be natural numbers. Every partition $\{X_1, \ldots, X_k\}$ of $[\omega]^n$ into k pieces has an infinite homogeneous set.*

Equivalently, for every $F: [\omega]^n \to \{1, \ldots, k\}$ there exists an infinite $H \subseteq \omega$ such that F is constant on $[H]^n$.

Proof. By induction on n. If $n = 1$, the lemma is trivial, so we assume that it holds for n and prove it for $n + 1$. Let F be a function from $[\omega]^{n+1}$ into

$\{1, \ldots, k\}$. For each $a \in \omega$, let F_a be the function on $[\omega - \{a\}]^n$ defined as follows:

$$F_a(X) = F(\{a\} \cup X)$$

By the induction hypothesis, there exists for each $a \in \omega$ an infinite set $H_a \subseteq \omega - \{a\}$ such that F_a is constant on $[H_a]^n$. We construct an infinite sequence $\{a_i : i = 0, 1, 2, \ldots\}$: We let $a_0 = 0$, and

$$a_{i+1} = \text{the least element of } H_{a_i} \text{ bigger than } a_i$$

It is clear that for each $i \in \omega$, the function F_{a_i} is constant on $[\{a_m : m > i\}]^n$; let $G(a_i)$ be its value. Now there is an infinite subset $H \subseteq \{a_i : i \in \omega\}$ such that G is constant on H. It follows that F is constant on $[H]^{n+1}$; this is because for $x_1 < \cdots < x_{n+1}$ in H we have $F(x_1, \ldots, x_{n+1}) = F_{x_1}(x_2, \ldots, x_{n+1})$. ∎

To facilitate our investigation of generalizations of Ramsey's theorem, I shall now introduce the *arrow notation.* Let κ and λ be infinite cardinal numbers, let n be a natural number and let m be a (finite or infinite) cardinal. The symbol

$$\kappa \to (\lambda)^n_m \tag{29.2}$$

(read: κ *arrows* λ) denotes the following *partition property*: Every partition of $[\kappa]^n$ into m pieces has a homogeneous set of size λ. In other words, every $F: [\kappa]^n \to m$ is constant on $[H]^n$ for some $H \subseteq \kappa$ such that $|H| = \lambda$. Using the arrow notation, Ramsey's theorem is expressed as follows:

$$\aleph_0 \to (\aleph_0)^n_k \qquad (n, k \in \omega) \tag{29.3}$$

The subscript m (in 29.2) is usually deleted when $m = 2$, and so

$$\kappa \to (\lambda)^n$$

is the same as $\kappa \to (\lambda)^n_2$.

The relation $\kappa \to (\lambda)^n_m$ stays true if κ is made larger or if λ or m are made smaller. A moment's reflection is sufficient to see that the relation also remains true when n is made smaller.

Obviously, the relation (29.2) makes any sense only if $\kappa \geq \lambda$ and $\kappa > m$; if $m = \kappa$, then it is clearly false. Thus we always assume $2 \leq m < \kappa$ and $\lambda \leq \kappa$. If $n = 1$, then (29.2) holds just in case either $\kappa > \lambda$, or $\kappa = \lambda$ and $\operatorname{cf} \kappa > m$. We shall concentrate on the nontrivial case: $n \geq 2$.

We start with two negative partition relations

Lemma 29.2. *For all* κ *and* λ,

$$2^\kappa \not\to (\lambda)^2_\kappa$$

In other words, there is a partition of 2^κ *into* κ *pieces that does not have an infinite homogeneous set.*

Proof. In fact, our partition has no homogeneous set of size ≥ 3. Let $S = {}^\kappa\{0, 1\}$ and let $F: [S]^2 \to \kappa$ be defined by $F(\{f, g\}) =$ the least $\alpha < \kappa$ such that $f(\alpha) \neq g(\alpha)$. If f, g, h are distinct elements of S, it is impossible to have $F(\{f, g\}) = F(\{f, h\}) = F(\{g, h\})$. ■

Lemma 29.3. *For every* κ,

$$2^\kappa \nrightarrow (\kappa^+)^2_2$$

(Thus the obvious generalization of Ramsey's theorem, namely $\aleph_1 \rightarrow (\aleph_1)^2_2$, is false.)

To construct a partition of $[2^\kappa]^2$ that violates the partition property, let us consider the linearly ordered set $(P, <)$ where $P = {}^\kappa\{0, 1\}$, and $f < g$ if and only if $f(\alpha) < g(\alpha)$ where α is the least α such that $f(\alpha) \neq g(\alpha)$ (the lexicographic ordering of P).

Lemma 29.4. *The lexicographically ordered set* ${}^\kappa\{0, 1\}$ *has no increasing or decreasing* κ^+*-sequence.*

Proof. Assume that $W = \{f_\alpha : \alpha < \kappa^+\} \subset {}^\kappa 2$ is such that $f_\alpha < f_\beta$ whenever $\alpha < \beta$ (the decreasing case is similar). Let $\gamma \leq \kappa$ be the least γ such that the set $\{f_\alpha | \gamma : \alpha < \kappa^+\}$ has size κ^+, and let $Z \subset W$ be such that $|Z| = \kappa^+$ and $f | \gamma \neq g | \gamma$ for $f, g \in Z$. We may as well assume that $Z = W$, so let us do so.

For each $\alpha < \kappa^+$, let ξ_α be such that $f_\alpha | \xi_\alpha = f_{\alpha+1} | \xi_\alpha$ and $f_\alpha(\xi_\alpha) = 0$, $f_{\alpha+1}(\xi_\alpha) = 1$; clearly $\xi_\alpha < \gamma$. Hence there exists $\xi < \gamma$ such that $\xi = \xi_\alpha$ for κ^+ elements f_α of W. However, if $\xi = \xi_\alpha = \xi_\beta$ and $f_\alpha | \xi = f_\beta | \xi$, then $f_\beta < f_{\alpha+1}$ and $f_\alpha < f_{\beta+1}$; hence $f_\alpha = f_\beta$. Thus the set $\{f_\alpha | \xi : \alpha < \kappa^+\}$ has size κ^+, contrary to the minimality assumption on γ. ■

Proof of Lemma 29.3. Let $2^\kappa = \lambda$ and let $\{f_\alpha : \alpha < \lambda\}$ be an enumeration of the set $P = {}^\kappa\{0, 1\}$. Let $\prec$ be a linear ordering of λ induced by the lexicographic ordering of P: $\alpha \prec \beta$ if $f_\alpha < f_\beta$.

Now we define a partition F: $[\lambda]^2 \rightarrow \{0, 1\}$ by letting $F(\{\alpha, \beta\}) = 1$ when the ordering $\prec$ of $\{\alpha, \beta\}$ agrees with the natural ordering; and letting $F(\{\alpha, \beta\}) = 0$ otherwise. If $H \subseteq \lambda$ is a homogeneous set of order type κ^+, then $\{f_\alpha : \alpha \in H\}$ constitutes an increasing or decreasing κ^+-sequence in $(P, <)$; a contradiction. ■

By Lemma 29.3, the relation $\kappa \rightarrow (\aleph_1)^2$ is false if $\kappa \leq 2^{\aleph_0}$. On the other hand, if $\kappa > 2^{\aleph_0}$, then $\kappa \rightarrow (\aleph_1)^2$ is true, as follows from this more general theorem:

Theorem 69 (*Erdös–Rado Partition Theorem*).

$$\beth_n^+ \rightarrow (\aleph_1)^{n+1}_{\aleph_0}$$

In particular, $(2^{\aleph_0})^+ \rightarrow (\aleph_1)^2_{\aleph_0}$.

Proof. We shall first prove the case $n = 1$ since the induction step parallels closely this case. Thus let $\kappa = (2^{\aleph_0})^+$ and let F: $[\kappa]^2 \rightarrow \omega$ be a partition of $[\kappa]^2$ into $\aleph_0$ pieces. We want to find a homogeneous $H \subset \kappa$ of size $\aleph_1$.

For each $a \in \kappa$, let F_a be a function on $\kappa - \{a\}$ defined by $F_a(x) = F(\{a, x\})$.

We shall first prove the following claim: There exists a set $A \subset \kappa$ such that $|A| = 2^{\aleph_0}$ and such that for every countable $C \subset A$ and every $u \in \kappa - C$ there exists $v \in A - C$ such that F_v agrees with F_u on C.

To prove the claim, we construct an ω_1-sequence $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots$, $\alpha < \omega_1$, of subsets of κ, each of size $2^{\aleph_0}$, as follows: Let A_0 be arbitrary, and for each limit α, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Given A_α, there exists a set $A_{\alpha+1} \supseteq A_\alpha$ of size $2^{\aleph_0}$ such that for each countable $C \subset A_\alpha$ and every $u \in \kappa - C$ there exists $v \in A_{\alpha+1} - C$ such that F_v agrees with F_u on C (because the number of such functions is $\leq 2^{\aleph_0}$). Then we let $A = \bigcup_{\alpha<\omega_1} A_\alpha$, and clearly A has the required property.

Next we choose some $a \in \kappa - A$, and construct a sequence $\{x_\alpha : \alpha < \omega_1\}$ in A as follows: Let x_0 be arbitrary, and given $\{x_\beta : \beta < \alpha\} = C$, let x_α be some $v \in A - C$ such that F_v agrees with F_a on C. Let $X = \{x_\alpha : \alpha < \omega_1\}$.

Now we consider the function $G\colon X \to \omega$ defined by $G(x) = F_a(x)$. It is clear that if $\alpha < \beta$, then $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha) = F_a(x_\alpha) = G(x_\alpha)$. Since the range of G is countable, there exists $H \subseteq X$ of size $\aleph_1$ such that G is constant on H. It follows that F is constant on $[H]^2$.

Thus we have proved the theorem for $n = 1$. The general case is proved by induction. Let us assume that $\beth_{n-1}^+ \to (\aleph_1)^n_{\aleph_0}$ and let $F\colon [\kappa]^{n+1} \to \omega$, where $\kappa = \beth_n^+$. For each $a \in \kappa$, let $F_a\colon [\kappa - \{a\}]^n \to \omega$ be defined by $F_a(x) = F(x \cup \{a\})$. As in the case $n = 1$, there exists a set $A \subset \kappa$ of size $\beth_n$ such that for every $C \subset A$ of size $|C| \leq \beth_{n-1}$ and every $u \in \kappa - C$ there exists $v \in A - C$ such that F_v agrees with F_u on $[C]^n$.

Next we choose $a \in \kappa - A$ and construct a set $X = \{x_\alpha : \alpha < \beth_{n-1}^+\} \subset A$ such that for each α, F_{x_α} agrees with F_a on $[\{x_\beta : \beta < \alpha\}]^n$.

Then we consider $G\colon [X]^n \to \omega$ where $G(x) = F_a(x)$. As before, if $\alpha_1 < \cdots < \alpha_{n+1}$, then $F(\{x_{\alpha_1}, \ldots, x_{\alpha_{n+1}}\}) = G(\{x_{\alpha_1}, \ldots, x_{\alpha_n}\})$. By the induction hypothesis, there exists $H \subseteq X$ of size $\aleph_1$ such that G is constant on $[H]^n$. It follows that F is constant on $[H]^{n+1}$. ■

It can be proved that for each n, the partition property $\beth_n^+ \to (\aleph_1)^{n+1}_{\aleph_0}$ is best possible. The property also generalizes easily to larger cardinals. For each κ, let $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

Exercise 29.1. For every κ, $(\exp_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$.

In particular, we have $(2^\kappa)^+ \to (\kappa^+)^2$.

Both Ramsey's theorem and the Erdös–Rado theorem have numerous applications in model theory. The following easy application of the Erdös–Rado theorem in point set topology is of some interest to us. We recall that if $(P, <)$ and $(Q, <)$ are two partially ordered sets such that $\operatorname{sat}(P) = \operatorname{sat}(Q) = \aleph_1$ (that is, both satisfy the countable chain condition) then $\operatorname{sat}(P \times Q)$ is not necessarily $\aleph_1$; see Exercise 22.4, and also Theorem 53 and the discussion preceding Lemma 20.3.

Exercise 29.2. If P and Q are partially ordered sets such that $\mathrm{sat}(P) = \mathrm{sat}(Q) = \aleph_1$, then $\mathrm{sat}(P \times Q) \leq (2^{\aleph_0})^+$.

[Let $\kappa = (2^{\aleph_0})^+$ and let $\{x_\alpha : \alpha < \kappa\}$ be an incompatible subset of $P \times Q$. For all α, $\beta < \kappa$, let $F(\{\alpha, \beta\}) = 0$ or 1 according to whether the first coordinates of x_α and x_β are incompatible or compatible. Apply the Erdös–Rado theorem and get either $\aleph_1$ incompatible elements of P or $\aleph_1$ incompatible elements of Q.]

A natural generalization of the partition property (29.2) is when we allow λ to be a limit ordinal, not just a cardinal. Let κ, n and m be as in (29.2) and let $\alpha > 0$ be a limit ordinal. The symbol

$$\kappa \to (\alpha)^n_m \tag{29.4}$$

stands for: For every $F\colon [\kappa]^n \to m$ there exists $H \subseteq \kappa$ of order-type α such that F is constant on $[H]^n$.

There are various results about the partition relation (29.4). For instance, it has been proved that $\aleph_1 \to (\alpha)^2$ for all $\alpha < \omega_1$. The analogous case for $\aleph_2$ is unclear: If $2^{\aleph_0} = \aleph_1$, then $\aleph_2 \to (\omega_1)^2$ (by Erdös–Rado). However, it is consistent (with $2^{\aleph_0} = \aleph_1$) that $\aleph_2 \not\to (\omega_1 + \omega)^2$ (either by forcing or in L, using the principle W in (24.16)). Moreover, it is not known whether $\aleph_2 \to (\omega_1 + \omega)^2$ is consistent.

In the positive results given by the Erdös–Rado theorem, the size of the homogeneous set is smaller than the size of the set being partitioned. A natural question arises, whether the relation $\kappa \to (\kappa)^2$ can hold for cardinals other than $\kappa = \omega$.

A cardinal κ is *weakly compact* if it is uncountable and satisfies the partition property $\kappa \to (\kappa)^2$. The reason for the name "weakly compact" is that these cardinals satisfy a certain compactness theorem for infinitary languages; we shall investigate weakly compact cardinals in some detail in Section 32.

Lemma 29.5. *Every weakly compact cardinal is inaccessible.*

Proof. Let κ be a weakly compact cardinal. To show that κ is regular, let us assume that κ is the disjoint union $\bigcup \{A_\gamma : \gamma < \lambda\}$ such that $\lambda < \kappa$ and $|A_\gamma| < \kappa$ for each $\gamma < \lambda$. We define a partition $F\colon [\kappa]^2 \to \{0, 1\}$ as follows: $F(\{\alpha, \beta\}) = 0$ just in case α and β are in the same A_γ. Obviously, this partition does not have a homogeneous set $H \subseteq \kappa$ of size κ.

That κ is a strong limit cardinal follows from Lemma 29.3: If $\kappa \leq 2^\lambda$ for some $\lambda < \kappa$, then because $2^\lambda \not\to (\lambda^+)^2$, we have $\kappa \not\to (\lambda^+)^2$ and hence $\kappa \not\to (\kappa)^2$. ■

We shall prove in Section 32 that every weakly compact cardinal κ is the κth inaccessible cardinal.

I shall now give an equivalent formulation of weak compactness. Recall that an *Aronszajn tree* is a tree of height ω_1 all of whose levels are at most countable and which has no uncountable branches (see Lemma 22.3). For any regular uncountable cardinal κ, let us call an *Aronszajn κ-tree* a tree of height κ with levels of size $< \kappa$ and with no branches of size κ. I shall look into this generalization of Aronszajn trees later in this section; for now let us just say that κ has the *tree property* if there is no Aronszajn κ-tree, that is, if every tree of height κ with small levels has a branch of size κ.

Lemma 29.6.

(a) *If κ is weakly compact, then κ has the tree property.*
(b) *If κ is inaccessible and has the tree property, then κ is weakly compact, and in fact $\kappa \to (\kappa)^2_m$ for every $m < \kappa$.*

Proof. (a) Let κ be weakly compact and let $(T, <_T)$ be a tree of height κ such that each level of T has size $< \kappa$. Since κ is inaccessible, $|T| = \kappa$ and we may assume that $T = \kappa$. We extend the partial ordering $<_T$ of κ to a linear ordering $\prec$: If $\alpha <_T \beta$, then we let $\alpha \prec \beta$; if α and β are incomparable and if ξ is the first level where the predecessors α_ξ, β_ξ of α and β are distinct, we let $\alpha \prec \beta$ if and only if $\alpha_\xi < \beta_\xi$.

Let F: $[\kappa]^2 \to \{0, 1\}$ be the partition defined by $F(\{\alpha, \beta\}) = 1$ just in case $\prec$ agrees with $<$ on $\{\alpha, \beta\}$. By weak compactness, let $H \subseteq \kappa$ be homogeneous for F, $|H| = \kappa$.

We now consider the set $B \subseteq \kappa$ of all $x \in \kappa$ such that $\{\alpha \in H : x <_T \alpha\}$ has size κ. Since every level has size $< \kappa$, it is clear that at each level there is at least one x in B. Thus if we show that any two elements of B are $<_T$-comparable, we shall have proved that B is a branch in T of size κ.

Thus assume that x, y are incomparable elements of B; let $x \prec y$. Since both x and y have κ successors in H, there exist $\alpha < \beta < \gamma$ in H such that $x <_T \alpha$, $y <_T \beta$, and $x <_T \gamma$. By the definition of $\prec$, we have $\alpha \prec \beta$ and $\gamma \prec \beta$. Thus $F(\{\alpha, \beta\}) = 1$ and $F(\{\gamma, \beta\}) = 0$, contrary to the homogeneity of H.

(b) Let κ be an inaccessible cardinal with the tree property, and let F: $[\kappa]^2 \to I$ be a partition such that $|I| < \kappa$. We shall find a homogeneous $H \subseteq \kappa$ of size κ.

We construct a tree $(T, \subseteq)$ whose elements are some functions t: $\gamma \to I$, $\gamma < \kappa$. We construct T by induction: At step $\alpha < \kappa$, we put into T one more element t, calling it t_α. Let $t_0 = \emptyset$. Having constructed $t_0, \ldots, t_\beta, \ldots, \beta < \alpha$, let us construct t_α as follows, by induction on ξ. Having constructed $t_\alpha | \xi$, we look first whether $t_\alpha | \xi$ is among the t_β, $\beta < \alpha$ (note that for $\xi = 0$ we have $t_\alpha | 0 = t_0$). If not, then we consider t_α constructed: $t_\alpha = t_\alpha | \xi$. If $t_\alpha | \xi = t_\beta$ for some $\beta < \alpha$, then we let $t_\alpha(\xi) = i$, where $i = F(\{\beta, \alpha\})$.

$(T, \subseteq)$ is a tree of size κ; and since κ is inaccessible, each level of T has size $< \kappa$ and the height of T is κ. It follows from the construction that if $t_\beta \subset t_\alpha$, then $\beta < \alpha$ and $F(\{\beta, \alpha\}) = t_\alpha(\text{length}(t_\beta))$. By the assumption, T has a branch B of size κ. If we now let, for each $i \in I$,

$$H_i = \{\alpha : t_\alpha \in B \text{ and } t_\alpha{}^\frown i \in B\} \tag{29.5}$$

then each H_i is clearly homogeneous for the partition F, and at least one H_i has size κ. ■

It should be mentioned that an argument similar to the one above, only more complicated, shows that if κ is inaccessible and has the tree property, then $\kappa \to (\kappa)^n_m$ for all $n \in \omega$, $m < \kappa$. We have proved that every measurable cardinal κ is inaccessible and is in fact the κth inaccessible cardinal. We shall now improve on this as follows:

Lemma 29.7. *Every measurable cardinal κ is weakly compact. Moreover, if D is a normal measure on κ, then the set $\{\alpha < \kappa : \alpha$ is weakly compact$\}$ is in D.*

Proof. Let κ be a measurable cardinal. To show that κ is weakly compact, it suffices to prove the tree property. Let $(T, <)$ be a tree of height κ with levels of size $< \kappa$. We consider a nonprincipal κ-complete ultrafilter U over T. Let B be the set of all $x \in T$ such that the set of all successors of x is in U. It is clear that B is a branch in T and it is easy to verify that each level of T has one element in B; thus B is a branch of size κ.

Let D be a normal measure on κ and consider the elementary embedding j_D: $V \to M$ where $M = Ult_D(V)$. Since $P^M(\kappa) = P(\kappa)$, it follows that κ is weakly compact in M, and since $[d]_D = \kappa$, we have $\{\alpha : \alpha$ is weakly compact$\} \in D$. ■

We shall consider a generalization of the partition properties introduced so far. But first let us consider this: If A is an infinite set of ordinals and α an ordinal, let $[A]^\alpha$ denote the set of all increasing α-sequences in A. The symbol

$$\kappa \to (\lambda)^\alpha \tag{29.6}$$

denotes: For every partition F: $[\kappa]^\alpha \to \{0, 1\}$ of $[\kappa]^\alpha$ into two pieces, there exists a set H of order type λ such that F is constant on $[H]^\alpha$.

Exercise 29.3. For all infinite cardinals κ, $\kappa \not\to (\omega)^\omega$.

[For s, $t \in [\kappa]^\omega$ let $s \equiv t$ iff $\{n : s(n) \neq t(n)\}$ is finite. Pick a representative in each equivalence class. Let $F(s) = 0$ if s differs from the representative of its class at an even number of places; let $F(s) = 1$ otherwise. F has no infinite homogeneous set.]

Let me introduce now the following partition property: Let κ be an infinite cardinal, let α be an infinite limit ordinal, $\alpha \leq \kappa$, and let m be a cardinal, $2 \leq m < \kappa$. The symbol

$$\kappa \to (\alpha)^{<\omega}_m \tag{29.7}$$

denotes the property that for every partition F of the set $[\kappa]^{<\omega} = \bigcup_{n=0}^{\infty} [\kappa]^n$ into m pieces, there exists a set $H \subseteq \kappa$ of order type α such that for each $n \in \omega$, F is constant on $[H]^n$. (Again, the subscript m is deleted when $m = 2$.) We say that H is *homogeneous* for F.

Exercise 29.4. $\omega \not\to (\omega)^{<\omega}$.

[For $x \in [\omega]^{<\omega}$, let $F(x) = 1$ if $|x| \in x$, and $F(x) = 0$ otherwise. If $H \subseteq \omega$ is infinite, pick $n \in H$ and show that F is not constant on $[H]^n$.]

We shall see later that the least κ such that $\kappa \nrightarrow (\omega)^{<\omega}$ is inaccessible, and greater than the least weakly compact cardinal. But first let us investigate the stronger property.

$$\kappa \to (\kappa)^{<\omega} \tag{29.8}$$

The cardinals that satisfy the partition relation (29.8) are called *Ramsey cardinals*. We shall show that every measurable cardinal is Ramsey and use this property to improve considerably on Scott's theorem (Theorem 67). Note that since $\aleph_0$ is not Ramsey, every Ramsey cardinal is weakly compact and hence inaccessible.

Theorem 70. *Let κ be a measurable cardinal, let D be a normal measure on κ, and let F be a partition of $[\kappa]^{<\omega}$ into less than κ pieces. Then there exists a set $H \in D$ homogeneous for F. Hence every measurable cardinal is Ramsey.*

Proof. Let D be a normal measure on κ, and let F be a partition of $[\kappa]^{<\omega}$ into less than κ pieces. It suffices to show that for each $n = 1, 2, \ldots$, there is $H_n \in D$ such that F is constant on $[H_n]^n$; then $H = \bigcap_{n=1}^{\infty} H_n$ is homogeneous for F.

We prove, by induction on n, that every partition of $[\kappa]^n$ into less than κ pieces is constant on $[H]^n$ for some $H \in D$. The assertion is trivial for $n = 1$, so we assume that it is true for n and prove that it holds also for $n + 1$. Let $F\colon [\kappa]^{n+1} \to I$, where $|I| < \kappa$. For each $\alpha < \kappa$, we define F_α on $[\kappa - \{\alpha\}]^n$ by $F_\alpha(x) = F(\{\alpha\} \cup x)$.

By the induction hypothesis, there exists for each $\alpha < \kappa$ a set $X_\alpha \in D$ such that F_α is constant on $[X_\alpha]^n$; let i_α be its constant value. Let X be the diagonal intersection

$$X = \{\alpha < \kappa : \alpha \in \bigcap_{\gamma < \alpha} X_\gamma\}$$

We have $X \in D$ since D is normal; also, if $\gamma < \alpha_1 < \cdots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\gamma]^n$ and so $F(\{\gamma, \alpha_1, \ldots, \alpha_n\}) = F_\gamma(\{\alpha_1, \ldots, \alpha_n\}) = i_\gamma$. Now, there is $i \in I$ and $H \subseteq X$ in D such that $i_\gamma = i$ for all $\gamma \in H$. It follows that $F(x) = i$ for all $x \in [H]^{n+1}$. ■

Exercise 29.5. If D is a normal measure on κ, then the set of all Ramsey cardinals below κ has D-measure 1.

[As in Lemma 29.7.]

Exercise 29.6. Let D be a normal measure on κ and let $f\colon [\kappa]^{<\omega} \to \kappa$ be such that $f(x) = 0$ or $f(x) < \min(x)$ for all $x \in [\kappa]^{<\omega}$ ($\min(x)$ is the least element of x). Then there is $H \in D$ such that for each n, f is constant on $[H]^n$.

[By induction, as in Theorem 70. Given f on $[\kappa]^{n+1}$, let $f_\alpha(s) = f(\{\alpha\} \cup s)$ for $\alpha < \min(s)$; f_α is constant on $[X_\alpha]^n$ with value $\gamma_\alpha < \alpha$. Let X be the diagonal intersection of X_α, $\alpha < \kappa$, and let γ and $H \subseteq X$ be such that $H \in D$ and $\gamma_\alpha = \gamma$ for all $\alpha \in H$.]

Let U be a κ-complete nonprincipal ultrafilter over κ. For each $n = 1, 2, \ldots$, we define an ultrafilter U_n over $[\kappa]^n$ as follows: If $X \subseteq [\kappa]^{n+1}$, let

$$X_{(\alpha)} = \{x \in [\kappa]^n : \alpha < \min(x) \text{ and } \{\alpha\} \cup x \in X\} \tag{29.9}$$

We let $U_1 = U$, and define by induction

$$X \in U_{n+1} \quad \text{iff} \quad \{\alpha : X_{(\alpha)} \in U_n\} \in U \tag{29.10}$$

(Thus $X \in U_n$ iff for almost all α_1, for almost all $\alpha_2, \ldots$, for almost all α_n, $\{\alpha_1, \ldots, \alpha_n\} \in X$.) It is easy to verify that each U_n is a nonprincipal κ-complete ultrafilter over $[\kappa]^n$. Moreover, if $Z \in U$, then $[Z]^n \in U_n$.

Exercise 29.7. If U is normal and $X \in U_n$, then there is $H \in U$ such that $[H]^n \subseteq X$.

[Either X or $[\kappa]^n - X$ contains $[H]^n$ for some $H \in U$, by Theorem 70. It must be X since $[H]^n \in U_n$.]

Partitions and Models

Partition theorems play an important role in model theory, and I shall have more opportunity to present some applications in the next section. For now, let us consider a model

$$\mathfrak{A} = \langle A, P^{\mathfrak{A}}, \ldots, F^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots \rangle \tag{29.11}$$

of a (not necessarily countable) language $\mathcal{L}$

$$\mathcal{L} = \{P, \ldots, F, \ldots, c, \ldots\} \tag{29.12}$$

Let κ be an infinite cardinal and let us assume that the universe A of the model $\mathfrak{A}$ contains all ordinals $\alpha < \kappa$: $A \supseteq \kappa$.

A set $I \subseteq \kappa$ is a *set of indiscernibles* for the model $\mathfrak{A}$ if for every $n \in \omega$, and every formula $\varphi(v_1, \ldots, v_n)$,

$$\mathfrak{A} \vDash \varphi[\alpha_1, \ldots, \alpha_n] \quad \text{iff} \quad \mathfrak{A} \vDash \varphi[\beta_1, \ldots, \beta_n] \tag{29.13}$$

whenever $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are two increasing sequences of elements of I.

Exercise 29.8. Let $F: [\kappa]^{<\omega} \to \{0, 1\}$, and let $F_n = F \upharpoonright [\kappa]^n$, for each n. Let I be a set of indiscernibles for the model $\langle \kappa, <, F_1, F_2, F_3, \ldots \rangle$. Then I is homogeneous for the partition F.

[Consider the formula $F_n(\alpha_1, \ldots, \alpha_n) = F_n(\beta_1, \ldots, \beta_n)$.]

Thus the partition properties are, in a way, a special case of existence of indiscernibles. We shall now show that on the other hand, partition properties imply existence of indiscernibles in models:

Lemma 29.8. *Let κ be an infinite cardinal and assume that*

$$\kappa \to (\alpha)^{<\omega}_{2^\lambda}$$

where α is a limit ordinal and λ is an infinite cardinal. Let $\mathscr{L}$ be a language of size $\leq \lambda$ and let $\mathfrak{A}$ be a model of $\mathscr{L}$ such that $A \supseteq \kappa$. Then $\mathfrak{A}$ has a set of indiscernibles of order type α.

Proof. Let Φ be the set of all formulas of the language $\mathscr{L}$. We consider the function $F\colon [\kappa]^{<\omega} \to P(\Phi)$ defined as follows: If $x \in [\kappa]^n$ and $x = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1 < \cdots < \alpha_n$, then

$$F(x) = \{\varphi(v_1, \ldots, v_n) \in \Phi : \mathfrak{A} \models \varphi[\alpha_1, \ldots, \alpha_n]\}$$

The function F is a partition into at most 2^λ pieces and thus has a homogeneous set $I \subseteq \kappa$ of order-type α. It is now easy to verify that I is a set of indiscernibles for $\mathfrak{A}$. ■

We shall see in Section 32 that for a given limit ordinal α, the least κ that satisfies $\kappa \to (\alpha)^{<\omega}$ is inaccessible and satisfies $\kappa \to (\alpha)^{<\omega}_\lambda$ for all $\lambda < \kappa$. Now we shall prove this for Ramsey cardinals.

Lemma 29.9. *If $\kappa \to (\kappa)^{<\omega}$ and if λ is a cardinal less than κ, then $\kappa \to (\kappa)^{<\omega}_\lambda$.*

Corollary. *If κ is a Ramsey cardinal and if $\mathfrak{A} \supseteq \kappa$ is a model of a language of size $< \kappa$, then $\mathfrak{A}$ has a set of indiscernibles of size κ.*

Proof. Let $F\colon [\kappa]^{<\omega} \to \lambda$ be a partition into $\lambda < \kappa$ pieces. We consider the following partition G of $[\kappa]^{<\omega}$ into two pieces: If $\alpha_1 < \cdots < \alpha_k < \alpha_{k+1} < \cdots < \alpha_{2k}$ are elements of κ and if $F(\{\alpha_1, \ldots, \alpha_k\}) = F(\{\alpha_{k+1}, \ldots, \alpha_{2k}\})$, then we let $G(\{\alpha_1, \ldots, \alpha_{2k}\}) = 1$; for all other $x \in [\kappa]^{<\omega}$, we let $G(x) = 0$.

Now, let $H \subseteq \kappa$ be a homogeneous set for G, $|H| = \kappa$. We claim that for each k and each $x \in [H]^{2k}$, $G(x) = 1$: This is because $|H| = \kappa > \lambda$, and therefore we can find $\alpha_1 < \cdots < \alpha_k < \alpha_{k+1} < \cdots < \alpha_{2k}$ in H such that $F(\{\alpha_1, \ldots, \alpha_k\}) = F(\{\alpha_{k+1}, \ldots, \alpha_{2k}\})$.

It follows that H is homogeneous for F: If $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are two sequences in H, we choose a sequence $\gamma_1 < \cdots < \gamma_n$ in H such that both $\alpha_n < \gamma_1$ and $\beta_n < \gamma_1$. Then

$$G(\{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n\}) = G(\{\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n\}) = 1$$

and hence

$$F(\{\alpha_1, \ldots, \alpha_n\}) = F(\{\gamma_1, \ldots, \gamma_n\}) = F(\{\beta_1, \ldots, \beta_n\}). \quad ■$$

The combinatorial methods introduced in this section will now be employed to obtain a result on measurable cardinals considerably stronger than Scott's theorem in the preceding section. Let me first make a few observations about models with definable Skolem function.

Let $\mathfrak{A}$ be a model of a language $\mathscr{L}$ such that $\mathfrak{A} \supseteq \kappa$ and let $I \subseteq \kappa$ be a set of indiscernibles for $\mathfrak{A}$. Let us assume that the model $\mathfrak{A}$ has *definable Skolem* *functions*; i.e., for every formula* $\varphi(u, v_1, \ldots, v_n)$ there exists an n-ary function h_φ in $\mathfrak{A}$ such that:

(i) h_φ is *definable* in $\mathfrak{A}$, i.e., there is a formula ψ such that

$$y = h_\varphi(x_1, \ldots, x_n) \qquad \text{iff} \qquad \mathfrak{A} \vDash \psi[y, x_1, \ldots, x_n]$$

for all $y, x_1, \ldots, x_n \in A$; and

(ii) h_φ is a *Skolem function for* φ (see Section 10).

Let $\mathfrak{B} \subseteq \mathfrak{A}$ be the closure of I under all functions in $\mathscr{L}$ and all the functions h_φ, φ a formula. $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, and in fact is the least elementary submodel of $\mathfrak{A}$ that includes the set I; let us call $\mathfrak{B}$ the *Skolem hull* of I and let us say that I *generates* $\mathfrak{B}$.

We augment the language of $\mathfrak{A}$ by adding function symbols for all the Skolem functions h_φ and call *Skolem terms* the terms built up from variables and constant symbols (which are 0-ary functions) by applications of functions in $\mathscr{L}$ and the Skolem functions. Since $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, the interpretation of each Skolem term t is the same in $\mathfrak{B}$ as in $\mathfrak{A}$. For every element $x \in \mathfrak{B}$ there is a Skolem term t and indiscernibles $\gamma_1 < \cdots < \gamma_n$ (elements of I) such that

$$x = t^{\mathfrak{A}}[\gamma_1, \ldots, \gamma_n] = t^{\mathfrak{B}}[\gamma_1, \ldots, \gamma_n] \tag{29.14}$$

Now if ψ is a formula of the augmented language, i.e., if ψ also contains the Skolem terms, it still does not distinguish between the indiscernibles: If $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are two sequences in I, then $\psi(\alpha_1, \ldots, \alpha_n)$ holds (either in $\mathfrak{A}$ or in $\mathfrak{B}$) if and only if $\psi(\beta_1, \ldots, \beta_n)$ holds.

Theorem 71 (Rowbottom). *If κ is a Ramsey cardinal, then the set of all constructible reals is countable. More generally, if λ is an infinite cardinal less than κ, then $|P^L(\lambda)| = \lambda$.*

Proof. Let κ be a Ramsey cardinal and let $\lambda < \kappa$. Since κ is inaccessible, we have $P^L(\lambda) \subset L_\kappa$. Consider the model

$$\mathfrak{A} = \langle L_\kappa, \in, P^L(\lambda), \alpha \rangle_{\alpha \leq \lambda} \tag{29.15}$$

$\mathfrak{A}$ is a model of the language $\mathscr{L} = \{\in, Q, c_\alpha\}_{\alpha \leq \lambda}$ where Q is a one-place predicate (interpreted in $\mathfrak{A}$ as $P(\lambda) \cap L$) and c_α, $\alpha \leq \lambda$, are constant symbols (interpreted as ordinals less than or equal to λ). Since κ is Ramsey, there exists a set I of size κ of indiscernibles for $\mathfrak{A}$.

The model $\mathfrak{A}$ has definable Skolem functions: Since κ is inaccessible, L_κ is a model of ZFC + $V = L$ and therefore has a definable well-ordering (see also Exercise 13.14). Thus let $\mathfrak{B} \subseteq L_\kappa$ be the elementary submodel of $\mathfrak{A}$ generated by the set I. Every element $x \in \mathfrak{B}$ is expressible as $x = t(\gamma_1, \ldots, \gamma_n)$ where t is a Skolem term and $\gamma_1 < \cdots < \gamma_n$ are elements of I.

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* $n \geq 0$.

We shall now show that the set $S = P^L(\lambda) \cap \mathfrak{B}$ has at most λ elements. Since S is the interpretation in $\mathfrak{B}$ of the one-place predicate Q, it suffices to show that there are at most λ elements $x \in \mathfrak{B}$ such that $\mathfrak{B} \models Q(x)$.

Let t be a Skolem term. Let us consider the truth value of the formula

$$t(\alpha_1, \ldots, \alpha_n) = t(\beta_1, \ldots, \beta_n) \tag{29.16}$$

for a sequence of indiscernibles $\alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots < \beta_n$. The formula (29.16) is either true for all increasing sequences in I or false for all increasing sequences in I. If (29.16) is true, then it is true for any two sequences $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$, in I: Pick $\gamma_1, \ldots, \gamma_n$ bigger than both α_n and β_n and then $t(\alpha_1, \ldots, \alpha_n) = t(\gamma_1, \ldots, \gamma_n) = t(\beta_1, \ldots, \beta_n)$. If (29.16) is false, then we choose κ increasing sequences

$$\alpha_1^0 < \cdots < \alpha_n^0 < \alpha_1^1 < \cdots < \alpha_n^1 < \cdots < \alpha_1^\xi < \cdots < \alpha_n^\xi < \cdots \qquad (\xi < \kappa)$$

in I and then $t(\alpha_1^\xi, \ldots, \alpha_n^\xi) \neq t(\alpha_1^\eta, \ldots, \alpha_n^\eta)$ whenever $\xi \neq \eta$. In conclusion, the set

$$\{t(\alpha_1, \ldots, \alpha_n) : \alpha_1 < \cdots < \alpha_n \text{ are in } I\} \tag{29.17}$$

has either one or κ elements.

Now we apply this to evaluate the size of the set S. We know that $|S| < \kappa$ because $S \subseteq P^L(\lambda) \subseteq P(\lambda)$ and κ is inaccessible. If t is a Skolem term for which the set (29.17) has size κ, then $t(\alpha_1, \ldots, \alpha_n)$ is not in S, for any $\alpha_1 < \cdots < \alpha_n$ in I; by indiscernibility, $Q(t(\alpha_1, \ldots, \alpha_n))$ is true or false simultaneously for all increasing sequences in I. Thus if $t(\alpha_1, \ldots, \alpha_n) \in S$, the set (29.17) has only one element.

However, since $|\mathscr{L}| \leq \lambda$, there are at most λ Skolem terms. And since every $x \in \mathfrak{B}$ has the form $t(\alpha_1, \ldots, \alpha_n)$ for some Skolem term and $\alpha_1 < \cdots < \alpha_n$ in I, it follows that $|S| \leq \lambda$.

Thus we have proved that $S = Q^{\mathfrak{B}} = P^L(\lambda) \cap \mathfrak{B}$ has at most λ elements. Now $\mathfrak{B} \prec L_\kappa$ and $|\mathfrak{B}| = \kappa$; hence the transitive collapse of $\mathfrak{B}$ is L_κ and we have an isomorphism

$$\pi : \mathfrak{B} \simeq L_\kappa$$

Since each $\alpha \leq \lambda$ has a name in $\mathfrak{A}$, we have $\lambda \cup \{\lambda\} \subseteq \mathfrak{B}$ and so $\pi(X) = X$ for each $X \subseteq \lambda$ in $\mathfrak{B}$. In particular $\pi(X) = X$ for all $X \in S$; and since $Q^{\pi[\mathfrak{B}]} = \pi[S] = S$, we have

$$S = P^L(\lambda) \cap \pi[\mathfrak{B}] = P^L(\lambda) \cap L_\kappa = P^L(\lambda)$$

This completes the proof: On the one hand, we proved that $|S| \leq \lambda$; and on the other hand, $|P^L(\lambda)| \geq \lambda$; thus $|P^L(\lambda)| = \lambda$. ■

Exercise 29.9. If κ is Ramsey, then $\aleph_1$ is inaccessible in L.

[Show that $P^{L[x]}(\omega)$ is countable for every $x \subseteq \omega$.]

Exercise 29.10. If κ is Ramsey, and $\lambda < \kappa$, then every set definable in $\langle L_\kappa, \in, \alpha \rangle_{\alpha \leq \lambda}$ has size either κ or $\leq \lambda$.

The proof of Theorem 71 is an important application of the Ramsey property of κ and can be used in a more general situation. Below we give an abstract version of the result just proved.

Lemma 29.10. *Let κ be a Ramsey cardinal, and let λ be an infinite cardinal less than κ. Let $\mathfrak{A} = \langle A, \ldots \rangle$ be a model of a language $\mathcal{L}$ such that $|\mathcal{L}| \leq \lambda$, and let $A \supseteq \kappa$. If $P \subseteq A$ is such that $|P| < \kappa$ then A has an elementary submodel $\mathfrak{B} = \langle B, \ldots \rangle$ such that $|B| = \kappa$ and $|P \cap B| \leq \lambda$.*

Moreover, if $X \subseteq A$ is of size at most λ, then we can find $\mathfrak{B}$ such that $X \subseteq B$.

Moreover, if κ is a measurable cardinal and D is a normal measure on κ, then we can find $\mathfrak{B}$ such that $B \cap \kappa \in D$.

Proof. First we add to the language $\mathcal{L}$ one unary predicate whose interpretation is the set P; we also add constant symbols for all $x \in X$. Next we find some Skolem functions h_φ (in $\langle A, \ldots, P, x \rangle_{x \in X}$) for every formula φ, and extend the language further by adding function symbols for the functions h_φ.

Next we find a set of indiscernibles $I \subseteq \kappa$, of size κ, for the expanded model $\mathfrak{A}'$; if κ is measurable and D is a normal measure, we find $I \in D$. We let B be the elementary submodel of $\mathfrak{A}'$ generated by I. As in the proof of Theorem 71, one proves that if $|P \cap B| < \kappa$ then $|P \cap B| \leq \lambda$. ■

Let us consider now models of a countable language $\mathcal{L}$ with a designated one-place predicate Q. A model $\mathfrak{A} = \langle A, Q^{\mathfrak{A}}, \ldots \rangle$ of $\mathcal{L}$ is said to have *type* (κ, λ) if $|A| = \kappa$ and $|Q^{\mathfrak{A}}| = \lambda$. One of the favorite subjects of model theorists is investigation of possible types of models of a given theory—so-called two-cardinal problems. Many of these problems are either independent of ZFC or lead to large cardinals.

Let us call a cardinal $\kappa > \aleph_1$ a *Rowbottom cardinal* if for every uncountable $\lambda < \kappa$, every model of type (κ, λ) has an elementary submodel of type $(\kappa, \aleph_0)$.

By Lemma 29.10, every Ramsey cardinal is Rowbottom. Using the argument from Exercise 29.9, one can see that if there is a Rowbottom cardinal, then $\aleph_1$ is inaccessible in L.

Let us call an infinite cardinal κ a *Jónsson cardinal* if every model $\mathfrak{A}$ (of a countable language) of size κ has a proper elementary submodel $\mathfrak{B}$ of size κ (i.e., $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B} \neq \mathfrak{A}$). Clearly every Rowbottom cardinal is Jónsson.

Both these properties can be formulated in terms of partitions. For instance:

Exercise 29.11.* An infinite cardinal κ is a Jónsson cardinal if and only if for every $F\colon [\kappa]^{<\omega} \to \kappa$ there exists $H \subseteq \kappa$ of size κ such that the image of $[H]^{<\omega}$ under F is not the whole set κ.

[To show that the condition is necessary, consider the model $\langle \kappa, <, F_1, F_2, \ldots \rangle$ where $F_n = F \mid [\kappa]^n$. To show that the condition is sufficient, let $\mathfrak{A} = \langle \kappa, \ldots \rangle$ be a model. Let $\{h_n : n < \omega\}$ be a set of Skolem functions for $\mathfrak{A}$, closed under composition and arranged so that each h_n is n-ary. For each $x \in [\kappa]^n$, let $F(x) = h_n(x)$. If $H \subseteq \kappa$, the image of $[H]^{<\omega}$ under F is an elementary submodel of $\mathfrak{A}$.]

We shall see in Section 32 that existence of Jónsson cardinals has the same strong consequences we stated in Theorem 71; in particular, there are no Jónsson cardinals if $V = L$.

Exercise 29.12. $\aleph_0$ is not a Jónsson cardinal.

[Let $\mathfrak{A} = \langle \omega, f \rangle$ where $f(n) = n - 1$ for all $n > 0$.]

It has been proved among other things that no $\aleph_n$, $n < \omega$, is a Jónsson cardinal and that the least Jónsson cardinal is either weakly inaccessible or has cofinality ω. While it is consistent (relative to ZFC + there exists a measurable cardinal) that all Jónsson cardinals are Ramsey cardinals, there is also a model in which a Rowbottom (and hence Jónsson) cardinal exists and has cofinality ω. One of the still open questions is whether $\aleph_\omega$ can be Jónsson.

Somewhat more is known about Rowbottom cardinals:

Exercise 29.13. If κ is a Rowbottom cardinal, then either κ is weakly inaccessible or $\operatorname{cf} \kappa = \omega$.

[To show that $\kappa = \lambda^+$ is not Rowbottom, let f_α be a one-to-one mapping of α onto λ, for each α, such that $\lambda \le \alpha < \kappa$. Let $\mathfrak{A} = \langle \kappa, \lambda, <, R \rangle$ where $R(\alpha, \beta, \gamma)$ iff $f_\alpha(\beta) = \gamma$. If $\langle B, B \cap \lambda, <, R \cap B^3 \rangle \prec A$ and $|B| = \kappa$, let α be the λth element of B; then $f_\alpha[B \cap \alpha] \subseteq B \cap \lambda$ and hence $|B \cap \lambda| = \lambda > \aleph_0$.

To show that κ is not Rowbottom if $\kappa > \operatorname{cf} \kappa = \lambda > \aleph_0$, let f be a nondecreasing function of κ onto λ and use f to produce a counterexample.]

A related problem is *Chang's conjecture*: Every model of type $(\aleph_2, \aleph_1)$ has an elementary submodel of type $(\aleph_1, \aleph_0)$. Chang's conjecture is consistent relative to the theory ZFC + "there exists a Ramsey cardinal." We shall show in Section 32 that Chang's conjecture has the strong consequences stated in Theorem 71.

For the following exercise, we recall that functions f and g on ω_1 are *almost disjoint* if $f(\alpha) \ne g(\alpha)$ for all $\alpha \ge$ some α_0.

*Exercise 29.14.** If there is a family $\mathscr{F}$ of $\aleph_2$ almost disjoint functions $f\colon \omega_1 \to \omega$ then Chang's conjecture fails.

[Consider a model $\mathfrak{A}$ with the universe $\mathscr{F} \cup \omega_1$ and the designated predicate ω_1. If $\langle \mathscr{G} \cup B, B, \ldots \rangle \prec \mathfrak{A}$ and $|\mathscr{G}| = \aleph_1$, $|B| = \aleph_0$, then $B \subseteq \alpha$ for some $\alpha < \omega_1$. Show that $f(\alpha) \ne g(\alpha)$ for all $f, g \in \mathscr{G}$, a contradiction.]

Trees

We dealt with trees of height ω_1 quite extensively in Sections 22 and 24. We shall now look at trees of larger height. In Lemma 29.6 we proved that for an inaccessible cardinal κ, nonexistence of an Aronszajn κ-tree is equivalent to weak compactness. There is a close relationship between partitions and trees; the following equivalence can be extracted from the proof of Lemma 29.6: Let us call a tree T *binary* if each $x \in T$ has at most two immediate successors.

*Exercise 29.15.** Let κ be a cardinal and λ a regular cardinal. Then $\kappa \to (\lambda)^2$ holds if and only if every binary tree of size κ has a branch of length λ.

We have called a tree of height κ an *Aronszajn tree* if all its levels have size $< \kappa$ and if it has no branch of length κ; κ has the *tree property* if there is no Aronszajn tree of height κ. The tree property of cardinals has been studied in some detail; let me consider the case $\kappa = \aleph_2$. This case is representative of the general case when κ is the successor of a regular cardinal.

In Lemma 22.3 I presented a construction of an Aronszajn tree of height ω_1. This construction can be generalized to $\aleph_2$ provided $2^{\aleph_0} = \aleph_1$ (or any $\kappa = \lambda^+$ such that $\lambda^{<\lambda} = \lambda$). Instead of the set Q of all rational numbers, we consider a linearly ordered set Q of size $\aleph_1$ such that every $\alpha < \omega_2$ can be embedded in any interval of Q (e.g., let Q be the lexicographically ordered set of all finite sequences of countable ordinals).

We construct a tree T whose elements are certain bounded increasing sequences in Q of length $< \omega_2$. We proceed very much as in Lemma 22.3: We try to obey the same condition (22.4). The difference is at limit steps of cofinality ω; if $\operatorname{cf} \alpha = \omega$, then we simply extend *all* α-branches represented by bounded α-sequences in Q. By the assumption $2^{\aleph_0} = \aleph_1$ their number is only $\aleph_1$, and so the αth level has still size $\aleph_1$. At limit steps of cofinality ω_1 we do more or less the same as in Lemma 23.3. We extend only some branches so that the size of the limit level is $\aleph_1$ and that the condition (22.4) is satisfied.

The Aronszajn ω_2-tree constructed thus under the assumption $2^{\aleph_0} = \aleph_1$ is *special* (see Section 24): Its nodes are one-to-one functions from some $\alpha < \omega_2$ into a fixed set of size $\aleph_1$, and the ordering of the tree is by inclusion. Special Aronszajn trees are of interest to model theorists: There is a sentence that has a model of type (κ^+, κ) just in case there exists a special Aronszajn κ^+-tree.

There is a model of set theory in which $2^{\aleph_0} = \aleph_2$ and no special Aronszajn ω_2-trees exist. This consistency result therefore establishes consistency of the statement: "There exists a model of type $(\aleph_1, \aleph_0)$ that has no elementarily equivalent model of type $(\aleph_2, \aleph_1)$" (this statement is known to be false if the continuum hypothesis holds).

The above-mentioned consistency result is relative to the consistency of "there exists a Mahlo cardinal." This assumption is necessary since it has also been proved that "there is no special Aronszajn ω_2-tree" implies that $\aleph_2$ is a Mahlo cardinal in L. (It is easy to see that $\aleph_2$ is inaccessible in L: If $\aleph_2$ is a successor cardinal in L, then there exists $A \subseteq \omega_1$, such that $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$—see Exercise 15.8. In $L[A]$, $2^{\aleph_0} = \aleph_1$ holds and hence there exists a special Aronszajn ω_2-tree T. It is obvious that T is a special Aronszajn ω_2-tree in the universe.)

Similarly, if $\aleph_2$ has the tree property (that is, if there are no Aronszajn trees of height ω_2, special or otherwise), then $\aleph_2$ is weakly compact in L; and the statement "$\aleph_2$ has the tree property and $2^{\aleph_0} = \aleph_2$" is consistent relative to the existence of weakly compact cardinals.

Let me finally consider a generalization of Suslin trees. If κ is a regular cardinal, we call a *Suslin κ-tree* a tree whose height is κ and that has no branches of length κ and no antichains of size κ.

If $\mathfrak{M}$ is a model of ZFC + GCH and if $\kappa = \lambda^+$ is a successor of a regular cardinal (in $\mathfrak{M}$), then a straightforward generalization of the forcing notion (22.6) yields a generic extension $\mathfrak{M}[G]$ in which there is a Suslin κ-tree: The forcing conditions are normal α-trees, $\alpha < \kappa$, whose nodes are functions $t: \beta \to \lambda$. The generalization consists in adding one more requirement on the trees $T \in P$: If $\alpha < \text{height}(T)$ is a limit ordinal of cofinality $< \kappa$, then $b: \alpha \to \lambda$ that represents a branch in the tree $T_\alpha = \{t \in T : \text{dom}(t) < \alpha\}$ belongs to the αth level of T. Since $\lambda^{<\lambda} = \lambda$, it is possible to satisfy this requirement and still have $|T| \leq \lambda$. The proof that $\mathfrak{M}[G]$ has a Suslin κ-tree follows closely the case $\kappa = \aleph_1$.

The construction of a Suslin tree in L also generalizes to higher cardinals. As before, let me concentrate on the simplest generalization, the case $\kappa = \aleph_2$.

Let us consider the analog of the diamond principle (22.15) for $\aleph_2$; since we will distinguish between ordinals of cofinality ω and cofinality ω_1, let me be a little more general. Let E be a stationary subset of ω_2.

(29.18) $\diamondsuit(E)$ There exists a collection of sets $\{S_\alpha : \alpha \in E\}$ such that for every $X \subseteq \omega_2$, the set

$$\{\alpha \in E : X \cap \alpha = S_\alpha\}$$

is a stationary subset of ω_2.

The principle $\diamondsuit(E)$ is proved in L exactly as Lemma 22.9. The easiest construction of a Suslin ω_2-tree in L follows closely the proof of Lemma 22.8. We let $E = \{\alpha : \text{cf}\,\alpha = \omega_1\}$. We construct a tree by induction on levels. We do different things at limit levels of cofinality ω and at limit levels of cofinality ω_1. If $\text{cf}\,\alpha = \omega$, then we construct the αth level by extending *all* branches in T_α; since $2^{\aleph_0} = \aleph_1$, the αth level has still size $\aleph_1$. If $\text{cf}\,\alpha = \omega_1$, then we use $\diamondsuit(E)$ to destroy potential antichains of size $\aleph_2$. Note that since all branches are extended at cofinality ω levels, the tree T_α for $\text{cf}\,\alpha = \omega_1$ has a plenty of α-branches: Each $x \in T_\alpha$ has an α-branch going through x. The proof that the resulting tree is a Suslin ω_2-tree follows closely Lemma 22.8.

There is another, more sophisticated construction of a Suslin ω_2-tree in L (which, unlike the construction above, generalizes to successors of singular cardinals). Let me recall the box principle $\square$ (cf. 24.18) which is true in L. The construction uses the box principle and the principle $\diamondsuit(E)$ for every stationary set of ordinals of cofinality ω. Roughly speaking, the $\diamondsuit$ principle is used to destroy antichains, while $\square$ enables us to keep going at limit levels of cofinality ω_1. In this construction, not all branches are extended at cofinality ω levels.

Now a modification of $\diamondsuit(E)$ for $E = \{\alpha : \text{cf}\,\alpha = \omega\}$ has been proved from GCH alone. Since it is also known that $\square$ is true unless $\aleph_2$ is a Mahlo cardinal in L, it follows that if GCH holds and $\aleph_2$ is not Mahlo in L, then there exists a Suslin ω_2-tree.

There is a model of ZFC in which $2^{\aleph_0} = \aleph_1$ and there are no Suslin ω_2-trees.

The construction uses a measurable cardinal, and the model satisfies $2^{\aleph_1} = \aleph_3$. It is still open whether nonexistence of Suslin ω_2-trees is consistent with the GCH.

30. SILVER INDISCERNIBLES

We have shown in the previous two sections that existence of measurable cardinals contradicts the axiom of constructibility, and in fact the incompatibility of measurable cardinals with $V = L$ is quite substantial: If there is a measurable cardinal, then there are only countable many constructible reals and $\aleph_1$ is inaccessible in L. We shall now use more model theoretical methods to obtain quite striking consequences of existence of measurable cardinals.

Before stating the main theorem of this section, let me define (or recall) the notions involved. Let $\mathfrak{A} = \langle A, \ldots \rangle$ be a model (of a language $\mathscr{L}$) and let $X \subseteq A$. An element $a \in A$ is *definable from* X if there exists a formula $\varphi(u, v_1, \ldots, v_n)$ of $\mathscr{L}$, and some $x_1, \ldots, x_n \in X$ such that

(30.1) $\mathfrak{A} \models a$ is the unique u such that $\varphi(u, x_1, \ldots, x_n)$.

Let λ be a limit ordinal and let $I \subset \lambda$ be a set of ordinals in L_λ. The set I is a *set of indiscernibles* for L_λ if for every formula $\varphi(v_1, \ldots, v_n)$ of the language of set theory

$$(30.2) \qquad L_\lambda \models \varphi[\alpha_1, \ldots, \alpha_n] \quad \text{iff} \quad L_\lambda \models \varphi[\beta_1, \ldots, \beta_n]$$

whenever $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are two increasing sequences in I.

Theorem 72. *If there exists a measurable cardinal, then:*

(A) *If κ and λ are uncountable cardinals and $\kappa < \lambda$, then $\langle L_\kappa, \in \rangle$ is an elementary submodel of $\langle L_\lambda, \in \rangle$.*

(B) *There is a unique class* S *of ordinals containing all uncountable cardinals such that for each uncountable cardinal κ:*

  (a) S $\cap$ κ *has order type κ, and if κ is regular, then* S $\cap$ κ *is closed and unbounded in κ;*

  (b) S $\cap$ κ *is a set of indiscernibles for $\langle L_\kappa, \in \rangle$;*

  (c) *every $a \in L_\kappa$ is definable in L_κ from* S $\cap$ κ.

Elements of the class S are the *Silver indiscernibles* for L. The proof of Theorem 72 will constitute about half of this section. Thus I shall first state some consequences of the conclusion of Theorem 72.

By the reflection principle, if φ is a formula, then there exists an uncountable cardinal κ such that

$$L \models \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad L_\kappa \models \varphi(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in L_\kappa$. By (A), we have then

$$L \models \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad L_\lambda \models \varphi(x_1, \ldots, x_n)$$

for all cardinals $\lambda \geq \kappa$ and all $x_1, \ldots, x_n$. In view of this, we can define satisfaction in L for all formulas $\varphi \in Form$: If $\varphi(v_1, \ldots, v_n)$ is a formula of the language $L = \{\in\}$ and if $\langle a_1, \ldots, a_n \rangle$ is an n-termed sequence in L, we define

$$L \vDash \varphi[a_1, \ldots, a_n] \tag{30.3}$$

as follows: For every uncountable cardinal κ such that $a_1, \ldots, a_n \in L_\kappa$, $L_\kappa \vDash \varphi[a_1, \ldots, a_n]$.

Note that this gives us a truth definition for the constructible universe (see 14.17): For each sentence σ, let $\#\sigma$ be its Gödel number and let

$$T = \{\#\sigma : L_{\aleph_1} \vDash \sigma\}$$

Then given any actual (metamathematical) sentence σ of set theory, we have

$$\#\sigma \in T \quad \text{iff} \quad L \vDash \sigma \tag{30.4}$$

(Note that the set T is constructible but not definable in L: Otherwise, T would be a truth definition in L. Hence the (real) cardinal $\aleph_1$ is not definable in L.)

As a consequence of (A) we have

$$\langle L_\kappa, \in\rangle \prec \langle L, \in\rangle \tag{30.5}$$

for every uncountable cardinal κ. (We can unabashedly write (30.5) since we have defined satisfaction for L.)

Corollary 1. *Every constructible set definable in L is countable.*

Proof. If $x \in L$ is definable in L by a formula φ, then by (30.5) the same formula defines x in $L_{\aleph_1}$ and hence $x \in L_{\aleph_1}$. ■

In particular, every ordinal number definable in L is countable.

It follows from (B) and (30.5) that the Silver indiscernibles are indeed indiscernibles for L: If $\varphi(v_1, \ldots, v_n)$ is a formula, then

$$L \vDash \varphi[\alpha_1, \ldots, \alpha_n] \quad \text{iff} \quad L \vDash \varphi[\beta_1, \ldots, \beta_n] \tag{30.6}$$

whenever $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are increasing sequences in S. Moreover, every constructible set is definable from S in the following sense: If $a \in L$, there exists an increasing sequence $\langle \gamma_1, \ldots, \gamma_n \rangle$ of Silver indiscernibles and a formula φ such that

$$L \vDash a \text{ is the unique } x \text{ such that } \varphi(x, \gamma_1, \ldots, \gamma_n)$$

Corollary 2. *Every uncountable cardinal is inaccessible in L.*

Proof. (For each α, $\aleph_\alpha$ denotes the real $\aleph_\alpha$, not $\aleph_\alpha^L$.) Since $L \vDash \aleph_1$ is regular, we have

$$L \vDash \aleph_\alpha \text{ is regular}$$

for every $\alpha \geq 1$. Similarly, $L \vDash \aleph_\omega$ is a limit cardinal, and hence

$$L \vDash \aleph_\alpha \text{ is a limit cardinal}$$

for every $\alpha \geq 1$. Thus every uncountable cardinal (and in fact every $\gamma \in \mathrm{S}$) is an inaccessible cardinal in L. ■

Corollary 3. *Every uncountable cardinal is a Mahlo cardinal in L.*

Proof. By Corollary 2, every Silver indiscernible is an inaccessible cardinal in L. Since $\mathrm{S} \cap \omega_1$ is closed unbounded in ω_1, $\aleph_1$ is a Mahlo cardinal in L.

Corollary 4. *For every* $\alpha \geq \omega$, $|V_\alpha \cap L| \leq |\alpha|$. *In particular, the set of all constructible reals is countable.*

Proof. The set $V_\alpha \cap L$ is definable in L from α. By (30.5), $V_\alpha \cap L$ is also definable from α in L_κ where κ is the least cardinal $> \alpha$. Hence $V_\alpha \cap L \subset L_\beta$ for some β such that $|\alpha| = |\beta|$. However, $|L_\beta| = |\beta|$. ■

By (30.6), every formula $\varphi(v_1, \ldots, v_n)$ is either true or false in L for any increasing sequence $\langle \gamma_1, \ldots, \gamma_n \rangle$ of Silver indiscernibles; moreover, the truth value coincides with the truth value of

$$L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n] \tag{30.7}$$

since $L_{\aleph_\omega} \prec L$ and $\aleph_1, \ldots, \aleph_n$ are Silver indiscernibles. Thus let us define

$$0^\# = \{\varphi : L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n]\} \tag{30.8}$$

(*zero-sharp*). Later in this section we shall give another definition of the set $0^\#$, not depending on the conclusion of Theorem 72. We shall show that the (new) definition of $0^\#$ is absolute for transitive models of set theory containing all ordinals and that a set $0^\#$ satisfying the definition exists if and only if the conclusion of Theorem 72 holds (and then $0^\#$ is as in 30.8). With this equivalence in mind, I shall henceforth use the statement

"$0^\#$ exists"

to denote the conclusion of Theorem 72; i.e., (A) and (B).

The axiom "$0^\#$ exists" admits a considerably simpler formulation. We shall prove:

Theorem 73 (Kunen). *The following are equivalent:*

(i) $0^\#$ *exists.*
(ii) *For some limit ordinal* λ, *the model* $\langle L_\lambda, \in \rangle$ *has an uncountable set of indiscernibles.*
(iii) *There exists a nontrivial elementary embedding of L into L.*

It will also follow from the proof of Theorem 72 that the conclusion "$0^\#$ exists" follows not only from existence of a measurable cardinal but in fact from weaker assumptions, like "there exists a Ramsey cardinal."

Review of L

Constructible sets were introduced in Section 12, and we proved in Sections 12 and 13 that constructibility and the corresponding well-ordering $<_L$ are absolute for models of sufficiently many axioms of ZF. In particular, we have defined the function $\alpha \mapsto L_\alpha$ (in 12.2) and the canonical well-ordering of L (Section 13) and showed that both are absolute for all adequate transitive sets.

We shall need a somewhat finer property of constructibility, namely that both the function $\alpha \mapsto L_\alpha$ and the well-ordering $<_L$ are absolute for every L_λ, where λ is a limit ordinal. This does not necessarily follow from our definition of L_α in (12.2) and so we have to amend slightly the defining formula as to obtain this absoluteness result. (However, the sets L_α remain the same.)

First we weaken the definition (13.9) of adequacy: Let us call a transitive set M *adequate* if (a) M is closed, (b) if $U \in M$, then $\mathfrak{G}(U) \in M$ where

$$\mathfrak{G}(U) = \{\mathfrak{F}_i(x, y) : x, y \in U, i = 1, \ldots, 10\}$$

and (c) if $\alpha \in M$, then $\langle L_\beta : \beta < \alpha \rangle \in M$.

Then we prove the analog of Lemma 13.2: We show that the function $\alpha \mapsto L_\alpha$ is absolute for every adequate transitive M. The Gödel operations, as well as the operation $\mathfrak{G}$ are absolute and defined in M. Neither $\mathrm{cl}(X)$ nor $\mathrm{def}(X)$ is necessarily defined in M, but since the function $W(U, n)$ defined in (13.13) is absolute, we easily see that the formula $x \in \mathrm{def}(U)$ is absolute:

$$x \in \mathrm{def}(U) \quad \leftrightarrow \quad x \subseteq U \quad \text{and} \quad \exists n\, x \in W(U \cup \{U\}, n)$$

And hence $\mathrm{def}^M(U) = \mathrm{def}(U)$ for each $U \in M$.

Now, the standard definition of L_α by induction (13.14) is absolute for M, because for each α the function f in (13.14) is unique, $f = \langle L_\beta : \beta < \alpha \rangle$, and $\mathrm{def}(f(\xi)) = L_{\xi+1} \in M$ for each $\xi < \alpha$.

The second part of Lemma 13.2 also goes through: If M is an adequate transitive set and satisfies the axiom of constructibility, then $M = L_\lambda$ for some limit ordinal λ.

The definition (13.7)–(13.8) of the canonical well-ordering $<_L$ is easily seen to be absolute for adequate transitive sets.

Next we show that each L_λ where λ is a limit, is adequate. If $x, y \in L_\alpha$, then for each $i = 1, \ldots, 10$, $\mathfrak{F}_i(x, y) \subseteq L_{\alpha+2}$ and hence $\in L_{\alpha+3}$. Thus L_λ is closed. If $U \in L_\alpha$, then $G(U) \subseteq L_{\alpha+3}$ and is definable over $L_{\alpha+3}$, hence $\in L_{\alpha+4}$. Thus if $U \in L_\lambda$ then $G(U) \in L_\lambda$. Finally, we prove by induction on λ that if $\alpha \in L_\lambda$, then $\langle L_\beta : \beta < \alpha \rangle \in L_\lambda$. If $\alpha < \lambda$ is a limit ordinal, then L_α is adequate by the induction hypothesis and so $\langle L_\beta : \beta < \alpha \rangle$ is definable in L_α; hence $\langle L_\beta : \beta < \alpha \rangle \in L_{\alpha+1} \subseteq L_\lambda$. And clearly, if $\langle L_\beta : \beta < \alpha \rangle \in L_\lambda$ then $\langle L_\beta : \beta < \alpha + 1 \rangle \in L_\lambda$.

Finally, as in Lemma 13.3, adequacy is a syntactical property of a transitive set: There is a sentence Θ such that a transitive set M is adequate if and only if $M \vDash \Theta$. Thus we have:

Lemma 30.1.

(i) *The function* $\alpha \mapsto L_\alpha$ *and the canonical well ordering of* L *are absolute for every* L_λ, *where* λ *is a limit ordinal.*

(ii) *If* M *is transitive and* $\langle M, \in \rangle$ *is elementarily equivalent to some* L_λ, λ *a limit, then* $M = L_\alpha$ *for some* α. *In particular, if* $M \prec \langle L_\lambda, \in \rangle$ *where* λ *is a limit, then* M *is isomorphic to some* L_α, $\alpha \leq \lambda$. ■

We shall now use the canonical well-ordering of L to endow the models L_λ with definable Skolem functions. For each formula $\varphi(u, v_1, \ldots, v_n)$, let h_φ be the n-ary function defined as follows:

$$h_\varphi(v_1, \ldots, v_n) = \begin{cases} \text{the } <_L\text{-least } u \text{ such that } \varphi(u, v_1, \ldots, v_n) \\ \varnothing \text{ otherwise} \end{cases} \tag{30.9}$$

We call h_φ, $\varphi \in Form$, the *canonical Skolem functions.*

For each limit ordinal λ, $h_\varphi^{L_\lambda}$ is an n-ary function on L_λ, the L_λ-interpretation of h_φ, and is definable in $\langle L_\lambda, \in \rangle$.

When dealing with models $\langle L_\lambda, \in \rangle$, we shall freely use terms and formulas involving the h_φ since they as definable functions can be eliminated and the formulas can be replaced by $\in$-formulas. For each limit ordinal λ, the functions $h_\varphi^{L_\lambda}$, $\varphi \in Form$, are Skolem functions for $\langle L_\lambda, \in \rangle$ and so a set $M \subseteq L_\lambda$ is an elementary submodel of $\langle L_\lambda, \in \rangle$ if and only if M is closed under the $h_\varphi^{L_\lambda}$. If $X \subseteq L_\lambda$, then the closure of X under the $h_\varphi^{L_\lambda}$ is the smallest elementary submodel $M \prec L_\lambda$ such that $X \subseteq M$, and is the collection of all elements of L_λ definable in L_λ from X.

The fact that the well-ordering $<_\lambda$ of L_λ is definable in L_λ uniformly for all limit ordinals λ (by the same formula) now brings the following benefit:

Lemma 30.2. *If* α *and* β *are limit ordinals and if* $j\colon L_\alpha \to L_\beta$ *is an elementary embedding of* $\langle L_\alpha, \in \rangle$ *in* $\langle L_\beta, \in \rangle$, *then for each formula* φ *and all* $x_1, \ldots, x_n \in L_\alpha$,

$$h_\varphi^{L_\beta}(j(x_1), \ldots, j(x_n)) = j(h_\varphi^{L_\alpha}(x_1, \ldots, x_n)) \tag{30.10}$$

Hence j *remains elementary with respect to the enriched language* $\mathscr{L}^* = \{\in\} \cup \{h_\varphi : \varphi \in Form\}$.

Proof. The mapping j is elementary with respect to formulas involving $<_L$. ■

Models with Indiscernibles

The method of this section is based on a theorem of Ehrenfeucht and Mostowski in model theory, stating that every infinite model is elementarily equivalent to a model that has a set of indiscernibles of prescribed order-type. We shall deal only with models $\langle L_\lambda, \in \rangle$ (and models elementarily equivalent to these); Lemma 30.3 is a special case of the Ehrenfeucht–Mostowski theorem.

Let λ be a limit ordinal, and let $\mathfrak{A} = \langle A, E\rangle$ be a model elementarily equivalent to $\langle L_\lambda, \in\rangle$. The set $Ord^{\mathfrak{A}}$ of all ordinal numbers of the model $\mathfrak{A}$ is linearly ordered by E; let us use $x < y$ rather than $x\ E\ y$ for $x, y \in Ord^{\mathfrak{A}}$. A set $I \subseteq Ord^{\mathfrak{A}}$ is a *set of indiscernibles* for $\mathfrak{A}$ if for every formula φ,

$$(30.11) \qquad \mathfrak{A} \vDash \varphi[x_1, \ldots, x_n] \quad \text{iff} \quad \mathfrak{A} \vDash \varphi[y_1, \ldots, y_n]$$

whenever $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ are elements of I. Let $h_\varphi^{\mathfrak{A}}$ denote the $\mathfrak{A}$-interpretation of the canonical Skolem functions (30.9). Given a set $X \subseteq A$, let us denote $\mathscr{H}^{\mathfrak{A}}(X)$ the closure of X under all $h_\varphi^{\mathfrak{A}}$, $\varphi \in Form$. The set $\mathscr{H}^{\mathfrak{A}}(X)$ is the *Skolem hull* of X and is an elementary submodel of $\mathfrak{A}$.

If I is a set of indiscernibles for $\mathfrak{A}$, let $\Sigma(\mathfrak{A}, I)$ be the set of all formulas $\varphi(v_1, \ldots, v_n)$ true in $\mathfrak{A}$ for increasing sequences of elements of I:

$$(30.12) \quad \varphi(v_1, \ldots, v_n) \in \Sigma(\mathfrak{A}, I) \leftrightarrow \mathfrak{A} \vDash \varphi[x_1, \ldots, x_n] \text{ for some } x_1, \ldots, x_n \in I \text{ such that } x_1 < \cdots < x_n$$

A set of formulas Σ is called an *E.M. set* (Ehrenfeucht–Mostowski) if there exists a model $\mathfrak{A}$ elementarily equivalent to some L_λ, λ a limit ordinal, and an infinite set I of indiscernibles for $\mathfrak{A}$ such that $\Sigma = \Sigma(\mathfrak{A}, I)$.

Lemma 30.3. *If Σ is an E.M. set and α an infinite ordinal number, then there exists a model $\mathfrak{A}$ and a set of indiscernibles I for $\mathfrak{A}$ such that:*

(a) $\Sigma = \Sigma(\mathfrak{A}, I)$;
(b) *the order-type of I is α;*
(c) $\mathfrak{A} = \mathscr{H}^{\mathfrak{A}}(I)$.

Moreover, the pair $(\mathfrak{A}, I)$ is unique up to isomorphism.

Proof. We prove uniqueness first. Let $(\mathfrak{A}, I)$ and $(\mathfrak{B}, J)$ be two pairs, each satisfying (a), (b), (c). Since both I and J have order-type α, let π be the isomorphism between I and J. We shall extend π to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Since $\mathfrak{A}$ is the Skolem hull of I, there is for each $a \in \mathfrak{A}$ a Skolem term $t(v_1, \ldots, v_n)$ (a combination of the Skolem functions h_φ) such that $a = t^{\mathfrak{A}}[x_1, \ldots, x_n]$ for some $x_1 < \cdots < x_n$ in I; similarly for $\mathfrak{B}, J$. Thus we define

$$(30.13) \qquad \pi(t^{\mathfrak{A}}[x_1, \ldots, x_n]) = t^{\mathfrak{B}}[\pi(x_1), \ldots, \pi(x_n)]$$

for each Skolem term t and all $x_1, \ldots, x_n \in I$ such that $x_1 < \cdots < x_n$. Since $\Sigma(\mathfrak{A}, I) = \Sigma(\mathfrak{B}, J)$, we have

(30.14)

$$t_1^{\mathfrak{A}}[x_1, \ldots, x_n] = t_2^{\mathfrak{A}}[y_1, \ldots, y_n] \quad \text{iff} \quad t_1^{\mathfrak{B}}[\pi x_1, \ldots, \pi x_n] = t_2^{\mathfrak{B}}[\pi y_1, \ldots, \pi y_n]$$

$$t_1^{\mathfrak{A}}[x_1, \ldots, x_n]\ E^{\mathfrak{A}}\ t_2^{\mathfrak{A}}[y_1, \ldots, y_n] \quad \text{iff} \quad t_1^{\mathfrak{B}}[\pi x_1, \ldots, \pi x_n]\ E^{\mathfrak{B}}\ t_2^{\mathfrak{B}}[\pi y_1, \ldots, \pi y_n]$$

for any terms t_1, t_2 and indiscernibles x, y: Let $z_1, \ldots, z_{n+m}$ be the enumeration of the set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ in increasing order. Then the equality in (30.14) holds (simultaneously in $\mathfrak{A}$ and $\mathfrak{B}$) just in case $\varphi(v_1, \ldots, v_{n+m}) \in \Sigma$ where $\varphi(z_1, \ldots, z_{n+m})$ is the formula that says that $t_1[x_1, \ldots, x_n] = t_2[y_1, \ldots, y_n]$. Hence π is well-defined by (30.13) and is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ extending the order-isomorphism of I and J.

To prove existence of a model with indiscernibles with properties (a), (b), (c), we use the compactness theorem. Since Σ is an E.M. set, there exists $(\mathfrak{A}_0, I_0)$ such that $\Sigma = \Sigma(\mathfrak{A}_0, I_0)$. Let us extend the language $\{\in\}$ by adding α constant symbols c_ξ, $\xi < \alpha$. Let Δ be the following set of sentences:

$$
\begin{array}{lll}
(30.15) & c_\xi \text{ is an ordinal} & (\text{all } \xi < \alpha) \\
& c_\xi < c_\eta & (\text{all } \xi, \eta \text{ such that } \xi < \eta < \alpha) \\
& \varphi(c_{\xi_1}, \ldots, c_{\xi_n}) & (\text{all } \varphi \in \Sigma \text{ and all } \xi_1 < \cdots < \xi_n < \alpha)
\end{array}
$$

We shall show that every finite subset of Δ has a model. Let $D \subset \Delta$ be finite. There exist $\xi_1 < \cdots < \xi_k$ such that $c_{\xi_1}, \ldots, c_{\xi_k}$ are the only constants mentioned in D. Let $\sigma(c_{\xi_1}, \ldots, c_{\xi_k})$ be the sentence that is the conjunction of all sentences in D.

Since I_0 is infinite, there are $i_1, \ldots, i_k \in I_0$ such that $i_1 < \cdots < i_k$. Let us take the model $\mathfrak{A}_0$ and expand it by interpreting the constant symbols $c_{\xi_1}, \ldots, c_{\xi_k}$ as $i_1, \ldots, i_k$. Since $\Sigma = \Sigma(\mathfrak{A}_0, I_0)$ and $D \subset \Delta$, it is clear that $\mathfrak{A}_0 \vDash \sigma[i_1, \ldots, i_k]$ and hence the expansion $\langle \mathfrak{A}_0, i_1, \ldots, i_k \rangle$ is a model of σ, hence of D.

By the compactness theorem, the set Δ has a model $\mathfrak{M} = \langle M, E, c_\xi^{\mathfrak{M}} \rangle_{\xi<\alpha}$. Let $I = \{c_\xi^{\mathfrak{M}} : \xi < \alpha\}$. I is a set of ordinals of $\mathfrak{M}$ and has order-type α. It is clear that if $\varphi(v_1, \ldots, v_n)$ is an $\in$-formula and $\xi_1 < \cdots < \xi_n$, then $\langle M, E \rangle \vDash \varphi[c_{\xi_1}^{\mathfrak{M}}, \ldots, c_{\xi_n}^{\mathfrak{M}}]$ if and only if $\varphi \in \Sigma$. Thus I is a set of indiscernibles for $\langle M, E \rangle$. Now we let A be the Skolem hull of I in $\langle M, E \rangle$. Since $\mathfrak{A} = \langle A, E \rangle$ is an elementary submodel of $\langle M, E \rangle$, it follows that I is a set of indiscernibles for $\mathfrak{A}$, $\Sigma(\mathfrak{A}, I) = \Sigma$, and that $\mathscr{H}^{\mathfrak{A}}(I) = \mathscr{H}^{\langle M,E \rangle}(I) = A$. Hence $(\mathfrak{A}, I)$ satisfies (a), (b), (c). ■

For each E.M. set Σ and each ordinal α, let us call the (Σ, α)-*model* the unique pair $(\mathfrak{A}, I)$ given by Lemma 30.3. The uniqueness proof of Lemma 30.3 easily extends to give the following:

Lemma 30.4. *Let Σ be an E.M. set, let $\alpha \leq \beta$, and let $j: \alpha \to \beta$ be order-preserving. Then j can be extended to an elementary embedding of the (Σ, α)-model into the (Σ, β)-model.*

Proof. Extend j as in (30.13). ■

We shall eventually show that a measurable cardinal implies existence of an E.M. set Σ having a certain syntactical property (*remarkability*) and such that every (Σ, α)-model is well-founded. Let us investigate well-foundedness first.

Lemma 30.5. *The following are equivalent, for any given E.M. set* Σ:

(i) *For every ordinal* α, *the* (Σ, α)*-model is well-founded.*
(ii) *For some ordinal* $\alpha \geq \omega_1$, *the* (Σ, α)*-model is well-founded.*
(iii) *For every ordinal* $\alpha < \omega_1$, *the* (Σ, α)*-model is well-founded.*

Proof. (i) → (ii) is trivial.

(ii) → (iii): If $(\mathfrak{A}, I)$ is the (Σ, α)-model and if $\beta \leq \alpha$, let J be the initial segment of the first β elements of I; let $\mathfrak{B} = \mathscr{H}^{\mathfrak{A}}(J)$. Clearly, $(\mathfrak{B}, J)$ is the (Σ, β)-model. Since a submodel of a well-founded model is well-founded, it follows that if $\beta \leq \alpha$ and the (Σ, α)-model is well-founded, then the (Σ, β)-model is also well-founded, and thus (ii) implies (iii).

(iii) → (i): Let us assume that there is a limit ordinal α such that the (Σ, α)-model is not well-founded; let $(\mathfrak{A}, I)$ be the model. There is an infinite sequence $a_0, a_1, a_2, \ldots$ in $\mathfrak{A}$ such that $a_1 \mathrel{E} a_0, a_2 \mathrel{E} a_1$, etc. Each a_n is definable from I; that is, for each n there is a Skolem term t_n such that $a_n = t_n^{\mathfrak{A}}[x_1, \ldots, x_{k_n}]$ for some $x_1, \ldots, x_{k_n} \in I$. Therefore there is a countable subset I_0 of I such that $a_n \in \mathscr{H}^{\mathfrak{A}}(I_0)$ for all $n \in \omega$. The order-type of I_0 is a countable ordinal β and $(\mathscr{H}^{\mathfrak{A}}(I_0), I_0)$ is the (Σ, β)-model. This model is clearly non-well-founded since it contains all the a_n. Hence for some countable β, the (Σ, β)-model is not well-founded. ■

I shall now define remarkability. To avoid unpleasant surprises in the middle of a proof, let me from now on consider only (Σ, α)-models where α is an infinite limit ordinal.

Let us say that a (Σ, α)-model $(\mathfrak{A}, I)$ is *unbounded* if the set I is unbounded in the ordinals of $\mathfrak{A}$, that is, if for every $x \in Ord^{\mathfrak{A}}$ there is $y \in I$ such that $x < y$.

Lemma 30.6. *The following are equivalent, for any given E.M. set* Σ:

(i) *For all* α, *the* (Σ, α)*-model is unbounded.*
(ii) *For some* α, *the* (Σ, α)*-model is unbounded.*
(iii) *For every Skolem term* $t(v_1, \ldots, v_n)$ *the set* Σ *contains the formula* $\varphi_t(v_1, \ldots, v_{n+1})$ *stating*

(30.16) *if* $t(v_1, \ldots, v_n)$ *is an ordinal, then* $t(v_1, \ldots, v_n) < v_{n+1}$.

Proof. (i) → (ii) is trivial.

(ii) → (iii): Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where α is a limit ordinal, and assume that I is unbounded in $Ord^{\mathfrak{A}}$. To prove (iii), it suffices to show that for any t, (30.16) is true in $\mathfrak{A}$ for some increasing sequence $x_1 < \cdots < x_{n+1}$ in I. Let t be a Skolem term. Let us choose $x_1 < \cdots < x_n \in I$ and let $y = t^{\mathfrak{A}}[x_1, \ldots, x_n]$. If $y \notin Ord^{\mathfrak{A}}$, then (30.16) is vacuously true; if $y \in Ord^{\mathfrak{A}}$, then there exists $x_{n+1} \in I$ such that $y < x_{n+1}$, and we have $\mathfrak{A} \vDash t[x_1, \ldots, x_n] < x_{n+1}$.

(iii) → (i): Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where α is a limit ordinal, and assume (iii). To prove that I is unbounded in $Ord^{\mathfrak{A}}$, let $y \in Ord^{\mathfrak{A}}$. There is a

Skolem term t and $x_1 < \cdots < x_n \in I$ such that $y = t^{\mathfrak{A}}[x_1, \ldots, x_n]$. Now if x_{n+1} is any element of I greater than x_n, (iii) implies that $y < x_{n+1}$. ■

Thus we say that Σ is *unbounded* if it contains the formulas (30.16) for all Skolem terms t.

Let α be a limit ordinal, $\alpha > \omega$, and let $(\mathfrak{A}, I)$ be the (Σ, α)-model. For each $\xi < \alpha$, let i_ξ denote the ξth element of I. We say that $(\mathfrak{A}, I)$ is *remarkable* if it is unbounded and if every ordinal x of $\mathfrak{A}$ less than i_ω is in $\mathscr{H}^{\mathfrak{A}}(\{i_n : n \in \omega\})$.

Lemma 30.7. *The following are equivalent for any given unbounded E.M. set* Σ:

(i) *For all* $\alpha > \omega$, *the* (Σ, α)-*model is remarkable.*

(ii) *For some* $\alpha > \omega$, *the* (Σ, α)-*model is remarkable.*

(iii) *For every Skolem term* $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$, *the set* Σ *contains the formula* $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_n)$ *stating*

(30.17) *if* $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ *is an ordinal and*

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) < y_1,$$

then

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) = t(x_1, \ldots, x_m, z_1, \ldots, z_n).$$

Moreover, if $(\mathfrak{A}, I)$ *is a remarkable* (Σ, α)-*model and* $\gamma < \alpha$ *is a limit ordinal, then every ordinal* x *of* $\mathfrak{A}$ *less than* i_γ *is in* $\mathscr{H}^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$.

Proof. (i) → (ii) is trivial.

(ii) → (iii): Let $\alpha > \omega$ be a limit ordinal and let $(\mathfrak{A}, I)$ be a remarkable (Σ, α)-model. To prove (iii), it suffices to show that for any t, (30.17) is true in $\mathfrak{A}$ for some increasing sequence $x_1 < \cdots < x_m < y_1 < \cdots < y_n < z_1 < \cdots < z_n$ in I. Let t be a Skolem term. We let $x_1 < \cdots < x_m < y_1 < \cdots < y_n < z_1 < \cdots < z_n \in I$ be such that $x_1, \ldots, x_m$ are the first m members of I and that y_1 is the ωth member of I, $y_1 = i_\omega$. Now if $a = t^{\mathfrak{A}}(x_1, \ldots, y_m)$ is an ordinal of $\mathfrak{A}$ and less than y_1, we have, by remarkability of $(\mathfrak{A}, I)$, $a \in \mathscr{H}^{\mathfrak{A}}(\{i_n : n < \omega\})$. Hence there is $k < \omega$, $k \geq m$, and a term s such that

$$\mathfrak{A} \models t[x_1, \ldots, x_m, y_1, \ldots, y_n] = s[i_0, \ldots, i_k] \tag{30.18}$$

In other words (30.18) says that $\mathfrak{A}$ satisfies a certain formula $\varphi[i_0, \ldots, i_k, y_1, \ldots, y_n]$. By indiscernibility, $\mathfrak{A}$ also satisfies $\varphi[i_0, \ldots, i_k, z_1, \ldots, z_n]$, i.e.,

$$\mathfrak{A} \models t[x_1, \ldots, x_m, z_1, \ldots, z_n] = s[i_0, \ldots, i_k] \tag{30.19}$$

Therefore $t^{\mathfrak{A}}[x_1, \ldots, x_m, y_1, \ldots, y_n] = t^{\mathfrak{A}}[x_1, \ldots, x_m, z_1, \ldots, z_n]$.

(iii) → (i) and "moreover": Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where $\alpha > \omega$ is a limit ordinal, and assume (iii). Let $\gamma \geq \omega$ be a limit ordinal and let $x \in Ord^{\mathfrak{A}}$ be less than i_γ, the γth element of I. We shall show that $x \in \mathscr{H}^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$. Since $\mathfrak{A} = \mathscr{H}^{\mathfrak{A}}(I)$, there is a Skolem term t and $x_1 < \cdots < x_m < y_1 < \cdots < y_n \in I$ such that $y_1 = i_\gamma$ and $x = t^{\mathfrak{A}}[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Let us choose $w_1, \ldots, w_n$

and $z_1, \ldots, z_n$ in I such that

$$x_1 < \cdots < x_m < w_1 < \cdots < w_n < y_1 < \cdots < y_n < z_1 < \cdots < z_n$$

Now since $x < y_1$, it follows from (30.17) that

$$\mathfrak{A} \models t[x_1, \ldots, x_m, y_1, \ldots, y_n] = t[x_1, \ldots, x_m, z_1, \ldots, z_n]$$

However, by indiscernibility, this implies that

$$\mathfrak{A} \models t[x_1, \ldots, x_m, w_1, \ldots, w_n] = t[x_1, \ldots, x_m, z_1, \ldots, z_n]$$

and hence $x = t^{\mathfrak{A}}[x_1, \ldots, x_m, w_1, \ldots, w_n]$. Therefore $x \in \mathscr{H}^{\mathfrak{A}}(i_\xi : \xi < \gamma)$. ■

Thus we say that Σ is *remarkable* if it is unbounded and contains the formulas (30.17) for all Skolem terms t.

The important consequence of remarkability is the following: Let $(\mathfrak{A}, I)$ be a remarkable (Σ, α)-model and let $\gamma < \alpha$ be a limit ordinal. Let $J = \{i_\xi : \xi < \gamma\}$ and let $\mathfrak{B} = \mathscr{H}^{\mathfrak{A}}(J)$. Then (B, J) is the (Σ, γ)-model, and the ordinals of B form an initial segment of the ordinals of $\mathfrak{A}$.

Another consequence of remarkability is that the indiscernibles form a closed unbounded subset of the ordinals. Let $(\mathfrak{A}, I)$ be the (Σ, α)-model. We say that the set I is *closed* in $Ord^{\mathfrak{A}}$ if for every limit $\gamma < \alpha$, i_γ is the least upper bound (in the linearly ordered set $Ord^{\mathfrak{A}}$) of the set $\{i_\xi : \xi < \gamma\}$.

Lemma 30.8. *If $(\mathfrak{A}, I)$ is remarkable, then I is closed in $Ord^{\mathfrak{A}}$.*

Proof. Let $\gamma < \alpha$ be a limit ordinal. If x is an ordinal of $\mathfrak{A}$ less than i_γ, then by remarkability, x is in the (Σ, γ)-model $\mathfrak{B} = \mathscr{H}(\{i_\xi : \xi < \gamma\})$. However, since Σ is unbounded, $\mathfrak{B}$ is an unbounded (Σ, γ)-model and hence $x < i_\xi$ for some $\xi < \gamma$. Hence i_γ is the least upper bound of $\{i_\xi : \xi < \gamma\}$. ■

Proof of Theorem 72

Let us call an E.M. set Σ *well-founded* if every (Σ, α)-model is well founded, and let us consider the statement

(30.20) There exists a well-founded remarkable E.M. set.

We shall prove Theorem 72 in two steps: First we shall show that both (A) and (B) are consequences of the assumption that there exists a well-founded remarkable E.M. set, and then we shall show that if there exists a measurable cardinal, then (30.20) holds. (Note that by Lemma 30.5 it suffices to find a well-founded remarkable model with uncountably many indiscernibles.)

Thus let us assume that there is a well-founded remarkable E.M. set and let Σ be such set.

For every limit ordinal α, the (Σ, α)-model is a well-founded model elementarily equivalent to some L_λ, and so by Lemma 30.1 is (isomorphic to) some L_β.

Lemma 30.9. *If κ is an uncountable cardinal, then the universe of the (Σ, κ)-model is L_κ.*

Proof. The (Σ, κ)-model is (L_β, I) for some β; since $|I| = \kappa$, we clearly have $\beta \geq \kappa$. To prove that $\beta = \kappa$, assume that $\beta > \kappa$. Since I is unbounded in β and has order-type κ, there is a limit ordinal $\gamma < \kappa$ such that $\kappa < i_\gamma$. By remarkability, all ordinals less than i_γ are in the (Σ, γ)-model $\mathfrak{A} = \mathscr{H}(\{i_\xi : \xi < \gamma\})$. This is a contradiction since on the one hand we have $\kappa \subseteq \mathfrak{A}$, and on the other hand $|\mathfrak{A}| = |\gamma| < \kappa$. ■

For each uncountable cardinal κ, let I_κ be the unique subset of κ such that (L_κ, I_κ) is the (Σ, κ)-model. By Lemma 30.8, I_κ is closed and unbounded in κ.

Lemma 30.10. *If $\kappa < \lambda$ are uncountable cardinals, then $I_\lambda \cap \kappa = I_\kappa$, and $\mathscr{H}^{L_\lambda}(I_\kappa) = L_\kappa$.*

Proof. Let J be the set consisting of the first κ members of I_λ and let $\mathfrak{A} = \mathscr{H}^{L_\lambda}(J)$. Then $(\mathfrak{A}, J)$ is a (Σ, κ)-model and the ordinals of $\mathfrak{A}$ are an initial segment of λ, say $Ord^{\mathfrak{A}} = \beta$. Since $(\mathfrak{A}, J)$ is isomorphic to (L_κ, I_κ), it is clear that $\beta = \kappa$ and $J = I_\kappa$. Hence $I_\lambda \cap \kappa = I_\kappa$.

Now since $\mathfrak{A} \prec L_\lambda$, $\mathfrak{A}$ is closed under the definable function $\mathrm{F}(\alpha) =$ the αth set in the well-ordering $<_L$, and since $Ord^{\mathfrak{A}} = \kappa$, we have $A = \mathrm{F}[\kappa] = L_\kappa$. ■

Using this lemma, we can now prove both (A) and (B) of Theorem 72, except for the uniqueness of S. We let

$$\mathrm{S} = \bigcup \{I_\kappa : \kappa \text{ an uncountable cardinal}\} \tag{30.21}$$

For each uncountable cardinal κ, $\mathrm{S} \cap \kappa = I_\kappa$ is a closed and unbounded set of order type κ, and is a set of indiscernibles for $\langle L_\kappa, \in\rangle$; moreover, by Lemma 30.3c, every $a \in L_\kappa$ is definable in L_κ from $\mathrm{S} \cap \kappa$. Let $\kappa < \lambda$ be uncountable cardinals. Since I_λ is closed in L_λ and $I_\lambda \cap \kappa = I_\kappa$, it follows that $\kappa \in I_\lambda$; hence S contains all uncountable cardinals. Also, since $L_\kappa = \mathscr{H}^{L_\lambda}(I_\kappa)$, we have $L_\kappa \prec L_\lambda$.

In order to prove uniqueness of Silver indiscernibles, we first observe:

Lemma 30.11. *There is at most one well-founded remarkable E.M. set.*

Proof. Assuming that there is one such Σ, we define the class S as in (30.21). Now since $L_{\aleph_\omega}$ is the $(\Sigma, \aleph_\omega)$-model and $\aleph_n \in \mathrm{S}$ for each $n \geq 1$, we have

$$\varphi(v_1, \ldots, v_n) \in \Sigma \quad \text{iff} \quad L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n] \tag{30.22}$$

which proves that Σ is unique. ■

We therefore define $0^\#$ (*zero-sharp*):

(30.23) $0^\#$ is the unique well-founded remarkable E.M. set if it exists.

Uniqueness of S now follows from:

Lemma 30.12. *For every regular uncountable cardinal κ there is at most one closed unbounded set of indiscernibles I for L_κ such that $L_\kappa = \mathscr{H}^{L_\kappa}(I)$.*

Proof. Let $\Sigma = \Sigma(L_\kappa, I)$. Since I is closed unbounded, it follows that Σ is remarkable; since κ is uncountable, Σ is well-founded (by Lemma 30.5). Hence $\Sigma = 0^\#$ and since (L_κ, I) is the (Σ, κ)-model, we have $I = \mathrm{S} \cap \kappa$. ■

Thus we have proved (A) and (B) of Theorem 72 under the assumption that $0^\#$ exists (i.e., (30.20)). On the other hand, if (B) holds, then $0^\#$ exists because, e.g., $(L_{\omega_1}, \mathrm{S} \cap \omega_1)$ is a remarkable well-founded model with $\aleph_1$ indiscernibles. To complete the proof of Theorem 72, it remains to show that if there is a measurable cardinal, then $0^\#$ exists. That will follow from:

Lemma 30.13. *Let κ be an uncountable cardinal. If there exists a limit ordinal λ such that $\langle L_\lambda, \in\rangle$ has a set of indiscernibles of order-type κ, then there exists γ and $I \subseteq \gamma$ of order type κ such that (L_γ, I) is remarkable.*

Then if κ is a measurable cardinal, κ is Ramsey; and by the corollary of Lemma 29.9 $\langle L_\kappa, \in\rangle$ has a set of indiscernibles of order-type κ. By Lemma 30.13, there exists a remarkable model $\langle L_\gamma, I\rangle$ where I has order-type κ. By Lemma 30.5, $\Sigma(L_\gamma, I)$ is well-founded and remarkable and hence $0^\#$ exists. Note that Lemma 30.13 also gives a proof of (i)$\leftrightarrow$(ii) in Theorem 73.

Proof. Let λ be the least limit ordinal such that $\langle L_\lambda, \in\rangle$ has a set of indiscernibles of order-type κ. We shall show first that there is a set of indiscernibles $I \subseteq \lambda$ for L_λ, of order-type κ, such that $\mathscr{H}^{L_\lambda}(I) = L_\lambda$. Let J be any set of indiscernibles for L_λ of order type κ, and let $\mathfrak{A} = \mathscr{H}^{L_\lambda}(J)$. Then $\mathfrak{A} \prec L_\lambda$ and hence $\mathfrak{A}$ is isomorphic to some L_β, $\beta \le \lambda$, by the collapsing map π. Now $I = \pi[J]$ is a set of indiscernibles for L_β, and $\mathscr{H}^{L_\beta}(I) = L_\beta$. By the minimality of λ, we have $\beta = \lambda$ and hence I is as claimed.

Next we show that any such set I is unbounded in λ. If not, there is a limit ordinal $\alpha < \lambda$ such that $I \subseteq \alpha$. There is a Skolem term t and $\gamma_1 < \cdots < \gamma_n \in I$ such that $\alpha = t^{L_\lambda}[\gamma_1, \ldots, \gamma_n]$. I claim that the set $J = \{i \in I : i > \gamma_n\}$ is a set of indiscernibles for $\langle L_\alpha, \in\rangle$. If $\varphi(v_1, \ldots, v_k)$ is a formula, then for any $i_1 < \cdots < i_k \in J$, L_α satisfies $\varphi[i_1, \ldots, i_k]$ if and only if L_κ satisfies the formula

$$L_\alpha \vDash \varphi[i_1, \ldots, i_k] \tag{30.24}$$

The formula (30.24) is a formula about $\alpha, i_1, \ldots, i_k$, and since $\alpha = t^{L_\lambda}[\gamma_1, \ldots, \gamma_n]$, there is a formula $\psi(u_1, \ldots, u_n, v_1, \ldots, v_k)$ such that L_κ satisfies (30.24) if and only if

$$L_\kappa \vDash \psi[\gamma_1, \ldots, \gamma_n, i_1, \ldots, i_k] \tag{30.25}$$

By indiscernibility of I, the truth of (30.25) is independent of the choice of $i_1 < \cdots < i_k$ in I provided $\gamma_n < i_1$. Hence the truth of (30.24) is independent of the choice of $i_1 < \cdots < i_k$ in J. Hence J is a set of indiscernibles for L_α, and this contradicts the minimality of λ since $\alpha < \lambda$ and the order type of J is κ.

Finally, let I be a set of indiscernibles for λ of order-type κ such that $\mathscr{H}^{L_\lambda}(I) = L_\lambda$, and that i_ω, the ωth element of I is least possible. I shall show that (L_λ, I) is remarkable.

Let us assume that (L_λ, I) is not remarkable. Then there is a Skolem term $t(x_1, \dots, x_m, y_1, \dots, y_n)$ such that the following holds in L_λ for any $x_1 < \cdots < x_m < y_1 < \cdots < y_n < z_1 < \cdots < z_n$:

$$t(x_1, \dots, x_n, y_1, \dots, y_n) < y_1 \tag{30.26}$$

and

$$t(x_1, \dots, x_m, y_1, \dots, y_n) \neq t(x_1, \dots, x_m, z_1, \dots, z_n) \tag{30.27}$$

We now consider the following increasing n-termed sequences in I: Let u_0 be the sequence of first n indiscernibles after x_m, let u_1 be the first n indiscernibles after u_0, etc.; for each $\alpha < \kappa$, let u_α be the first n indiscernibles after all u_β, $\beta < \alpha$. For each $\alpha < \kappa$, let

$$\gamma_\alpha = t(x_1, \dots, x_m, u_\alpha)$$

By indiscernibility, applied to the formula (30.27), we have $\gamma_\alpha \neq \gamma_\beta$ whenever $\alpha \neq \beta$. In fact, in (30.27) we have either $<$ or $>$ (in place of $\neq$); but $>$ is impossible since that would mean that $\gamma_\alpha > \gamma_\beta$ whenever $\alpha < \beta$. Thus $\langle \gamma_\alpha : \alpha < \kappa \rangle$ is an increasing sequence of ordinals.

I claim that $J = \{\gamma_\alpha : \alpha < \kappa\}$ is a set of indiscernibles for L_λ. This is so because for any formula φ, the truth value of $\varphi(\gamma_{\alpha_1}, \dots, \gamma_{\alpha_k})$ in L_λ does not depend on the choice of $\gamma_{\alpha_1} < \cdots < \gamma_{\alpha_k}$ in J because by the definition of the u_α, the truth value of

$$\varphi(t(x_1, \dots, x_m, u_{\alpha_1}), \dots, t(x_1, \dots, x_m, u_{\alpha_k}))$$

does not depend on the choice of $\alpha_1 < \cdots < \alpha_k$.

Hence $\{\gamma_\alpha : \alpha < \kappa\}$ is a set of indiscernibles for L_λ. Since i_ω is the first member of u_ω, it follows by (30.26) that $\gamma_\omega < i_\omega$. Now if $A = \mathscr{H}(J)$ and π is the transitive collapse of A, then, as we proved in the first paragraph, $\pi[A] = L_\lambda$ and $K = \pi[J]$ is a set of indiscernibles for L_λ of order-type κ such that $\mathscr{H}^{L_\lambda}(K) = L_\lambda$. However, $\pi(\gamma_\omega) \leq \gamma_\omega < i_\omega$, and so the ωth member of K is smaller than i_ω, contrary to our assumption. Hence (L_λ, I) is remarkable. ■

This completes the proof of Theorem 72.

Absoluteness of $0^\#$

The set $0^\#$ is not only unique if it exists, but is the unique well-founded remarkable E.M. set in every transitive model of ZF that contains it. And conversely, if $\mathfrak{M}$ is a transitive model of ZF containing all ordinals and $\Sigma \in \mathfrak{M}$ is such that $\mathfrak{M} \vDash$ (Σ is a well-founded remarkable E.M. set), then $\Sigma = 0^\#$. This follows from the fact that $\Sigma = 0^\#$ can be expressed as a Π_1 property and from the Lévy–Shoenfield absoluteness lemma.

Lemma 30.14. *There is a Π_1 formula $\Phi(\Sigma)$ such that $\Phi(\Sigma)$ holds if and only if Σ is a well-founded remarkable E.M. set.*

Corollary. *The property "Σ is a well-founded remarkable E.M. set" is absolute for every transitive model $\mathfrak{M}$ of ZF containing all ordinals. Hence $\mathfrak{M} \vDash (0^{\#}$ exists) if and only if $0^{\#} \in \mathfrak{M}$, in which case $(0^{\#})^{\mathfrak{M}} = 0^{\#}$.*

The corollary is a consequence of Theorem 36′ (the parametric version of the Lévy–Shoenfield absoluteness lemma). Although the theorem is stated for $Z \subseteq \omega$, it is applicable here since a set of formulas Σ can be replaced by a subset of ω, the set of all Gödel numbers of $\varphi \in \Sigma$.

Proof. A set of formulas Σ is a well-founded remarkable E.M. set if:

(30.28) (a) Σ is an E.M. set;
(b) Σ is remarkable;
(c) for every α, the (Σ, α)-model is well-founded.

Since (a) does not readily admit a nice formulation, we shall first replace it by another condition.

Let $\hat{\mathscr{L}}$ be the language $\{\in, c_1, c_2, \ldots, c_n, \ldots\}$ where c_n, $n < \omega$, are constant symbols. For every $\in$-formula $\varphi(v_1, \ldots, v_n)$ let $\hat{\varphi}$ be the sentence $\varphi(c_1, \ldots, c_n)$ of $\hat{\mathscr{L}}$. For each set of formulas Σ, let $\hat{\Sigma}$ be the set containing (i) all $\hat{\varphi}$ for $\varphi \in \Sigma$, (ii) the sentence "c_1 is an ordinal and $c_1 < c_2$," and (iii) the sentence "$\varphi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_n})$" for every $\varphi \in \Sigma$ and any $i_1 < \cdots < i_n$, $j_1 < \cdots < j_n$, (iv) all axioms of ZFC + $V = L$. Let

(30.28) (a′) $\hat{\Sigma}$ is consistent.

Clearly if Σ is an E.M. set, then $\hat{\Sigma}$ is consistent for we simply interpret the constants c_n, $n < \omega$, as some Silver indiscernibles. Conversely, if $\hat{\Sigma}$ is consistent, then $\hat{\Sigma}$ has a model and that model provides us with a (Σ, ω)-model (with indiscernibles c_n, $n < \omega$) and the proof of Lemma 30.3 goes through. Therefore if Σ satisfies (a′), (b), and (c), we proceed as in the proof of Theorem 72 and prove that $\Sigma = 0^{\#}$.

Thus Σ is a well-founded remarkable E.M. set if and only if it satisfies (30.28)(a′), (b), (c). Now (a′) and (b) are syntactical conditions, and so the conjunction of (a′) and (b) is a Δ_0-property over HF ($= V_\omega$), the collection of all hereditarily finite sets.

We shall now show that (c) can be written as a Π_1 property of Σ. If Σ satisfies (a′) and (b), then for every limit ordinal α there is a unique (up to isomorphism) (Σ, α)-model and we can find one $(\langle A, E\rangle, I)$ such that $I = \alpha$ and that $<^{\mathfrak{A}}$ (i.e., E) agrees with $<$ on α. If $(\langle A, E\rangle, \alpha)$ is such, we say that "$(\langle A, E\rangle, \alpha)$ is a (Σ, α)-model." This last property is a Δ_0 property of Σ, $\langle A, E\rangle$ and α (with parameter V_ω). Then Σ satisfies (c) if and only if:

(30.28) (c′) $\forall\alpha\ \forall\langle A, E\rangle$ (if $(\langle A, E\rangle, \alpha)$ is a (Σ, α)-model, then $\langle A, E\rangle$ is well-founded).

Now "$\langle A, E\rangle$ is well-founded" is Δ_1 (see Section 14) and hence the conjunction of (a′), (b), and (c′) is Π_1. ■

Exercise 30.1. The property "Σ is a well-founded remarkable E.M. set" is Π_1 over HC.

[Note that "$\forall\alpha$ the (Σ, α)-model is well-founded" can be replaced by $\forall\alpha \in \text{HC}$ the (Σ, α)-model is well-founded.]

Exercise 30.2. Let $\mathfrak{M}$ be a model of ZFC, let $B \in \mathfrak{M}$ be a complete Boolean algebra in $\mathfrak{M}$ and let G be an $\mathfrak{M}$-generic ultrafilter on B. If $\mathfrak{M} \vDash 0^\#$ does not exist, then $\mathfrak{M}[G] \vDash 0^\#$ does not exist.

[All cardinals $\geq |B^+|$ remain cardinals in $\mathfrak{M}[G]$. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_n < \cdots < \gamma_\omega$ be an increasing sequence of cardinals in $\mathfrak{M}$ such that $\gamma_1 \geq |B^+|$. If $0^\#$ exists in $\mathfrak{M}[G]$, then $0^\# = \{\varphi : L_{\gamma_\omega} \vDash \varphi[\gamma_1, \ldots, \gamma_n]\}$ and hence $0^\# \in \mathfrak{M}$.]

Elementary Embeddings of L

In Section 28 we proved that a well-founded ultrapower of the universe induces an elementary embedding $j_U\colon V \to Ult$, and conversely, if $j\colon V \to M$ is a nontrivial elementary embedding, then (28.21) defines a normal measure on the least ordinal moved by j.

Let j be a nontrivial elementary embedding of the universe, and let M be a transitive model of ZFC, containing all ordinals. Let $N = j(M) = \bigcup_{\alpha \in Ord} j(M \cap V_\alpha)$. Then N is a transitive model of ZF and $j\colon M \to N$ is elementary:

$$(30.29) \qquad M \vDash \varphi(a_1, \ldots, a_n) \quad \text{iff} \quad N \vDash \varphi(j(a_1), \ldots, j(a_n))$$

((30.29) is proved by induction on the complexity of φ). In particular, if $M = L$, then $j(V) \vDash$ (N is the constructible universe), and so $N = L$, and $j|L$ is an elementary embedding of L in L. Note that by Scott's theorem, the function $j|L$ is not a class in L; thus if there is an elementary embedding of L (into L), then $V \neq L$.

If $0^\#$ exists, then there are nontrivial elementary embeddings of L. In fact, let j be any order-preserving function from the class S of all Silver indiscernibles into S. Then j can be extended to an elementary embedding of L; we simply let

$$(30.30) \qquad j(t^L[\gamma_1, \ldots, \gamma_n]) = t^L[j(\gamma_1), \ldots, j(\gamma_n)]$$

for every Skolem term t and any Silver indiscernibles $\gamma_1 < \cdots < \gamma_n$. We shall prove that the converse is true, that if there is a nontrivial elementary embedding of L, then $0^\#$ exists. But let us first look into the duality of elementary embeddings and ultrapowers, in the case when the model being embedded is not the whole universe.

Let M be a transitive model of ZFC containing all ordinals, and let $j\colon M \to N$ be an elementary embedding; let γ be the least ordinal moved by j. Let D be the collection of subsets X of γ, $X \in M$, defined as follows:

$$(30.31) \qquad X \in D \quad \text{iff} \quad \gamma \in j(x)$$

We have $D \subseteq P^M(\gamma)$ but D is not necessarily in M. D clearly has the attributes of an ultrafilter:

(30.32) (i) $\gamma \in D$ and $\varnothing \notin D$;
(ii) if $X \in D$ and $Y \in D$, then $X \cap Y \in D$;
(iii) if $X \in D$ and $Y \in M$ is such that $X \subseteq Y$, then $Y \in D$;
(iv) for every $X \subseteq \gamma$ such that $X \in M$, either X or $\gamma - X$ is in D.

We shall call every D satisfying (30.32) an *M-ultrafilter*. The M-ultrafilter D defined in (30.31) is also *M-γ-complete* and *M-normal*: If $\alpha < \gamma$ and $\{X_\xi : \xi < \alpha\} \in M$ is such that $X_\xi \in D$ for all $\xi < \alpha$, then $\bigcap_{\xi<\alpha} X_\xi \in D$; and if $f \in M$ is a regressive function on $X \in D$, then f is constant on some $Y \in D$. The proofs are the same as in Lemma 28.5.

If D is an M-ultrafilter over γ, one can construct the ultrapower of M mod D as follows: Consider, *in M* (!), the class of all functions f with domain γ (that is $\{f \in M : \operatorname{dom}(f) = \gamma\}$). Using D, define an equivalence relation $=^*$ and $\in^*$ as usual;

$$(30.33) \qquad f =^* g \quad \text{iff} \quad \{\alpha < \gamma : f(\alpha) = g(\alpha)\} \in D$$

(note that the set on the right-hand side of (30.33) is in M, and D decides whether $f =^* g$ or $f \neq^* g$). Then define the equivalence classes and the model

$$Ult = Ult_D(M)$$

The fundamental theorem of ultraproducts is easily verified:

$$Ult \vDash \varphi([f_1], \ldots, [f_n]) \quad \text{iff} \quad \{\alpha < \gamma : M \vDash \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in D$$

If for each $a \in M$, c_a denotes the constant function with value a, then

$$j_D(a) = [c_a]$$

defines an elementary embedding of M in Ult (the natural embedding).

The ultrapower of M modulo an M-ultrafilter D is not necessarily well-founded, even if D is M-countably-complete. However, if D is obtained from an elementary embedding by (30.31), then $Ult_D(M)$ is well-founded because we have the commutative diagram (as in Lemma 28.6):

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[d, "j_D"'] & N \\ Ult \arrow[ur, "k"'] & \end{tikzcd}$$

FIGURE 30.1

(If $[f_0] \ni^* [f_1] \ni^* \cdots$ were a descending sequence in Ult, then $k[f_0] \ni k[f_1] \ni \cdots$ would be a descending sequence in N.)

Proof of Theorem 73

If $0^\#$ exists, then there are nontrivial elementary embeddings of L; each order-preserving $j\colon \mathbf{S} \to \mathbf{S}$ generates one. It was proved in Lemma 30.13 that if some L_λ has uncountably many indiscernibles, then there exists a *remarkable* (L_γ, I) with uncountably many indiscernibles, and by Lemma 30.5, $0^\#$ exists (and $0^\# = \Sigma(L_\gamma, I)$). We shall now show that if there exists an elementary embedding of L, then there is L_λ with $\aleph_1$ indiscernibles and hence $0^\#$ exists.

Let $j\colon L \to L$ be an elementary embedding. We shall first replace j by a more manageable embedding. We let D be the L-ultrafilter defined by

$$X \in D \quad \text{iff} \quad \gamma \in j(X)$$

where γ is the least ordinal moved by j. The ultrapower $Ult_D(L)$ is well-founded (see Fig. 30.1) and so we identify Ult with its transitive collapse L; let j_D be the associated natural embedding, $j_D\colon L \to L$. The least ordinal moved by j_D is γ (because D is L-γ-complete).

Lemma 30.15. *If κ is a limit cardinal such that* $\operatorname{cf} \kappa > \gamma$, *then* $j_D(\kappa) = \kappa$.

Proof. Every constructible function $f\colon \gamma \to \kappa$ is bounded by some $\alpha < \kappa$ and hence $[f] < [c_\alpha]$ (where c_α is the constant function with value α). Thus $j_D(\kappa) = \lim_{\alpha \to \kappa} j_D(\alpha)$. Now if $\alpha < \kappa$, then $|j_D(\alpha)| \le |({}^\gamma\alpha)^L|$, hence $j_D(\alpha) < \kappa$. It follows that $j_D(\kappa) = \kappa$. ∎

Let me drop the subscript D and simply assume that $j\colon L \to L$ is an elementary embedding, that γ is the least ordinal moved, and that $j(\kappa) = \kappa$ for every limit cardinal κ such that $\operatorname{cf} \kappa > \gamma$.

Let $\mathbf{U}_0$ be the class of all limit cardinals κ with $\operatorname{cf} \kappa > \gamma$; by transfinite induction we define a sequence of classes $\mathbf{U}_0 \supset \mathbf{U}_1 \supset \cdots \supset \mathbf{U}_\alpha \cdots$ as follows:

$$\text{(30.34)} \qquad \mathbf{U}_{\alpha+1} = \{\kappa \in \mathbf{U}_\alpha : |\mathbf{U}_\alpha \cap \kappa| = \kappa\}$$

$$\mathbf{U}_\lambda = \bigcap_{\alpha < \lambda} \mathbf{U}_\alpha \qquad (\lambda \text{ a limit})$$

(That is, $\mathbf{U}_{\alpha+1}$ consists of fixed points of the increasing enumeration of $\mathbf{U}_\alpha$.) Each $\mathbf{U}_\alpha$ is nonempty, and in fact a proper class. To see this, verify, by induction on α, that each $\mathbf{U}_\alpha$ is a proper class and is *δ-closed*, for each δ with $\operatorname{cf} \delta > \gamma$; that is, whenever $\langle \kappa_\xi : \xi < \delta \rangle$ is an increasing sequence in $\mathbf{U}_\alpha$, then $\lim_{\xi \to \delta} \kappa_\xi \in \mathbf{U}_\alpha$. Hence each $\mathbf{U}_\alpha$ is nonempty, and we choose a cardinal $\kappa \in \mathbf{U}_{\omega_1}$.

Thus κ is such that $\operatorname{cf} \kappa > \gamma$ and κ is the κth element of each $\mathbf{U}_\alpha$, $\alpha < \omega_1$. We shall find a set of $\aleph_1$ indiscernibles for $\langle L_\kappa, \in \rangle$.

Since $j\colon L \to L$ is elementary and $j(\kappa) = \kappa$, it is clear that the mapping $i = j \restriction L_\kappa$ is an elementary embedding of $\langle L_\kappa, \in \rangle$ into $\langle L_\kappa, \in \rangle$. We shall use i

and the sets $\mathrm{U}_\alpha \cap \kappa$, $\alpha < \omega_1$, to produce indiscernibles γ_α, $\alpha < \omega_1$, for L_κ. Let $X_\alpha = \mathrm{U}_\alpha \cap \kappa$ for each $\alpha < \omega_1$, and recall that γ is the least ordinal moved by i.

For each $\alpha < \omega_1$, we let

$$M_\alpha = \mathscr{H}^{L_\kappa}(\gamma \cup X_\alpha) \tag{30.35}$$

M_α is an elementary submodel of L_κ.

If π_α is the transitive collapse of M_α, then because $|X_\alpha| = \kappa$, we have $\pi_\alpha[M_\alpha] = L_\kappa$. Thus if we denote $i_\alpha = \pi_\alpha^{-1}$, then i_α is an elementary embedding of L_κ in L_κ. Let

$$\gamma_\alpha = i_\alpha(\gamma) \tag{30.36}$$

Lemma 30.16.

(a) *The ordinal* γ_α *is the least ordinal* $> \gamma$ *in* M_α.
(b) *If* $\alpha < \beta$ *and* $x \in M_\beta$, *then* $i_\alpha(x) = x$. *In particular,* $i_\alpha(\gamma_\beta) = \gamma_\beta$.
(c) *If* $\alpha < \beta$, *then* $\gamma_\alpha < \gamma_\beta$.

Proof. (a) Since $\gamma \subset M_\alpha$, $i_\alpha(\gamma)$ is the least ordinal in M_α greater than or equal to γ; thus it suffices to show that $\gamma \notin M_\alpha$. If $x \in M_\alpha$, then $x = t[\eta_1, \ldots, \eta_n]$ where t is a Skolem term and the η's are either $< \gamma$ or elements of X_α. For all such η, $i(\eta) = \eta$ and hence $i(x) = i(t(\eta_1, \ldots, \eta_n)) = t(i(\eta_1), \ldots, i(\eta_n)) = x$. However, $i(\gamma) \neq \gamma$ and so $\gamma \notin M_\alpha$.

(b) Each $x \in M_\beta$ is of the form $t[\eta_1, \ldots, \eta_n]$ where the η's are either $< \gamma$ or in X_β. If $\eta < \gamma$, then clearly $i_\alpha(\eta) = \eta$. If $\eta \in X_\beta$, then because $\alpha < \beta$, we have $|X_\alpha \cap \eta| = \eta$ and hence $\pi_\alpha(\eta) = \eta$; in other words, $i_\alpha(\eta) = \eta$. Therefore $i_\alpha(x) = x$.

(c) If $\alpha < \beta$, then $M_\alpha \supseteq M_\beta$ and hence $\gamma_\alpha \leq \gamma_\beta$. To see that $\gamma_\alpha \neq \gamma_\beta$, note that because $\gamma_\alpha > \gamma$, we have $i_\alpha(\gamma_\alpha) > i_\alpha(\gamma) = \gamma_\alpha$, while $i_\alpha(\gamma_\beta) = \gamma_\beta$.

Lemma 30.17. *If* $\alpha < \beta$, *then there is an elementary embedding* $i_{\alpha\beta}\colon L_\kappa \to L_\kappa$ *such that for every* ξ *that is either* $< \alpha$ *or* $> \beta$ *we have* $i_{\alpha\beta}(\gamma_\xi) = \gamma_\xi$, *and* $i_{\alpha\beta}(\gamma_\alpha) = \gamma_\beta$.

Proof. Let $M_{\alpha\beta} = \mathscr{H}^{L_\kappa}(\gamma_\alpha \cup X_\beta)$, and let $i_{\alpha\beta} = \pi_{\alpha\beta}^{-1}$ where $\pi_{\alpha\beta}$ is the transitive collapse of $M_{\alpha\beta}$. The mapping $i_{\alpha\beta}$ is an elementary embedding of L_κ in L_κ.

If $\eta < \gamma_\alpha$, then clearly $i_{\alpha\beta}(\eta) = \eta$; in particular $i_{\alpha\beta}(\gamma_\xi) = \gamma_\xi$ if $\xi < \alpha$. If $x \in M_{\beta+1}$, then $x = t(\eta_1, \ldots, \eta_n)$ where the η's are either $< \gamma$ or $\in X_{\beta+1}$. If $\eta \in X_{\beta+1}$, then $|X_\beta \cap \eta| = \eta$ and therefore $i_{\alpha\beta}(\eta) = \eta$. Hence $i_{\alpha\beta}(x) = x$ for every $x \in M_{\beta+1}$, and in particular $i_{\alpha\beta}(\gamma_\xi) = \gamma_\xi$ if $\xi > \beta$.

Now we shall show that $i_{\alpha\beta}(\gamma_\alpha) = \gamma_\beta$. Since $M_{\alpha\beta} \supseteq M_\beta$, we have $\gamma_\beta \in M_{\alpha\beta}$; and since $\gamma_\alpha \subset M_{\alpha\beta}$, $i_{\alpha\beta}(\gamma_\alpha)$ is the least ordinal in $M_{\alpha\beta}$ greater than or equal to γ_α; hence we have $\gamma_\alpha \leq i_{\alpha\beta}(\gamma_\alpha) \leq \gamma_\beta$.

Thus it suffices to show that there is no ordinal $\delta \in M_{\alpha\beta}$ such that $\gamma_\alpha \leq \delta < \gamma_\beta$. Otherwise there is some $\delta = t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k)$ such that the ξ's are $< \gamma_\alpha$ and the η's are in X_β (and t is a Skolem term) and that $\gamma_\alpha \leq \delta < \gamma_\beta$. Thus we have:

$$\langle L_\kappa, \in \rangle \vDash \exists \xi < \gamma_\alpha \text{ such that } \gamma_\alpha \leq t(\xi, \eta) < \gamma_\beta. \tag{30.37}$$

The formula in (30.37) is a formula φ about γ_α, η, and γ_β. At this point, we apply the elementary embedding $i_\alpha: L_\kappa \to L_\kappa$ backward. That is, γ_α, the η's and γ_β are all in the range of i_α: $\gamma_\alpha = i_\alpha(\gamma)$, $\eta = i_\alpha(\eta)$, and $\gamma_\beta = i_\alpha(\gamma_\beta)$; and since $L_\kappa \vDash \varphi[i_\alpha(\gamma), i_\alpha(\eta), i_\alpha(\gamma_\beta)]$, we conclude that $L_\kappa \vDash \varphi[\gamma, \eta, \gamma_\beta]$, namely

$$(30.38)\quad \langle L_\kappa, \in\rangle \vDash \exists\xi < \gamma \text{ such that } \gamma \le t(\xi, \eta) < \gamma_\beta .$$

Thus pick some ξ's $< \gamma$ such that $\gamma \le t(\xi, \eta) < \gamma_\beta$. Since $\xi \in \gamma$ and $\eta \in X_\beta$, we have $t(\xi, \eta) \in M_\beta$, which means that $t(\xi, \eta)$ is an ordinal in M_β between γ and γ_β, and that contradicts Lemma 30.16a. ■

The proof of Theorem 73 will be complete when we show:

Lemma 30.18. *The set* $\{\gamma_\alpha : \alpha < \omega_1\}$ *is a set of indiscernibles for* $\langle L_\kappa, \in\rangle$.

Proof. Let φ be a formula and let $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ be two sequences of countable ordinals. We wish to show that

$$(30.39)\qquad L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \quad \text{iff} \quad L_\kappa \vDash \varphi[\gamma_{\beta_1}, \ldots, \gamma_{\beta_n}]$$

Let us pick $\delta_1 < \cdots < \delta_n$ such that $\alpha_n < \delta_1$ and $\beta_n < \delta_1$. First we apply the elementary embedding i_{α_n,δ_n} and get

$$L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_{n-1}}, \gamma_{\alpha_n}] \quad \text{iff} \quad L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_{n-1}}, \gamma_{\delta_n}]$$

because $i_{\alpha_n,\delta_n}(\gamma_{\alpha_n}) = \gamma_{\delta_n}$, and preserves the other γ's. Then we apply $i_{\alpha_{n-1},\delta_{n-1}}$ with a similar effect, and by a successive application of $i_{\alpha_{n-2},\delta_{n-2}}, \ldots, i_{\alpha_1,\delta_1}$ we get

$$L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \quad \text{iff} \quad L_\kappa \vDash \varphi[\gamma_{\delta_1}, \ldots, \gamma_{\delta_n}]$$

Then we do the same for the β's and δ's as we did for the α's and δ's, and (30.39) follows. ■

This completes the proof of Theorem 73.

Exercise 30.3. Let κ be an uncountable regular cardinal. If $0^\#$ exists, then for every constructible set $X \subseteq \kappa$, either X or $\kappa - X$ contains a closed unbounded subset.

[Let $X = t(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ where $\alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots < \beta_m$ are Silver indiscernibles such that $\alpha_n < \kappa \le \beta_1$. Show that either X or $\kappa - X$ contains all Silver indiscernibles γ such that $\alpha_n < \gamma < \kappa$: The truth value of $\gamma \in t(\alpha, \beta)$ is the same for all such γ.]

Exercise 30.4. Let us assume that for some uncountable regular cardinal κ, every constructible $X \subseteq \kappa$ either contains or is disjoint from a closed unbounded set. Then $0^\#$ exists.

[Let D be the collection of all constructible subsets of κ containing a closed unbounded subset. D is an L-ultrafilter and every intersection of less than κ elements of D is nonempty; hence the ultrapower $Ult_D(L)$ is well-founded and gives an elementary embedding of L in L.]

Exercise 30.5. Assume that $0^\#$ exists. If $A \subseteq \omega_1$ is such that $A \cap \alpha \in L$ for every $\alpha < \omega_1$, then $A \in L$.

[For every $\alpha \in S \cap \omega_1$ there is t_α such that $A \cap \alpha = t_\alpha(\gamma_1^\alpha, \ldots, \gamma_{n(\alpha)}^\alpha, \alpha, \delta_1^\alpha, \ldots, \delta_{k(\alpha)}^\alpha)$.

Clearly, $A \cap \alpha = t_\alpha(\gamma_1^\alpha, \ldots, \alpha, \aleph_2, \ldots, \aleph_{k(\alpha)+1})$. Since there are only countably many Skolem terms, and by the Fodor theorem, there is a stationary subset X of $\mathrm{S} \cap \omega_1$ and t, $\gamma_1, \ldots, \gamma_n$ such that for all $\alpha \in X$, $A \cap \alpha = t(\gamma_1, \ldots, \gamma_n, \alpha, \aleph_2, \ldots, \aleph_{k+1})$. Show that $A = t(\gamma_1, \ldots, \gamma_n, \aleph_1, \aleph_2, \ldots, \aleph_{k+1})$.]

The Sharps

All results about $0^\#$ and Silver indiscernibles for L proved in the present section can be relativized to obtain similar results for the models $L[x]$, where $x \subseteq V_\omega$.

In particular, under the assumption of existence of measurable (or just Ramsey) cardinals, there is for every $x \subseteq V_\omega$ a unique class S_x containing all uncountable cardinals such that for each uncountable cardinal κ, $\mathrm{S}_x \cap \kappa$ is a set of indiscernibles for the model $\langle L_\kappa[x], \in, x\rangle$ and all elements of $L_\kappa[x]$ are definable in the model from $\mathrm{S}_x \cap \kappa$. Here x is considered a one-place predicate. Also, for each regular uncountable cardinal κ, $\mathrm{S}_x \cap \kappa$ is closed unbounded in κ.

The proof of the relativization of Theorem 72 uses models with indiscernibles $(\mathfrak{A}, I)$ where $\mathfrak{A}$ is elementary equivalent to some $\langle L_\lambda[x], \in, x\rangle$ where $\lambda > \omega$ is a limit ordinal (note that $x \in L_\lambda[x]$). If κ is a Ramsey cardinal, then $\langle L_\kappa[x], \in, x\rangle$ has a set of indiscernibles of size κ, and the theorem follows.

We define $x^\#$ as the unique set $\Sigma = \Sigma(\langle L_\lambda[x], \in, x\rangle, I)$ that is well-founded and remarkable. If $x^\#$ exists, then we have

$$x^\# = \{\varphi : \langle L_{\aleph_\omega}[x], \in, x\rangle \models \varphi[\aleph_1, \ldots, \aleph_n]\}$$

Here φ is a formula of the language $\{\in, P\}$ where P is a one-place predicate symbol (interpreted as x). Note that x is definable in the model $\langle L_\lambda[x], \in, x\rangle$ (by the formula $P(v)$).

The property (of Σ) "Σ is $x^\#$" is Π_1 with a parameter x. Thus $x^\#$ is absolute for all transitive models $\mathfrak{M}$ or ZF containing all ordinals such that $x^\# \in \mathfrak{M}$.

Also, "$x^\#$ exists" is equivalent to the existence of a nontrivial elementary embedding $j: L[x] \to L[x]$.

Jensen's Covering Theorem

I shall state without proof an important theorem which decides many problems about singular cardinals if $0^\#$ does not exist. In particular, the theorem implies that unless $0^\#$ exists, the singular cardinals hypothesis is true and hence all cardinal exponentiation is determined by the continuum function on regular cardinals (see Section 8). It follows that in order to establish independence of the singular cardinals hypothesis (or of related problems), one needs large cardinals, and that such independence results cannot be accomplished by generic extensions of the constructible universe. (By more recent results, in order to violate the singular cardinal hypothesis, one needs to assume existence of measurable cardinals.)

The proof of Theorem 74 is rather long and uses recursion theoretic methods (the fine structure of L).

Theorem 74 (*Jensen's Covering Theorem*). *If $0^{\#}$ does not exist, then for every uncountable set X of ordinals there exists a constructible set $Y \supseteq X$ such that $|Y| = |X|$.*

The conclusion of Theorem 74 is in fact equivalent to nonexistence of $0^{\#}$: If $0^{\#}$ exists, it is easy to find a counterexample. For instance, let $X = \{\aleph_\alpha : \alpha < \omega_1\}$; since $\aleph_{\omega_1}$ is a regular cardinal in L, every constructible $Y \supseteq X$ has size at least $\aleph_{\omega_1}$ while $|X| = \aleph_1$.

The assumption that X is uncountable cannot be dropped:

Exercise 30.6. It is consistent that $0^{\#}$ does not exist and $\mathrm{cf}(\omega_2^L) = \omega$ while $|\omega_2^L| = \aleph_1$ (then there is a countable $X \subset \omega_2^L$ that cannot be covered by a countable constructible set).

[Example II in Section 26 gives a generic extension of L with these properties; $0^{\#}$ does not exist in a generic extension of L (Exercise 30.2).]

However, note that the theorem implies that a countable X can be covered by constructible Y of size $\aleph_1$.

I shall prove some consequences of the covering theorem. The most important is Corollary 1 which states that unless $0^{\#}$ exists, the singular cardinals hypothesis is true. It follows that if $0^{\#}$ does not exist, then cardinal exponentiation is determined by the continuum function on regular cardinals, and κ^λ always has the least possible value (see Lemma 8.1); in particular, $2^\kappa = \kappa^+$ for all strong limit singular cardinals.

Corollary. *Assume that $0^{\#}$ does not exist. Then:*

(1) *For every singular cardinal κ, if $2^{\mathrm{cf}\,\kappa} < \kappa$, then $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$. In particular, if κ is a strong limit singular cardinal, then $2^\kappa = \kappa^+$.*

(2) *If κ is a singular cardinal and if there exists a nonconstructible subset of κ, then some $\alpha < \kappa$ has a nonconstructible subset.*

(3) *If λ is a regular cardinal in L, then either $|\lambda| \le \aleph_1$ or $\mathrm{cf}\,\lambda = |\lambda|$. In particular, λ is not a singular cardinal.*

(4) *If κ is a singular cardinal, then $(\kappa^+)^L = \kappa^+$.*

Before giving a proof of the corollary, let me make a few comments. Result (2) suggests that in the absence of $0^{\#}$, singular cardinals behave like large cardinals: We shall show in Section 32 that every weakly compact cardinal has this property. Results (3) and (4) shed light on the difficulties with collapsing cardinals and changing cofinalities by forcing: The standard method of collapsing κ^+ onto κ does not work if κ is singular; and the only method we have dealt with so far of changing the cofinality of a regular cardinal λ such that $\mathrm{cf}\,\lambda < |\lambda|$ in the generic extension is Example II in Section 26, where $\mathrm{cf}\,\lambda = \omega$ and $|\lambda| = \aleph_1$.

Proof of the Corollary. (1) Let κ be such that $2^{\operatorname{cf}\kappa} < \kappa$, and let

$$A = [\kappa]^{\operatorname{cf}\kappa}$$

be the set of all subsets of κ of size cf κ. We shall show that $|A| \leq \kappa^+$. By the covering theorem, for every $X \in A$ there exists a constructible $Y \subseteq \kappa$ such that $X \subseteq Y$ and $|Y| = \lambda$ where $\lambda = \aleph_1 \cdot \operatorname{cf}\kappa$. Thus

$$A \subseteq \bigcup \{[Y]^{\operatorname{cf}\kappa} : Y \in C\} \tag{30.40}$$

where

$$C = \{Y \subseteq \kappa : |Y| = \lambda \text{ and } Y \in L\}$$

If $Y \in C$, then

$$|[Y]^{\operatorname{cf}\kappa}| = \lambda^{\operatorname{cf}\kappa} = (\aleph_1 \cdot \operatorname{cf}\kappa)^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa} < \kappa$$

Since

$$|C| \leq |P^L(\kappa)| = |(\kappa^+)^L| \leq \kappa^+$$

it follows from (30.40) that $|A| \leq \kappa^+$.

(2) Let κ be a singular cardinal and assume that each $\alpha < \kappa$ has only constructible subsets; we shall show that every subset of κ is constructible. It suffices to show that each subset of κ of size cf κ is constructible: If $A \subseteq \kappa$, let $\{\alpha_\nu : \nu < \operatorname{cf}\kappa\}$ be such that $\lim_\nu \alpha_\nu = \kappa$; then $\mathscr{A} = \{A \cap \alpha_\nu : \nu < \operatorname{cf}\kappa\}$ is a subset of L_κ of size $\leq \operatorname{cf}\kappa$ and hence constructible. It follows that A is constructible.

Let $X \subseteq \kappa$ be such that $|X| \leq \operatorname{cf}(\kappa)$. By the covering theorem, there exists a constructible set of ordinals $Y \supseteq X$ such that $|Y| < \kappa$. Let π be the isomorphism between Y and its order-type α; the function π is constructible and one-to-one. Since $|\alpha| = |Y| < \kappa$, we have $\alpha < \kappa$.

Let $Z = \pi[X]$. Then $Z \subseteq \alpha$ is constructible by the assumption, and hence $X = \pi^{-1}[Z]$ is also constructible.

(3) Let λ be a limit ordinal such that $\lambda \geq \omega_2$ and that λ is a regular cardinal in L. Let X be an unbounded subset of λ such that $|X| = \operatorname{cf}\lambda$. By the covering theorem, there exists a constructible set Y such that $X \subseteq Y \subseteq \lambda$ and that $|Y| = |X| \cdot \aleph_1$. Since Y is unbounded in λ and λ is a regular cardinal in L, we have $|Y| = |\lambda|$. This gives us $|\lambda| = \aleph_1 \cdot \operatorname{cf}\lambda$ and since we assume that $|\lambda| \geq \aleph_2$, we have $|\lambda| = \operatorname{cf}\lambda$.

(4) Let κ be a singular cardinal and let λ be the successor cardinal of κ in L; we want to show that $\lambda = \kappa^+$. If not, then $|\lambda| = \kappa$; and since κ is singular, we have $\operatorname{cf}\lambda < \kappa$. However, this means that $\operatorname{cf}\lambda < |\lambda|$ which contradicts (3). ∎

I shall conclude with the following application of Jensen's covering theorem. In Section 21, Example IV, we have seen a model of ZF without the axiom of choice in which $\aleph_1$ is singular. A still unsolved problem is to construct a model in which all uncountable alephs are singular. The following exercise shows that some large cardinal assumption is necessary for such consistency result:

Exercise 30.7. If both ω_1 and ω_2 are singular, then $0^\#$ exists.

[Let $\kappa = \omega_1$ and let λ be the successor cardinal of κ in L. Since cf κ = cf $\lambda = \omega$, there are sets $X \subseteq \kappa$ and $Y \subseteq \lambda$, both of order type ω such that sup $X = \kappa$ and sup $Y = \lambda$. Let $M = L[X, Y]$; M is a model of ZFC and in M, κ is a singular cardinal, and λ is not a cardinal. By Corollary 4, $0^\#$ exists in M.]

31. THE MODEL $L[U]$

The results of preceding sections show that measurable cardinals are inconsistent with the axiom of constructibility, and in fact existence of measurable cardinals implies that very few sets are constructible. This eliminates the constructible universe L with its nice structural properties as a tool for consistency results; thus if we want to show consistency of, say, the continuum hypothesis relative to existence of measurable cardinals, we have to look for models with nonconstructible sets.

It turns out that the theory ZFC + "there is a measurable cardinal" has a canonical model that shares with L many of its useful properties. If κ is a measurable cardinal, let U be any κ-complete nonprincipal ultrafilter over κ and let us consider the model $L[U]$ of all sets constructible from U. We shall prove that the model $L[U]$ is unique; it is the same model regardless of our choice of U. Moreover, in $L[U]$ κ is the only measurable cardinal and κ has only one normal measure; and $L[U]$ is the smallest model in which κ is measurable. Every set in $L[U]$ is the least model in which κ is measurable. Every set in $L[U]$ is constructible from the unique normal measure and hence all sets in $L[U]$ are ordinal definable and $L[U]$ has a definable well-ordering of the universe. We shall also prove that if U' is a κ'-complete nonprincipal ultrafilter over some $\kappa' \neq \kappa$, then the models $L[U]$ and $L[U']$ are elementarily equivalent. Besides uniqueness of the model $L[U]$, its most important property is that it satisfies the generalized continuum hypothesis; in that respect, $L[U]$ behaves like L. In fact, the model $L[U]$ has a "fine structure" analogous to the "fine structure" of L, and so $L[U]$ satisfies many combinatorial principles that hold in L.

We shall first investigate the model $L[D]$ where D is some normal measure. In Theorem 75 we prove that $L[D]$ satisfies GCH. The proof is not very difficult and is very much like the proof of GCH in L. They key step is the use of Rowbottom's theorem (Lemma 29.10). The uniqueness theorem for $L[U]$ and further results on $L[U]$ will require a new technique, the method of iterated ultrapowers.

Let me first recall some facts about relative constructibility (Section 15). If A is a set, the hierarchy $L_\alpha[A]$ is defined in analogy with the L_α-hierarchy, except that

$$L_{\alpha+1}[A] = \mathrm{def}_A(L_\alpha[A])$$

where $\operatorname{def}_A(M)$ is the Gödel closure of the set $M \cup \{M\} \cup \{A \cap M\}$. Equivalently, $\operatorname{def}_A(M)$ is the collection of all $X \subseteq M$ definable over the model $\langle M, \in, A \cap M\rangle$ (where $A \cap M$ is taken as a one-place predicate). If we denote $\bar{A} = A \cap L[A]$, then $L[A] = L[\bar{A}]$, and the model $L[A]$ satisfies "$V = L[\bar{A}]$."

By Lemma 15.6, there is a sentence Θ^* such that if M is a transitive model of Θ^* and A is a set in M, then the function $\alpha \mapsto L_\alpha[A]$ is absolute for M. The sentence Θ^* holds in $L_\lambda[A]$ for every infinite cardinal λ and any A. It follows that if M is an elementary submodel of $\langle L_\lambda[A], \in\rangle$ and $A \in M$, then M is isomorphic to some $L_\alpha[\pi(A)]$ where π is the transitive collapse of M.

Also, if M is a transitive model of Θ^* and if A is such that $A \cap M \in M$, then for every $\alpha \in M$,

$$L_\alpha[A] = L_\alpha[A \cap M] = (L_\alpha[A \cap M])^M \tag{31.1}$$

This is proved in Lemma 15.4.

In Section 15 we dealt mostly with $L[A]$ where A is a set of ordinals; among others, we have shown (Lemma 15.7) that if $V = L[A]$ and if λ is an infinite cardinal such that $A \subseteq \lambda$, then $2^\lambda = \lambda^+$. In the present situation we want to investigate $L[U]$ where U is an ultrafilter, thus a set of sets of ordinals. The following lemma is a straightforward generalization of Lemma 15.7:

Lemma 31.1. *If $V = L[A]$, and if $A \subseteq P(\omega_\alpha)$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.*

Proof. Let X be a subset of ω_α. Let λ be a cardinal such that $A \in L_\lambda[A]$ and $X \in L_\lambda[A]$. Let M be an elementary submodel of $\langle L_\lambda[A], \in\rangle$ such that $\omega_\alpha \subseteq M$, $A \in M$, $X \in M$, and $|M| = \aleph_\alpha$. Let π be the transitive collapse of M, and let $N = \pi[M]$. Since $\omega_\alpha \subseteq M$, we have $\pi(Z) = Z$ for every $Z \subseteq \omega_\alpha$ that is in M and in particular $\pi(X) = X$; also, $\pi(A) = \pi[A \cap M] = A \cap N$. Now $N = L_\gamma[A \cap N]$ for some γ, and hence, by (31.1), $N = L_\gamma[A]$. Since $|N| = \aleph_\alpha$, we have $\gamma < \omega_{\alpha+1}$ and hence $X \in L_{\omega_{\alpha+1}}[A]$. It follows that every subset of ω_α is in $L_{\omega_{\alpha+1}}[A]$ and therefore $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. ■

Let κ be a measurable cardinal and let U be a κ-complete nonprincipal ultrafilter over κ. Let us consider the model $L[U]$. To start with, $L[U] = L[\bar{U}]$, where $\bar{U} = U \cap L[U]$.

Lemma 31.2. *In $L[U]$, $\bar{U}$ is a κ-complete nonprincipal ultrafilter over κ. Moreover, if U is normal, then $L[U] \models \bar{U}$ is normal.*

Proof. A straightforward verification. For instance if U is normal and $f \in L[U]$ is a regressive function on κ, then for some $\gamma < \kappa$, the set $X = \{\alpha : f(\alpha) = \gamma\}$ is in U; since $X \in L[U]$, $L[U] \models f$ is constant on some $X \in \bar{U}$. ■

Thus the model $L[U]$ satisfies that κ is a measurable cardinal and that $V = L[\bar{U}]$ for some κ-complete nonprincipal ultrafilter over κ. The following theorem shows that if D is normal, then $L[D]$ satisfies the GCH.

Theorem 75 (Silver). *If $V = L[D]$ where D is a normal measure on a measurable cardinal κ, then the generalized continuum hypothesis holds.*

Proof. If $\lambda \geq \kappa$, then $D \subseteq P(\lambda)$ and hence $2^\lambda = \lambda^+$ by Lemma 31.1. Thus it suffices to show that $2^\lambda = \lambda^+$ for any infinite cardinal $\lambda < \kappa$. Let $\lambda < \kappa$ and let us assume that there are more than λ^+ subsets of λ; we shall reach a contradiction. If $2^\lambda > \lambda^+$, then there exists a set $X \subseteq \lambda$ that is the λ^+th subset of λ in the canonical well-ordering $<_{L[D]}$ of $L[D]$. Let α be the least ordinal such that $X \in L_\alpha[D]$. Since the well-ordering $<_{L[D]}$ has the property that each $L_\xi[D]$ is an initial segment of $<_{L[D]}$ every subset of λ preceding X is also in $L_\alpha[D]$ and hence the set $P(\lambda) \cap L_\alpha[D]$ has size at least λ^+.

We shall now apply Lemma 29.10. Let η be a cardinal such that $\eta > \alpha$ and that $D \in L_\eta[D]$, and consider the model $\mathfrak{A} = \langle A, \in \rangle$ where $A = L_\eta[D]$. We have $\kappa \subseteq A$, and we consider the set $P = P(\lambda) \cap A$. Since $2^\lambda < \kappa$, we have $|P| < \kappa$. By Lemma 29.10, there is an elementary submodel $\mathfrak{B} \prec \mathfrak{A}$ such that $\lambda \cup \{D, X, \alpha\} \subseteq \mathfrak{B}$, $\kappa \cap \mathfrak{B} \in D$ and $|P \cap \mathfrak{B}| \leq \lambda$. Let π be the transitive collapse of B onto a transitive set M; we have $M = L_\gamma[\pi(D)]$ for some γ.

Using normality of D, we show that $\pi(D) = D \cap M$. Clearly, $\pi(\kappa) = \kappa$ because $|\kappa \cap \mathfrak{B}| = \kappa$. The function π is one-to-one, and for every $\xi < \kappa$, $\pi(\xi) \leq \xi$. Since D is normal, there is a set $Z \in D$ such that $\pi(\xi) = \xi$ for all $\xi \in Z$. Hence if $Y \in \mathfrak{B}$ is a set in D, then $\pi(Y) \supseteq \pi(Y \cap Z) = Y \cap Z$, and so $\pi(Y)$ is also in D; similarly, if $Y \in B$ and $\pi(Y) \in D$, then $Y \in D$. It follows that $\pi(D) = \pi[D \cap \mathfrak{B}] = D \cap M$.

Hence by (31.1), $M = L_\gamma[D]$. Since $\lambda \subseteq \mathfrak{B}$, π maps every subset of λ onto itself, and so $P(\lambda) \cap M = P(\lambda) \cap \mathfrak{B}$. In particular, we have $\pi(X) = X$ and so $X \in L_\gamma[D]$. By the minimality assumption on α, we have $\alpha \leq \gamma$, and this is a contradiction since on the one hand $|P(\lambda) \cap L_\alpha[D]| \geq \lambda^+$, and on the other hand $|P(\lambda) \cap L_\gamma[D]| \leq \lambda$. ■

One proves rather easily that the model $L[D]$ has only one measurable cardinal:

Lemma 31.3. *If $V = L[D]$ and D is a normal measure on κ, then κ is the only measurable cardinal.*

Proof. Let us assume that there is a measurable cardinal $\lambda \neq \kappa$ and let us consider the elementary embedding $j_U \colon V \to M$ where U is some nonprincipal λ-complete ultrafilter over λ. We shall prove that $M = L[D] = V$ thus getting a contradiction since $U \notin M$ by Lemma 28.9b.

Since j is elementary, it is clear that $M = L[j(D)]$. If $\lambda > \kappa$, then $j(D) = D$ and so $M = L[D]$. Thus assume that $\lambda < \kappa$.

Since κ is measurable, the set $Z = \{\alpha < \kappa : \alpha \text{ is inaccessible and } \alpha > \lambda\}$ belongs to D. By Lemma 28.9e, $j(\kappa) = \kappa$ and $j(\alpha) = \alpha$ for all $\alpha \in Z$. I shall show that $j(D) = D \cap M$. It suffices to show that $j(D) \subseteq D \cap M$ since $j(D)$ is (in M) an ultrafilter. Let $X \in j(D)$ be represented by $f \colon \lambda \to D$. Let $Y = \bigcap_{\xi < \lambda} f(\xi)$; we have $Y \in D$, and clearly $j(Y) \subseteq X$. Now if $\alpha \in Y \cap Z$, then $j(\alpha) = \alpha$ and so $X \supseteq j(Y) \supseteq j[Y \cap Z] = Y \cap Z \in D$ and hence $X \in D$.

Thus $j(D) = D \cap M$, and we have (by Lemma 15.4)

$$M = L[j(D)] = L[D \cap M] = L[D]. \quad \blacksquare$$

I shall now introduce the method of iterated ultrapowers and use it to obtain further results on the model $L[U]$. But first I shall make a few remarks on the model $L[U]$ related to the results above.

The proof of Silver's theorem, in particular the use of Lemma 29.10, gives more information on the construction of $L[U]$. One can show (using normality of D) that the set of indiscernibles $I \in D$ for $L_\eta[D]$ has the property that $\pi[I]$ is a closed unbounded subset of κ. One can deduce from this that every subset of κ in $L_\alpha[D]$ contains or is disjoint from a closed unbounded set. This fact can be used to show that the canonical well-ordering of $P(\omega) \cap L[D]$ does not depend on D and that it is in fact Σ_2 over HC.

Another application of the method of proof of Theorem 75 is the proof of Jensen's principle $\diamondsuit$ in $L[D]$. The proof of $\diamondsuit$ in L, combined with the proof of Silver's theorem, shows that $L[D]$ satisfies $\diamondsuit$.

As a last remark, if j is an elementary embedding of the universe, there is a least model M, containing all ordinals, such that $j \mid M$ is a class in M and is an elementary embedding. (This has to be formulated in Gödel–Bernays theory of sets and classes rather than in ZF.) This model, denoted $L(j)$, is obtained by constructing a hierarchy $L_\alpha(j)$, using Gödel operations and in addition closing under the function j. The model $L(j)$ is not necessarily equal to the model $L[U]$, and it is not known whether it has some nice properties like GCH.

Iterated Ultrapowers

Let κ be a measurable cardinal and let U be a κ-complete nonprincipal ultrafilter over κ. Using U, we construct an ultrapower of V modulo U; and since the ultrapower is well-founded, we identify the ultrapower with its transitive collapse, a transitive model $M = Ult_U(V)$. Let us denote this transitive model $Ult_U^{(1)}(V)$ or just $Ult^{(1)}$. Let $j = j_U$ be the natural embedding of V in $Ult^{(1)}$, and let $\kappa^{(1)} = j(\kappa)$ and $U^{(1)} = j(U)$.

In the model $Ult^{(1)}$, the ordinal $\kappa^{(1)}$ is a measurable cardinal and $U^{(1)}$ is a $\kappa^{(1)}$-complete nonprincipal ultrafilter over $\kappa^{(1)}$. Thus, working inside $Ult^{(1)}$, we can construct an ultrapower modulo $U^{(1)}$: $Ult_{U^{(1)}}(Ult^{(1)})$. Let us denote this ultrapower $Ult^{(2)}$, and let $j^{(1)}$ be the natural embedding of $Ult^{(1)}$ in $Ult^{(2)}$ given by this ultrapower. Let $\kappa^{(2)} = j^{(1)}(\kappa^{(1)})$ and $U^{(2)} = j^{(1)}(U^{(1)})$.

We can continue this procedure and obtain transitive models

$$Ult^{(1)}, \quad Ult^{(2)}, \quad \ldots, \quad Ult^{(n)}, \quad \ldots \qquad (n < \omega)$$

[That we can indeed construct such a sequence of classes follows from the observation that for each α, the initial segment $V_\alpha \cap Ult^{(n)}$ of each ultrapower in the sequence is defined from an initial segment V_β of the universe (where β is something like $\kappa + \alpha + 1$).]

Thus we get a sequence of models $Ult^{(n)}$, $n < \omega$. For any $n < m$, we have an elementary embedding i_{nm}: $Ult^{(n)} \to Ult^{(m)}$ which is just the composition of the embeddings $j^{(n)}, j^{(n+1)}, \ldots, j^{(m-1)}$:

$$(31.2) \qquad i_{nm}(x) = j^{(m-1)}j^{(m-2)} \cdots j^{(n)}(x) \qquad (x \in Ult^{(n)})$$

For $n = m$, let i_{nm} be the identity mapping on $Ult^{(n)}$.

Also, let $Ult^{(0)}$ denote the universe V.

These embeddings form a commutative system; that is,

$$(31.3) \qquad i_{mk} \circ i_{nm} = i_{nk} \qquad (m < n < k)$$

We also let $\kappa^{(n)} = i_{0n}(\kappa)$, and $U^{(n)} = i_{0n}(U)$. Note that $\kappa^{(0)} < \kappa^{(1)} < \cdots < \kappa^{(n)} < \cdots$, and $Ult^{(0)} \supset Ult^{(1)} \supset \cdots \supset Ult^{(n)} \supset \cdots$. See Fig. 31.1.

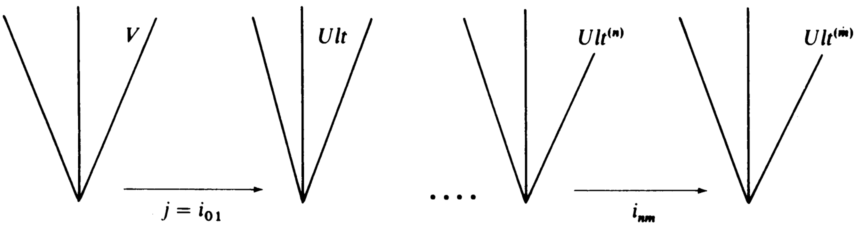

FIGURE 31.1

We shall now define a *direct limit* of the system

$$(31.4) \qquad \{(Ult^{(n)}, \in); i_{nm}\}_{n,m<\omega}$$

a model (M, E), together with elementary embeddings $i_{n\omega}$: $Ult^{(n)} \to M$ which commute with all the embeddings i_{nm}. [Recall the construction of the direct limit of a sequence of complete Boolean algebras. The underlying idea in the present situation is the same (and can be formulated algebraically): If the embeddings i_{nm} were inclusion mappings, i.e., $i_{nm}(x) = x$ for all x, then we would take M to be the union of the $Ult^{(n)}$, $n < \omega$.]

Let us call a *thread* a function t whose domain is the set $\{k, k+1, \ldots, n, \ldots\}_{k \le n < \omega}$ and such that $t(k) \in Ult^{(k)}$ and for all $n \ge k$, $t(n) = i_{kn}(t(k))$. By abuse of notation, let us use

$$(31.5) \qquad \langle x_k, x_{k+1}, \ldots \rangle$$

where $x = t(k)$, etc., to denote the thread t. Let us introduce an equivalence on threads: Two threads $\langle x_{k+1}, x_{k+1}, \ldots \rangle$ and $\langle y_l, y_{l+1}, \ldots \rangle$ (where $k \le l$) are *equivalent* if $x_l = y_l$. Note that in each equivalence class of threads there exists one *maximal thread*; moreover, a thread (31.5) is maximal just in case there is no $x \in Ult^{(k-1)}$ such that $x_k = i_{k-1,k}(x)$.

Let M be the class of all maximal threads. We define a relation E over M as follows: If t and s are maximal threads, then

$$(31.6) \qquad t\ E\ s \quad \text{iff} \quad \text{for some } n,\ t(n) \in s(n)$$

It should be clear that (31.6) holds just in case $t(n) \in s(n)$ for all $n \geq k$ where k is the least element of $\text{dom}(t) \cap \text{dom}(s)$.

For each $n < \omega$, we define a mapping $i_{n\omega}$ from $Ult^{(n)}$ into M as follows: For each $x \in Ult^{(n)}$, consider the thread $\langle x_n, x_{n+1}, \ldots\rangle$ where for each $k \geq n$, $x_n = i_{nk}(x)$. Let $i_{n\omega}(x)$ be the maximal thread that extends $\langle x_n, x_{n+1}, \ldots\rangle$. See Fig. 31.2.

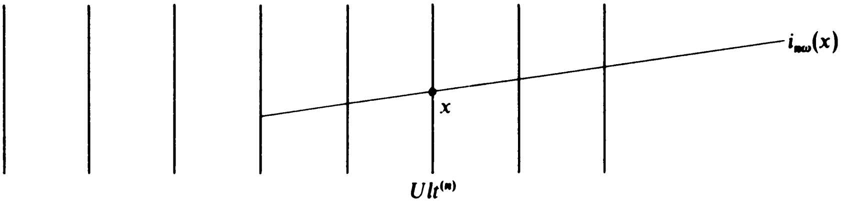

FIGURE 31.2

We shall show that for each n, the mapping $i_{n\omega}$: $Ult^{(n)} \to (M, E)$ is an elementary embedding. We prove this by induction on complexity of formulas, simultaneously for all n.

Clearly, $i_{n\omega}$ preserves both $=$ and $\in$:

$$(31.7) \qquad \begin{aligned} i_{n\omega}(x) = i_{n\omega}(y) \quad &\text{iff} \quad x = y \\ i_{n\omega}(x)\ E\ i_{n\omega}(y) \quad &\text{iff} \quad x \in y \end{aligned}$$

The induction step is easy to prove for propositional connectives. Thus consider a formula $\exists z\ \varphi(z, x, \ldots)$. Let $n < \omega$, let $x, \ldots \in Ult^{(n)}$, and assume that

$$(31.8) \qquad M \vDash \exists z\ \varphi(z, i_{n\omega}(x), \ldots)$$

We want to show that

$$(31.9) \qquad Ult^{(n)} \vDash \exists z\ \varphi(z, x, \ldots)$$

Let t be some maximal thread such that $M \vDash \varphi(t, \ldots)$. If $n \in \text{dom}(t)$, then we simply let $z = t(n)$, and have, by the induction hypothesis, $Ult^{(n)} \vDash \varphi(z, \ldots)$. Otherwise, we consider some $m > n$ such that $m \in \text{dom}(t)$. Since $i_{n\omega}(x) = i_{m\omega}(i_{nm}(x))$, we have

$$Ult^{(m)} \vDash \exists z\ \varphi(z, i_{nm}(x))$$

and since i_{nm}: $Ult^{(n)} \to Ult^{(m)}$ is elementary, (31.9) follows. ■

Thus (M, E) is a model of ZFC, elementarily equivalent to each $Ult^{(n)}$ (hence to V); the mappings $i_{n\omega}$ are elementary and commute with the i_{nm}.

Moreover, for each $x \in M$ there exists $n < \omega$ and $x_n \in Ult^{(m)}$ such that $x = i_{n\omega}(x_n)$. We call (M, E) the *direct limit* of the system (31.4):

$$(M, E) = \lim_{n \to \omega} \text{dir}\, \{Ult^{(n)};\, i_{nm}\} \tag{31.10}$$

It is clear that the above construction works in general, for any system of models and elementary embeddings

$$\{(M_n, E_n);\, i_{nm}\}_{n,m < \omega}$$

Moreover, there is no reason to restrict ourselves to an ω-sequence of models. Let us assume that we have a system

$$\{(M_\alpha, E_\alpha);\, i_{\alpha\beta}\}_{\alpha,\beta < \lambda}$$

of models and elementary embeddings $i_{\alpha\beta}$: $M_\alpha \to M_\beta$ where λ is a limit ordinal and the $i_{\alpha\beta}$ commute. Then we define the direct limit

$$(M_\lambda, E_\lambda) = \lim_{\alpha \to \lambda} \text{dir}\, \{M_\alpha;\, i_{\alpha\beta}\} \tag{31.11}$$

analogously: *Threads* are now functions t with domain $\{\alpha : \alpha_0 \le \alpha < \lambda\}$ and such that $t_\alpha \in M_\alpha$ for each $\alpha \in \text{dom}(t)$ and $t(\beta) = i_{\alpha\beta}(t(\alpha))$. Again, we obtain elementary embeddings $i_{\alpha\lambda}$: $M_\alpha \to M_\lambda$ that commute with the $i_{\alpha\beta}$ and each $x \in M_\lambda$ is in the range of some $i_{\alpha\lambda}$.

There is no reason to expect that the direct limit of well-founded models is a well-founded model. However, as we shall see shortly, the direct limit of $Ult^{(n)}$ is well-founded.

Exercise 31.1. Let κ be a measurable cardinal and j: $V \to M$ be the corresponding elementary embedding. Let $M_0 = V$, $M_1 = M$, and for each $n < \omega$, $M_{n+1} = j(M_n)$ and $i_{n,n+1} = j \restriction M_n$. The direct limit of $\{M_n;\, i_{nm}\}$ is not well founded.

[$i_{0\omega}(\kappa)$, $i_{1\omega}(\kappa)$, ..., $i_{n\omega}(\kappa)$, ... is a descending sequence of ordinals in the model.]

Let us return to iterated ultrapowers. We started with a κ-complete nonprincipal ultrafilter U over κ and defined the models $Ult^{(n)}$ and elementary embeddings i_{nm}. Let us denote by $(Ult^{(\omega)}, E^{(\omega)})$ (or just $Ult^{(\omega)}$) the direct limit

$$(Ult^{(\omega)}, E^{(\omega)}) = \lim_{n \to \omega} \text{dir}\, \{(Ult^{(n)}, \in);\, i_{nm}\} \tag{31.12}$$

Let $\kappa^{(\omega)} = i_{0\omega}(\kappa)$ and $U^{(\omega)} = i_{0\omega}(U)$. Since $Ult^{(\omega)}$ satisfies that $U^{(\omega)}$ is a $\kappa^{(\omega)}$-complete nonprincipal ultrafilter over $\kappa^{(\omega)}$, we can construct, working inside the model $Ult^{(\omega)}$, the ultrapower of $Ult^{(\omega)}$ modulo $U^{(\omega)}$ and the corresponding natural embedding $j^{(\omega)}$. [This is easy to understand if $Ult^{(\omega)}$ is a transitive model. If $(Ult^{(\omega)}, E^{(\omega)})$ is not well-founded, then the ultrapower and the mapping $j^{(\omega)}$ are classes in the sense of the non-well-founded model. In general, note that a "class" **C** in a nonstandard model (M, E) is just a subclass of M, namely $\{x \in M : M \vDash x$ belongs to $\mathbf{C}\}$, and similarly, a "function" **F** in (M, E) is a function from M to M: $\{(x, y) \in M \times M : M \vDash \mathbf{F}(x) = y\}$.]

Let us denote by $(Ult^{(\omega+1)}, E^{(\omega+1)})$ the ultrapower of $(Ult^{(\omega)}, E^{(\omega)})$ modulo $U^{(\omega)}$ and let $i_{\omega,\omega+1}$ be the corresponding natural embedding. For each $n<\omega$, let $i_{n,\omega+1} = i_{\omega,\omega+1} \circ i_{n\omega}$. [Another way of looking at $Ult^{(\omega+1)}$: Let B be the set algebra $\{ext_{E^{(\omega)}}(X) : Ult^{(\omega)} \vDash X \subseteq \kappa^{(\omega)}\}$ and let U^* be the ultrafilter $\{ext(X) : X\ E^{(\omega)}\ U^{(\omega)}\}$. For each f such that $Ult^{(\omega)} \vDash f$ is a function on $\kappa^{(\omega)}$, let $f^* = \{(x, y) : Ult^{(\omega)} \vDash f(x) = y\}$. Then $Ult^{(\omega+1)}$ is (isomorphic to) the ultrapower of $Ult^{(\omega)}$ using U^* and only the functions f^*.]

This procedure can be continued, and so we define the *iterated ultrapower* of V modulo U as follows:

$$(31.13)\qquad \begin{aligned} (Ult^{(0)}, E^{(0)}) &= (V, \in) \\ (Ult^{(\alpha+1)}, E^{(\alpha+1)}) &= Ult_{U^{(\alpha)}}(Ult^{(\alpha)}, E^{(\alpha)}) \\ (Ult^{(\lambda)}, E^{(\lambda)}) &= \lim_{\alpha\to\lambda} \operatorname{dir} \{(Ult^{(\alpha)}, E^{(\alpha)}); i_{\alpha\beta}\} \qquad (\lambda \text{ a limit}) \end{aligned}$$

where $U^{(\alpha)} = i_{0\alpha}(U)$, for each α. We do not know yet that all the models $Ult^{(\alpha)}$ are well-founded; but we make a convention that if $Ult^{(\alpha)}$ is well-founded, then we identify it with its transitive collapse.

The first lemma about iterated ultrapowers is the Factor Lemma below. If M is a transitive model of set theory and U is (in M) a κ-complete nonprincipal ultrafilter over κ, we can construct, within M, the iterated ultrapowers. Let us denote by $Ult_U^{(\alpha)}(M)$ the αth iterated ultrapower, constructed in M. (Thus $Ult^{(\alpha)} = Ult_U^{(\alpha)}(V)$).

Lemma 31.4 (Factor Lemma). *Let us assume that $Ult^{(\alpha)}$ is well-founded. Then for each β, the iterated ultrapower $Ult_{U^{(\alpha)}}^{(\beta)}(Ult^{(\alpha)})$ taken in $Ult^{(\alpha)}$ is isomorphic to the iterated ultrapower $Ult^{(\alpha+\beta)}$.*

Moreover, there is for each β an isomorphism $e_\beta^{(\alpha)}$ such that if for all ξ and η, $i_{\xi\eta}^{(\alpha)}$ denotes the elementary embedding of $Ult_{U^{(\alpha)}}^{(\xi)}(Ult^{(\alpha)})$ into $Ult_{U^{(\alpha)}}^{(\eta)}(Ult^{(\alpha)})$, then the following diagram (Fig. 31.3) commutes:

$$\begin{tikzcd} Ult_{U^{(\alpha)}}^{(\xi)}(Ult^{(\alpha)}) \arrow[r, "i_{\xi\eta}^{(\alpha)}"] \arrow[d, "e_\xi^{(\alpha)}"'] & Ult_{U^{(\alpha)}}^{(\eta)}(Ult^{(\alpha)}) \arrow[d, "e_\eta^{(\alpha)}"] \\ Ult_U^{(\alpha+\xi)} \arrow[r, "i_{\alpha+\xi,\alpha+\eta}"'] & Ult_U^{(\alpha+\eta)} \end{tikzcd}$$

FIGURE 31.3

Proof. Although the lemma sounds complicated, it is actually quite simple, and also easy to prove. The proof is by induction on β. If $\beta = 0$, then the 0th iterated ultrapower in $Ult^{(\alpha)}$ is $Ult^{(\alpha)}$; and we let $e_0^{(\alpha)}$ be the identity mapping. If $Ult_{U^{(\alpha)}}^{(\beta)}$ and $Ult_U^{(\alpha+\beta)}$ are isomorphic and $e_\beta^{(\alpha)}$ is the isomorphism, then $Ult_{U^{(\alpha)}}^{(\beta+1)}$ and

$Ult_U^{(\alpha+\beta+1)}$ are ultrapowers of $Ult_{U^{(\alpha)}}^{(\beta)}$ and $Ult_U^{(\alpha+\beta)}$, respectively, modulo $i_{0\beta}^{(\alpha)}(U^{(\alpha)})$ and $i_{0,\alpha+\beta}(U)$, respectively; and since $i_{0,\alpha+\beta}(U) = e_\beta^{(\alpha)}(i_{0\beta}^{(\alpha)}(U^{(\alpha)}))$, the isomorphism $e_\beta^{(\alpha)}$ induces an isomorphism $e_{\beta+1}^{(\alpha)}$ between $Ult_{U^{(\alpha)}}^{(\beta+1)}$ and $Ult_U^{(\alpha+\beta+1)}$.

If λ is a limit ordinal, then $Ult_{U^{(\alpha)}}^{(\lambda)}$ is the direct limit (in $Ult^{(\alpha)}$) of $\{Ult_{U^{(\alpha)}}^{(\beta)};\, i_{\beta\gamma}^{(\alpha)}\}_{\beta,\gamma<\lambda}$ and $Ult_U^{(\alpha+\lambda)}$ is the direct limit of $\{Ult_U^{(\alpha+\beta)};\, i_{\alpha+\beta,\alpha+\gamma}\}_{\beta,\gamma<\lambda}$. It is clear that the isomorphisms $e_\beta^{(\alpha)}$, $\beta < \lambda$, induce an isomorphism $e_\lambda^{(\alpha)}$ between $Ult_{U^{(\alpha)}}^{(\lambda)}$ and $Ult_U^{(\alpha+\lambda)}$. Throughout, the commutative diagram in Fig. 31.3 is valid. ■

We shall now prove a significant lemma on iterated ultrapowers:

Lemma 31.5. *Let U be a κ-complete nonprincipal ultrafilter over κ. Then for every α, the αth iterated ultrapower $Ult^{(\alpha)}$ is well-founded.*

Proof. Clearly, if $Ult^{(\alpha)}$ is well-founded, then $Ult^{(\alpha+1)}$ is well-founded. Thus if γ is the least γ such that $Ult^{(\gamma)}$ is not well-founded, then γ is a limit ordinal. The ordinals of the model $Ult^{(\gamma)}$ are not well-ordered; let ξ be the least ordinal such that the ordinals of $Ult^{(\gamma)}$ below $i_{0\gamma}(\xi)$ are not well-ordered; see Fig. 31.4.

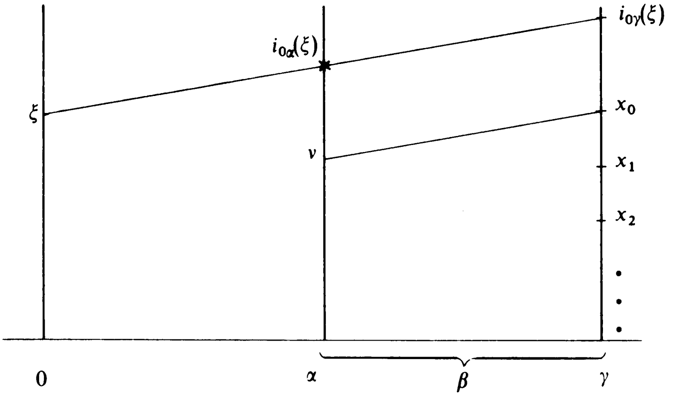

FIGURE 31.4

Let $x_0, x_1, x_2, \ldots$ be a descending sequence of ordinals in the model $Ult^{(\gamma)}$ such that x_0 is less than $i_{0\gamma}(\xi)$. Since $Ult^{(\gamma)}$ is the direct limit of $Ult^{(\alpha)}$, $\alpha < \gamma$, there is $\alpha < \gamma$ and an ordinal ν (less than $i_{0\alpha}(\xi)$) such that $x_0 = i_{\alpha\gamma}(\nu)$. Let β be such that $\alpha + \beta = \gamma$.

By our assumptions, the following is true (in V):

(31.14) $(\forall\gamma' \le \gamma)(\forall\xi' < \xi)$ the ordinals below $i_{0\gamma'}(\xi')$ in $Ult^{(\gamma')}$ are well-ordered.

When we apply the elementary embedding $i_{0\alpha}$ to (31.14), we get:

(31.15) $Ult^{(\alpha)} \vDash (\forall\gamma' \le i_{0\alpha}(\gamma))(\forall\xi' < i_{0\alpha}(\xi))$ *the ordinals below* $i_{0\gamma'}^{(\alpha)}(\xi')$ *in* $Ult_{U^{(\alpha)}}^{(\gamma')}$ *are well-ordered.*

Now $\beta \le \gamma \le i_{0\alpha}(\gamma)$, and $\nu < i_{0\alpha}(\xi)$. Hence if we let $\gamma' = \beta$ and $\xi' = \nu$ in (31.15), we get

(31.16) $Ult^{(\alpha)} \models$ the ordinals below $i^{(\alpha)}_{0\beta}(\nu)$ in $Ult^{(\beta)}_{U^{(\alpha)}}$ are well-ordered.

By the factor lemma, $Ult^{(\beta)}_{U^{(\alpha)}}$ is (isomorphic to) $Ult^{(\alpha+\beta)}$, and $i^{(\alpha)}_{0\beta}(\nu)$ is $i_{\alpha,\alpha+\beta}(\nu)$. Since $\alpha + \beta = \gamma$ and $i_{\alpha\gamma}(\nu) = x_0$, and since being well-ordered is absolute (for the transitive model $Ult^{(\alpha)}$), we have:

(31.17) The ordinals below x_0 in $Ult^{(\gamma)}$ are well-ordered.

But this is a contradiction since $x_1, x_2, x_3, \ldots$ is a descending sequence of ordinals below x_0 in $Ult^{(\gamma)}$. ∎

Thus for any given κ-complete nonprincipal ultrafilter U over κ we have a transfinite sequence of transitive models, the iterated ultrapowers $Ult^{(\alpha)}_U(V)$, and the elementary embeddings $i_{\alpha\beta}$: $Ult^{(\alpha)} \to Ult^{(\beta)}$. Let $\kappa^{(\alpha)} = i_{0\alpha}(\kappa)$ for each α; we shall show that the sequence $\kappa^{(\alpha)}$, $\alpha \in Ord$, is a normal sequence, and consequently obtain a characterization of the iterations $Ult^{(\alpha)}$ in the case $V = L[D]$.

Lemma 31.6.

(a) *If* $\gamma < \kappa^{(\alpha)}$, *then* $i_{\alpha\beta}(\gamma) = \gamma$ *for all* $\beta \ge \alpha$.
(b) *If* $X \subseteq \kappa^{(\alpha)}$ *and* $X \in Ult^{(\alpha)}$, *then* $X \subseteq i_{\alpha\beta}(X)$ *for all* $\beta \ge \alpha$; *in fact* $X = \kappa^{(\alpha)} \cap i_{\alpha\beta}(X)$.

Proof. By the factor lemma, it suffices to give the proof for $\alpha = 0$.

(a) As we know, $i_{01}(\gamma) = \gamma$ for all $\gamma < \kappa$. By induction on β, if $i_{0\beta}(\gamma) = \gamma$, then $i_{0,\beta+1}(\gamma) = i_{\beta,\beta+1}(\gamma) = \gamma$ because $\gamma < \kappa^{(\beta)}$; if λ is a limit and $i_{0\beta}(\xi) = \xi$ for all $\xi \le \gamma$ and $\beta < \lambda$, then $i_{0\lambda}(\gamma) = \gamma$.

(b) Follows from (a). ∎

Lemma 31.7. *The sequence* $\langle \kappa^{(\alpha)} : \alpha \in Ord \rangle$ *is normal; i.e., increasing and continuous.*

Proof. For each α, $\kappa^{(\alpha+1)} = i_{\alpha,\alpha+1}(\kappa^{(\alpha)}) > \kappa^{(\alpha)}$. To show that the sequence is continuous, let λ be a limit ordinal; we want to show that $\kappa^{(\lambda)} = \lim_{\alpha\to\lambda} \kappa^{(\alpha)}$. If $\gamma < \kappa^{(\lambda)}$, then $\gamma = i_{\alpha\lambda}(\delta)$ for some $\alpha < \lambda$ and $\delta < \kappa^{(\alpha)}$. Hence $\gamma = \delta$ and so $\gamma < \kappa^{(\alpha)}$. ∎

Lemma 31.8. *Let D be a normal measure on κ, and let for each α, $Ult^{(\alpha)}$ be the* αth *iterated ultrapower* mod D, $\kappa^{(\alpha)} = i_{0\alpha}(\kappa)$, *and* $D^{(\alpha)} = i_{0\alpha}(D)$. *Let λ be an infinite limit ordinal. Then for each* $X \in Ult^{(\lambda)}$, $X \subseteq \kappa^{(\lambda)}$,

$$X \in D^{(\lambda)} \quad \text{iff} \quad (\exists\alpha < \lambda) X \supseteq \{\kappa^{(\gamma)} : \alpha \le \gamma < \lambda\} \tag{31.18}$$

Proof. Since for no X can both X and its complement contain a final segment of the sequence $\langle \kappa^{(\gamma)} : \gamma < \lambda \rangle$, it suffices to show that if $X \in D^{(\lambda)}$, then there is α such that $\kappa^{(\gamma)} \in X$ for all $\gamma \ge \alpha$.

There exists $\alpha < \lambda$ such that $X = i_{\alpha\lambda}(Y)$ for some $Y \in D^{(\alpha)}$. Let me show that $\kappa^{(\gamma)} \in X$ for all γ, $\alpha \leq \gamma < \lambda$. Let $\gamma \geq \alpha$ and let $Z = i_{\alpha\gamma}(Y)$. Then $Z \in D^{(\gamma)}$ and since $D^{(\gamma)}$ is a normal measure on $\kappa^{(\gamma)}$ in $Ult^{(\gamma)}$, we have $\kappa^{(\gamma)} \in i_{\gamma,\gamma+1}(Z)$. However, $i_{\gamma,\gamma+1}(Z) \subseteq i_{\gamma+1,\lambda}(i_{\gamma,\gamma+1}(Z)) = X$ and hence $\kappa^{(\gamma)} \in X$. ■

Representation of Iterated Ultrapowers

We shall now give an alternative description of each of the models $Ult^{(\alpha)}$ by means of a single ultrapower of the universe modulo an ultrafilter on a certain Boolean algebra of subsets of ${}^{\alpha}\kappa$. This will enable us to obtain more precise information about the embeddings $i_{0\alpha}$: $V \to Ult^{(\alpha)}$ and also to generalize the iteration to the case when U is an M-ultrafilter for some model M.

We shall deal first with the finite iterations. Let U be a κ-complete nonprincipal ultrafilter over κ. Let us use the symbol $\forall^*\alpha$ for "for almost all $\alpha < \kappa$":

$$\forall^*\alpha\ \varphi(\alpha) \quad \text{iff} \quad \{\alpha < \kappa : \varphi(\alpha)\} \in U \tag{31.19}$$

If $X \subseteq {}^{n}\kappa$ and $\alpha < \kappa$, let

$$X_{(\alpha)} = \{\langle\alpha_1, \ldots, \alpha_{n-1}\rangle : \langle\alpha, \alpha_1, \ldots, \alpha_{n-1}\rangle \in X\} \tag{31.20}$$

We define ultrafilters U_n over ${}^{n}\kappa$, by induction on n:

$$U_1 = U \tag{31.21}$$

$$X \in U_{n+1} \quad \text{iff} \quad \forall^*\alpha\ X_{(\alpha)} \in U_n$$

Each U_n is a nonprincipal κ-complete ultrafilter over ${}^{n}\kappa$, and if $Z \in U$, then ${}^{n}Z \in U_n$. It is easy to see that for all $X \subseteq {}^{n}\kappa$,

$$X \in U_n \quad \text{iff} \quad \forall^*\alpha_0\ \forall^*\alpha_1 \cdots \forall^*\alpha_{n-1}\ \langle\alpha_0, \ldots, \alpha_{n-1}\rangle \in X$$

Although the present definition varies slightly from definition (29.10), it is practically the same because for each n, U_n concentrates on *increasing* n-sequences:

$$\{\langle\alpha_0, \ldots, \alpha_{n-1}\rangle \in {}^{n}\kappa : \alpha_0 < \cdots < \alpha_{n-1}\} \in U_n \tag{31.22}$$

(because $\forall\alpha_0(\forall\alpha_1 > \alpha) \cdots (\forall\alpha_{n-1} > \alpha_{n-2})[\alpha_0 < \cdots < \alpha_{n-1}]$).

Lemma 31.9. *For every n,*

$$Ult_{U_n}(V) = Ult^{(n)}(V)$$

and $j_{U_n} = i_{0n}$.

[Here j_{U_n} is the natural embedding j: $V \to Ult_{U_n}(V)$.]

Proof. By induction on n. The case $n = 1$ is trivial. Let us assume that the lemma is true for n and let us consider $Ult_{U_{n+1}}$. Let f be a function on ${}^{n+1}\kappa$. For each $t = \langle\alpha_0, \ldots, \alpha_{n-1}\rangle \in {}^{n}\kappa$, let $f_{(t)}$ be the function on κ defined by $f_{(t)}(\xi) = f(\alpha_0, \ldots, \alpha_{n-1}, \xi)$, and let F be a function on ${}^{n}\kappa$ such that $F(t) = f_{(t)}$ for all

$t \in {}^n\kappa$. In $Ult_{U_n} = Ult^{(n)}$, the function F represents a function on $j_{U_n}(\kappa) = \kappa^{(n)}$: Let $\tilde{f} = [F]_{U_n}$. This way we assign to each function f on ${}^{n+1}\kappa$ a function $\tilde{f} \in Ult^{(n)}$ on $\kappa^{(n)}$.

Conversely, if $h \in Ult^{(n)}$ is a function on $\kappa^{(n)}$, there is f on ${}^{n+1}\kappa$ such that $h = \tilde{f}$: There exists F on ${}^n\kappa$ such that $\tilde{f} = [F]_{U_n}$ and that for each $t \in {}^n\kappa$, $F(t)$ is a function on κ; thus we let $f(\alpha_0, \ldots, \alpha_n)$ be the value of $F(\alpha_0, \ldots, \alpha_{n-1})$ at α_n.

We shall show that the correspondence $[f]_{U_{n+1}} \mapsto [\tilde{f}]_{U^{(n)}}$ is an isomorphism between $Ult_{U_{n+1}}$ and $Ult^{(n+1)} = Ult_{U^{(n)}}(Ult^{(n)})$. We have

$$\begin{aligned}
[f]_{U_{n+1}} = [g]_{U_{n+1}} \quad &\text{iff} \quad \forall^*\alpha_0 \cdots \forall^*\alpha_{n-1}\, \forall^*\xi \\
&\phantom{\text{iff}} \quad f(\alpha_0, \ldots, \alpha_{n-1}, \xi) = g(\alpha_0, \ldots, \alpha_{n-1}, \xi) \\
&\text{iff} \quad \forall^* t \{\xi < \kappa : f_{(t)}(\xi) = g_{(t)}(\xi)\} \in U \\
&\text{iff} \quad Ult_{U_n} \vDash \{\xi < j_{U_n}(\kappa) : \tilde{f}(\xi) = \tilde{g}(\xi)\} \in j_{U_n}(U) \\
&\text{iff} \quad Ult^{(n)} \vDash \{\xi < \kappa^{(n)} : \tilde{f}(\xi) = \tilde{g}(\xi)\} \in U^{(n)} \\
&\text{iff} \quad [\tilde{f}]_{U^{(n)}} = [\tilde{g}]_{U^{(n)}}
\end{aligned}$$

and similarly for $\in$ in place of $=$.

Thus $Ult_{U_{n+1}} = Ult^{(n+1)}$. To show that $j_{U_{n+1}} = i_{0,n+1}$, let $f = c_x$ be the constant function on ${}^{n+1}\kappa$ with value x. It follows that $\tilde{f}$ is the constant function on $\kappa^{(n)}$ with value $i_{0n}(x)$, and therefore

$$j_{U_{n+1}}(x) = [c_x]_{U_{n+1}} = i_{n,n+1}(i_{0n}(x)) = i_{0,n+1}(x) \quad \blacksquare$$

Exercise 31.2. Show that if $m \le n$, then for each f on ${}^m\kappa$, $i_{mn}([f]_{U_m}) = [g]_{U_n}$ where g is the function on ${}^n\kappa$ defined by $g(\alpha_0, \ldots, \alpha_{n-1}) = f(\alpha_0, \ldots, \alpha_{m-1})$.

The infinite iterations are described with the help of ultrafilters U_E over ${}^E\kappa$, where E ranges over finite sets of ordinal numbers. If E is a finite set of ordinals, then the order isomorphism π between $n = |E|$ and E induces, in a natural way, an ultrafilter U_E corresponding to U_n:

$$U_E = \{\pi[X] : X \subseteq {}^n\kappa\} \tag{31.23}$$

where $\pi(\langle\alpha_0, \ldots, \alpha_{n-1}\rangle) = t \in {}^E\kappa$ where $t(\pi(k)) = \alpha_k$ for all $k = 0, \ldots, n-1$.

If S is any set of ordinals and $E \subseteq S$ is a finite set, we define a mapping $in_{E,S}$ (*an inclusion map*) of $P({}^E\kappa)$ into $P({}^S\kappa)$ as follows:

$$in_{E,S}(X) = \{t \in {}^S\kappa : t \restriction E \in X\} \qquad (\text{all } X \subseteq {}^E\kappa) \tag{31.24}$$

Lemma 31.10. *If $E \subseteq F$ are finite sets of ordinals, then for each $X \subseteq {}^E\kappa$,*

$$X \in U_E \quad \textit{iff} \quad in_{E,F}(X) \in U_F$$

Proof. By induction on (m, n) where $m = |E|$ and $n = |F|$. Let $E \subseteq F$ be finite sets of ordinals. Let a be the least element of F, and let us assume that $a \in E$ (if $a \notin E$, then the proof is similar). Let $E' = E - \{a\}$ and $F' = F - \{a\}$.

If $X \subseteq {}^{E}\kappa$, let us define for each $\alpha < \kappa$, the set $X_{(\alpha)} \subseteq {}^{E'}\kappa$ as follows: $X_{(\alpha)} = \{t \mid E' : t \in X \text{ and } t(a) = \alpha\}$; for $Z \subseteq {}^{F}\kappa$, let us define $Z_{(\alpha)} \subseteq {}^{F'}\kappa$ similarly (for all $\alpha < \kappa$). It should be clear that

$$(31.25) \quad X \in U_E \quad \text{iff} \quad \forall^*\alpha\; X_{(\alpha)} \in U_{E'}; \qquad Z \in U_F \quad \text{iff} \quad \forall^*\alpha\; Z_{(\alpha)} \in U_{F'}$$

Now we observe that if $Z = in_{E,F}(X)$, then $Z_{(\alpha)} = in_{E',F'}(X_{(\alpha)})$, and the lemma for E, F follows from (31.25) and the induction hypothesis. ■

Let us now consider an ordinal number α. If $E \subseteq \alpha$ is a finite set, let us say that a set $Z \subseteq {}^{\alpha}\kappa$ has *support* E if

$$Z = in_{E,\alpha}(X)$$

for some $X \subseteq {}^{E}\kappa$. Note that if Z has support E and $E \subseteq F$, then Z also has support F. Let B_α denote the collection of all subsets of ${}^{\alpha}\kappa$ that have finite support. $(B_\alpha, \subseteq)$ is a Boolean algebra.

Let U_α be the following ultrafilter on B_α: For each $Z \in B_\alpha$, if $Z = in_{E,\alpha}(X)$ where $X \subseteq {}^{E}\kappa$, let

$$(31.26) \qquad Z \in U_\alpha \quad \text{iff} \quad X \in U_E$$

By Lemma 31.10, the definition of U_α does not depend on the choice of support E of Z.

We shall now construct an ultrapower modulo U_α. If f is a function on ${}^{\alpha}\kappa$, let us say that f has a *finite support* $E \subseteq \alpha$ if $f(t) = f(s)$ whenever t and $s \in {}^{\alpha}\kappa$ are such that $t \mid E = s \mid E$. (In other words, there is g on ${}^{E}\kappa$ such that $f(t) = g(t \mid E)$ for all $t \in {}^{\alpha}\kappa$. Let us consider only functions f on ${}^{\alpha}\kappa$ with finite support and define

$$(31.27) \qquad \begin{aligned} f =_\alpha g \quad &\text{iff} \quad \{t : f(t) = g(t)\} \in U_\alpha \\ f\, E_\alpha\, g \quad &\text{iff} \quad \{t : f(t) \in g(t)\} \in U_\alpha \end{aligned}$$

The sets on the right-hand side of (31.27) have finite support, namely $E \cup F$ where E and F are respectively supports of f and g.

Let $(Ult_{U_\alpha}(V), E_\alpha)$ be the model whose elements are equivalence classes mod $=_\alpha$ of functions on ${}^{\alpha}\kappa$ with finite support.

Exercise 31.3. Prove the fundamental theorem:

$$(Ult_{U_\alpha}, E_\alpha) \vDash \varphi([f_1], \ldots, [f_n]) \quad \text{iff} \quad \{t : \varphi(f_1(t), \ldots, f_n(t)\} \in U_\alpha$$

We are now in a position to state the main lemma.

Lemma 31.11 (Representation Lemma). *For every* α, *the model* $(Ult_{U_\alpha}(V), E_\alpha)$ *is* (*isomorphic to*) *the* αth *iterated ultrapower* $Ult_U^{(\alpha)}(V)$, *and the natural embedding* $j_{U_\alpha}\colon V \to Ult_{U_\alpha}$ *is equal to* $i_{0\alpha}$. *Moreover, if* $\alpha \le \beta$ *and* $[f]_{U_\alpha} \in Ult^{(\alpha)}$, *then* $i_{\alpha\beta}([f]_{U_\alpha}) = [g]_{U_\beta}$ *where* g *is defined by* $g(t) = f(t \mid \alpha)$ *for all* $t \in {}^{\beta}\kappa$.

Proof. By induction on α. The induction step from α to $\alpha + 1$ follows closely the proof of Lemma 31.9; thus let me describe only how to assign to $[f]_{U_{\alpha+1}}$ the corresponding $[\tilde{f}]_{U^{(\alpha)}}$ in $Ult^{(\alpha+1)}$. Let f be a function on ${}^{\alpha+1}\kappa$ with support $E \cup \{\alpha\}$ where $E \subseteq \alpha$. For each $t \in {}^{\alpha}\kappa$ let $f_{(t)}(\xi) = f(t^\frown\xi)$ for all $\xi < \kappa$, and let F be a function on ${}^{\alpha}\kappa$ (with support E) such that $F(t) = f_t$ for all $t \in {}^{\alpha}\kappa$. Let $\tilde{f} = [F]_{U_\alpha}$; f is in $Ult^{(\alpha)}$ and is a function on $\kappa^{(\alpha)}$.

The induction step is rather simple for a limit ordinal λ. Let us assume that $Ult_{U_\alpha} = Ult^{(\alpha)}$ for all $\alpha < \lambda$. If f is a function on ${}^{\lambda}\kappa$ with a finite support E, we assign to f a thread in the direct limit of $\{Ult^{(\alpha)}\}_{\alpha<\lambda}$ as follows: There is g on ${}^{E}\kappa$ such that $f(t) = g(t \mid E)$ for all $t \in {}^{\lambda}\kappa$. Let α_0 be some ordinal greater than all elements of E. For each $\alpha \geq \alpha_0$, we let $f \mid \alpha$ be the function on ${}^{\alpha}\kappa$ defined by $f(t) = g(t \mid E)$; each $f \mid \alpha$ has support E. Let us assign to f the thread $\langle [f \mid \alpha]_{U_\alpha} : \alpha_0 \leq \alpha < \lambda \rangle$. It is not difficult to verify that this is a thread.

Conversely, given any thread $\langle x_\beta : \alpha \leq \beta < \lambda \rangle$ we have $x_\alpha = [f_\alpha]_{U_\alpha}$ where f_α is a function on ${}^{\alpha}\kappa$ with support $E \subseteq \alpha$. Let f be the function on ${}^{\lambda}\kappa$ such that $f(t) = f_\alpha(t \mid \alpha)$. Then for each β, $\alpha \leq \beta < \lambda$, we have $x_\beta = i_{\alpha\beta}(x_\alpha) = [f \mid \beta]_{U_\beta}$, and hence $\langle x_\beta : \alpha \leq \beta < \lambda \rangle$ is assigned to f.

I leave it to the reader to verify that the described correspondence is an isomorphism, and also the equality $j_{U_\alpha} = i_{0\alpha}$ and the "moreover" clause. ■

One advantage of this description of iterated ultrapowers is that we can define $Ult_U^{(\alpha)}(M)$ even if M is a model of ZFC and U is an M-ultrafilter over γ, provided U satisfies the following condition: Let us call U an *iterable M-ultrafilter* if, in addition to (30.32), U has the following property:

$$\text{(31.28)} \qquad \text{if} \quad \langle X_\alpha : \alpha < \gamma \rangle \in M, \qquad \text{then} \quad \{\alpha : X_\alpha \in U\} \in M$$

[Note that the M-ultrafilter defined in (30.31) from $j \colon M \to N$ is iterable, provided $P^N(\kappa) = P^M(\kappa)$.]

Let M be a transitive model of ZFC containing all ordinals. Given an iterable M-ultrafilter U over γ, we can define M-ultrafilters U_n, $n < \omega$, over ${}^{n}\gamma$ by induction:

$$\text{(31.29)} \qquad X \in U_{n+1} \qquad \text{iff} \qquad \{\alpha < \gamma : X_{(\alpha)}\} \in U$$

where $X_{(\alpha)} = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \alpha, \alpha_1, \ldots, \alpha_n \rangle \in X\}$. The definition (31.19) works because U_n is iterable, which is proved by induction on n.

Then one defines M-ultrafilters U_α as in (31.26), and the ultrapowers $Ult_{U_\alpha}(M)$ as in (31.27), using functions $f \in M$ with finite support. The ultrapowers Ult_{U_α} are not necessarily well-founded.

Exercise 31.4. Each U_α is an iterable M-ultrafilter on the algebra (in M) of subsets of ${}^{\alpha}\gamma$ with finite support.

Exercise 31.5 (***Fundamental Theorem***). Prove that

$$Ult_{U_\alpha}(M) \models \varphi([f_1], \ldots, [f_n]) \qquad \text{iff} \qquad \{t : M \models \varphi(f_1(t), \ldots, f_n(t))\} \in U_\alpha$$

Exercise 31.6. If λ is a limit ordinal, then $Ult_{U_\lambda}(M)$ is the direct limit of $\langle Ult_{U_\alpha}(M); i_{\alpha\beta}\rangle_{\alpha,\beta<\lambda}$.

If the model $Ult_{U_\alpha}(M)$ is well-founded, let us define

$$U^{(\alpha)} = \{[f]_{U_\alpha} : \{t : f(t) \in U\} \in U_\alpha\}$$

Exercise 31.7. If $N_\alpha = Ult_{U_\alpha}(M)$ is well-founded, then $U^{(\alpha)}$ is an iterable N_α-ultrafilter over $i_{0\alpha}(\gamma)$.

One can also prove the factor lemma for $Ult_{U_\alpha}(M)$ if this model is well-founded.

Exercise 31.8.* If $Ult_{U_\alpha}(M)$ is well-founded for all $\alpha < \omega_1$, then $Ult_{U_\alpha}(M)$ is well-founded for all α.

[Assume that $Ult_{U_\alpha}(M)$ is not well-founded and let $f_0, f_1, \ldots, f_n, \ldots$ constitute a counterexample. Each f_n has a finite support E_n. Let β be the order-type of $\bigcup_{n=0}^{\infty} E_n$; we have $\beta < \omega_1$. Produce a counterexample in $Ult_{U_\beta}(M)$.]

An important sufficient condition for well-foundedness of iterated ultrapowers:

Exercise 31.9.* If arbitrary countable intersections of elements of U are nonempty, then $Ult_{U_\alpha}(M)$ is well-founded for all α.

[Let $f_0, f_1, \ldots, f_n, \ldots$ be a counterexample, let $X_n = \{t \in {}^\alpha\gamma : f_n(t) \ni f_{n+1}(t)\}$. To reach a contradiction, find $t \in \bigcap_{n=0}^{\infty} X_n$. Construct t by induction such that for each $\nu < \alpha$ if $\alpha = \nu + \eta$, then $t \mid \nu$ has the property that for all n, $\{s \in {}^\eta\gamma : (t \mid \nu)^\frown s \in X_n\} \in U_\eta$: Given $t \mid \nu$, there is $t(\nu)$ such that the condition is satisfied for $t \mid (\nu + 1)$. Then $t \in \bigcap_{n=0}^{\infty} X_n$.]

As an application, consider the following:

Exercise 31.10. Assume that every constructible subset of ω_1 either contains or is disjoint from a closed unbounded set. Let F be the closed unbounded filter over ω_1. Then $D = F \cap L$ is an iterable L-ultrafilter and $Ult_{D_\alpha}(L)$ is well-founded (and hence equal to L) for all α.

Uniqueness of the Model $L[D]$

Theorem 76 (Kunen).

(a) *If $V = L[D]$ and D is a normal measure on κ, then κ is the only measurable cardinal and D is the only normal measure on κ.*

(b) *For every ordinal κ, there is at most one $D \subseteq P(\kappa)$ such that $D \in L[D]$ and*

$$L[D] \models D \text{ is a normal measure on } \kappa$$

(c) *If $\kappa_1 < \kappa_2$ are ordinals and if D_1, D_2, are such that*

$$L[D_i] \models D_i \text{ is a normal measure on } \kappa_i \qquad (i = 1, 2)$$

then $L[D_2]$ is an iterated ultrapower of $L[D_1]$; i.e., there is α such that $L[D_2] = Ult_{D_1}^{(\alpha)}(L[D_1])$, and $D_2 = i_{0\alpha}(D_1)$.

The proof of Theorem 76 uses iterated ultrapowers. The following lemma uses the representation of iterated ultrapowers (Lemma 31.11) (compare with Lemma 28.9).

Lemma 31.12. *Let U be a κ-complete nonprincipal ultrafilter over κ and let, for each α, $i_{0\alpha}: V \to Ult^{(\alpha)}$ be the natural embedding of V in its* αth *iterated ultrapower.*

(a) *If α is a cardinal and $\alpha > 2^\kappa$, then $i_{0\alpha}(\kappa) = \alpha$.*
(b) *If λ is a strong limit cardinal, $\lambda > \alpha$, and if* cf $\lambda > \kappa$, *then $i_{0\alpha}(\lambda) = \lambda$.*

Proof. It follows from the representation lemma that for all ξ, η, the ordinals below $i_{0\xi}(\eta)$ are represented by functions with finite support from ${}^{\xi}\kappa$ into η and hence $|i_{0\xi}(\eta)| \leq |\xi| \cdot |\eta|^\kappa$.

(a) We have $i_{0\alpha}(\kappa) = \lim_{\xi\to\alpha} i_{0\xi}(\kappa)$, and for each $\xi < \alpha$, $|i_{0\xi}(\kappa)| \leq |\xi| \cdot 2^\kappa < \alpha$. Hence $i_{0\alpha}(\kappa) = \alpha$.

(b) Since cf $\lambda > \kappa$, every function $f: {}^{\alpha}\kappa \to \lambda$ with finite support is bounded below λ: There exists $\gamma < \lambda$ such that $f(t) < \gamma$ for all $t \in {}^{\alpha}\kappa$. Hence $i_{0\alpha}(\lambda) = \lim_{\gamma\to\lambda} i_{0\alpha}(\gamma)$. Since λ is a strong limit, we have $|i_{0\alpha}(\gamma)| < \lambda$ for all $\gamma < \lambda$ and hence $i_{0\alpha}(\lambda) = \lambda$. ■

Remark. It is clear from the proof that in (b) it is enough to assume that $\gamma^\kappa < \lambda$ for all cardinals $\gamma < \lambda$, instead of that λ is a strong limit.

Let $U \subseteq P(\kappa)$. If θ is a cardinal and $U \in L_\theta[U]$, then by absoluteness of relative constructibility, every elementary submodel of $\langle L_\theta[U], \in\rangle$ that contains U and all ordinals $< \kappa$, is isomorphic to $M = L_\alpha[U]$ for some α. (If π is the transitive collapse of the submodel, then $\pi(U) = U \cap M \in M$, and $M = L_\alpha[U]$ follows by (31.1).) Let θ be a cardinal such that $U \in L_\theta[U]$ and let us consider the model $\langle L_\theta[U], \in, U\rangle$ where U is regarded as a constant. This model has a definable well-ordering, hence definable Skolem functions, and so we can talk about Skolem hulls of subsets of $L_\theta[U]$.

Lemma 31.13. *Let $V = L[D]$ where D is a normal measure on κ. Let A be a set of ordinals of size at least κ^+ and let θ be a cardinal such that $D \in L_\theta[D]$ and $A \subseteq L_\theta[D]$. Let*

$$M \prec \langle L_\theta[D], \in, D\rangle$$

be the Skolem hull of $\kappa \cup A$. Then M contains all subsets of κ.

Hence every $X \subseteq \kappa$ is definable in $L_\theta[D]$ from $\kappa \cup A \cup \{D\}$; in other words, there is a Skolem term t such that for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\gamma_1, \ldots, \gamma_m \in A$,

$$L_\theta[D] \vDash X = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, D]$$

Proof. Let π be the transitive collapse of M. We have $\pi[M] = L_\alpha[D]$ for some α, and since $A \subseteq M$, we have necessarily $\alpha \geq \kappa^+$. By Lemma 31.1, every $X \subseteq \kappa$ is in $L_{\kappa^+}[D]$, and since π is the identity on κ, we have $X \in M$ for all $X \subseteq \kappa$. ■

The following is the key lemma in the proof of uniqueness of $L[D]$:

Lemma 31.14. *Let $D \subseteq P(\kappa)$ be such that $D \in L[D]$ and*

$$L[D] \vDash D \text{ is a normal measure on } \kappa$$

For each α, let $Ult_D^{(\alpha)}(L[D])$ denote the αth iterated ultrapower, constructed inside $L[D]$, modulo D. Let $i_{0\alpha}$ be the corresponding natural embedding.

Let λ be a regular cardinal greater than κ^+, and let F be the closed unbounded filter over λ. Then:

(a) $i_{0\lambda}(D) = F \cap Ult_D^{(\lambda)}(L[D])$;
(b) $Ult_D^{(\lambda)}(L[D]) = L[F]$.

Proof. First, we have $i_{0\lambda}(\kappa) = \lambda$ by Lemma 31.12a because $\lambda > \kappa^+ \geq (\kappa^+)^{L[D]} = (2^\kappa)^{L[D]}$. Let $D^{(\lambda)} = i_{0\lambda}(D)$ and let $M = Ult_D^{(\lambda)}(L[D])$. If $X \in D^{(\lambda)}$, then by (31.18), X contains a closed unbounded subset and hence $X \in F$. Since $D^{(\lambda)}$ is an ultrafilter in M and F is a filter, it follows that $D^{(\lambda)} = F \cap M$.

As for (b), we have

$$M = Ult^{(\lambda)}(L[D]) = L[D^{(\lambda)}] = L[F \cap M] = L[F]$$

by Lemma 15.4. ■

We shall now prove statements (a) and (b) of Theorem 76. We already know by Lemma 31.3 that in $L[D]$ κ is the only measurable cardinal. Thus (a) and (b) follow from this lemma:

Lemma 31.15. *Let $D_1, D_2 \subseteq P(\kappa)$ be such that $D_1 \in L[D_1]$, $D_2 \in L[D_2]$ and*

$$L[D_i] \vDash D_i \text{ is a normal measure on } \kappa \qquad (i = 1, 2)$$

Then $D_1 = D_2$.

Proof. Let $D_1, D_2 \subseteq P(\kappa)$ be such that $L[D_i] \vDash D_i$ is a normal measure on κ, for $i = 1, 2$; we want to show that $D_1 = D_2$. By reason of symmetry, it suffices to show that if $X \subseteq \kappa$ is in D_1, then $X \in D_2$.

Let λ be a regular cardinal greater than κ^+ and let F be the closed unbounded filter over λ. Let us consider the λth iterated ultrapowers $M_i = Ult_{D_i}^{(\lambda)}(L[D_i])$ $(i = 1, 2)$, and the corresponding natural embeddings $i_{0\lambda}^1$, $i_{0\lambda}^2$.

By Lemma 31.14, $M_1 = M_2 = L[F]$, and $i_{0\lambda}^1(D) = i_{0\lambda}^2(D_2) = F \cap L[F]$. Let $G = F \cap L[F]$.

Let A be a set of ordinals, $|A| = \kappa^+$, such that all $\gamma \in A$ are bigger than λ and that $i_{0\lambda}^1(\gamma) = i_{0\lambda}^2(\gamma) = \gamma$ for all $\gamma \in A$; such a set exists by Lemma 31.12b. Let θ be a cardinal bigger than all $\gamma \in A$ such that $i_{0\lambda}^1(\theta) = i_{0\lambda}^2(\theta) = \theta$.

Now let X be a subset of κ such that $X \in D_1$. By Lemma 31.13, X is in the Skolem hull of $\kappa \cup A$ in $\langle L_\theta[D_1], \in, D_1\rangle$. Thus there is a Skolem term t such that for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\gamma_1, \ldots, \gamma_m \in A$,

$$(31.30) \qquad L_\theta[D_1] \vDash X = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, D_1]$$

Let $Y \in L_\theta[D_2]$ be such that

$$(31.31) \qquad L_\theta[D_2] \vDash Y = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, D_2]$$

We shall show that $Y \in D_2$ and $Y = X$, hence $X \in D_2$.

First we observe that $i^1_{0\lambda}(X) = i^2_{0\lambda}(Y)$: Let $Z_1 = i^1_{0\lambda}(X)$ and $Z_2 = i^2_{0\lambda}(Y)$. We have $i^{(1)}_{0\lambda}(\alpha) = \alpha$, $i^{(1)}_{0\lambda}(\gamma) = \gamma$, $i^{(1)}_{0\lambda}(\theta) = \theta$, and $i^1_{0\lambda}(D_1) = G$; and thus when we apply $i^1_{0\lambda}$ to (31.30), we get

$$(31.32) \qquad L_\theta[G] \vDash Z_1 = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n, G]$$

Similarly, when we apply $i^2_{0\lambda}$ to (31.30), we get (31.32) with Z_2 instead of Z_1. Thus $Z_1 = Z_2$.

Now, by Lemma 31.6b, we have $X = Z_1 \cap \kappa$ and $Y = Z_2 \cap \kappa$. Hence $X = Y$.

Finally, since $i^2_{0\lambda}(Y) \in F$, it follows that $i^2_{0\lambda}(Y) \in i^2_{0\lambda}(D_2)$ and hence $Y \in D_2$. Thus $X \in D_2$ and this completes the proof of $D_1 = D_2$. ■

The key lemma in the proof of the clause (c) in Theorem 76 is the following:

Lemma 31.16. *Let κ, D be such that $L[D] \vDash D$ is a normal measure on κ, and let γ be an ordinal such that $\kappa < \gamma < i_{01}(\kappa)$, where i_{01} is the natural embedding of $L[D]$ in $Ult_D(L[D])$. Then there is no $U \subseteq P(\gamma)$ such that $L[U] \vDash U$ is a normal measure on γ.*

Proof. Let us assume that on the contrary there is such a U. Let j be the natural embedding of $L[U]$ in $Ult_U(L[U])$. Let λ be a regular cardinal greater than $|\gamma|^+$, and let F be the closed unbounded filter over λ. Let $G = F \cap L[F]$.

Since $L[U] \vDash$ GCH, we have $j(\lambda) = \lambda$ (see the remark following Lemma 28.9). In $L[U]$, G is the λth iterate of U, and in $L[j(U)]$, G is the $j(\lambda)$th iterate of $j(U)$; hence $j(G) = G$.

Let $f\colon \kappa \to \kappa$ be a function in $L[D]$ such that f represents γ in $Ult_D(L[D])$. Since D is normal, the diagonal $d(\alpha) = \alpha$ represents κ, and thus we have $(i_{01}(f))(\kappa) = \gamma$. Let $i_{0\lambda}$ be the natural embedding of $L[D]$ in $Ult^{(\lambda)}_D(L[D]) = L[G]$. It is clear that $(i_{0\lambda}(f))(\kappa) = \gamma$.

Now let A be a set of ordinals such that $|A| = \kappa^+$, that all $\xi \in A$ are greater than λ, and that $i_{0\lambda}(\xi) = \xi$ and $j(\xi) = \xi$ for all $\xi \in A$. Let θ be a cardinal greater than all $\xi \in A$, such that $i_{0\lambda}(\theta) = \theta$ and $j(\theta) = \theta$.

By Lemma 31.13, the function f is definable in $L_\theta[D]$ from $A \cup \kappa \cup \{D\}$; thus $i_{0\lambda}(f)$ is definable in $L_\theta[G]$ from $A \cup \kappa \cup \{G\}$. Hence γ is definable in $L_\theta[G]$ from $A \cup \kappa \cup \{G\} \cup \{\kappa\}$, and so there is a Skolem term t such that

$$(31.33) \qquad L_\theta[G] \vDash \gamma = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_m, G, \kappa]$$

for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\xi_1, \ldots, \xi_m \in A$.

Now we apply the elementary embedding j to (31.33); and since $j(\theta) = \theta$, $j(G) = G$, $j(\xi) = \xi$ for $\xi \in A$, and $j(\alpha) = \alpha$ for all $\alpha < \gamma$ (hence $j(\kappa) = \kappa$), we have

$$L_\theta[G] \models j(\gamma) = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_n, G, \kappa]$$

which is a contradiction because $j(\gamma) > \gamma$. ■

Proof of Theorem 76c. Let $\kappa_1 < \kappa_2$ and let D_1, D_2 be such that $L[D_i] \models D$ is a normal measure on κ_i ($i = 1, 2$). Let $i_{0\alpha}$ denote the natural embedding of $L[D_1]$ in $Ult_{D_1}^{(\alpha)}(L[D_1])$ and let α be the unique α such that $i_{0\alpha}(\kappa_1) \leq \kappa_2 < i_{0,\alpha+1}(\kappa_1)$. By Lemma 31.16 (if we let $\kappa = i_{0\alpha}(\kappa_1)$, $D = i_{0\alpha}(D_1)$, and $\gamma = \kappa_2$), it is necessary that $\kappa_2 = i_{0\alpha}(\kappa_1)$.

Now the statement follows from the uniqueness of $i_{0\alpha}(D_1)$ (Theorem 76b). ■

Thus we have proved that the model $V = L[D]$ (where D is a normal measure on κ) is unique, has only one measurable cardinal and only one normal measure on κ, and satisfies the generalized continuum hypothesis. The next lemma completes the characterization of $L[D]$ by showing that for *any* κ-complete nonprincipal ultrafilter U over κ, $L[U]$ is equal to $L[D]$.

Lemma 31.17. *Let U be a nonprincipal κ-complete ultrafilter over κ. Then $L[U] = L[D]$ where D is the normal measure on κ in $L[D]$.*

Proof. By the absoluteness of $L[D]$, we have $L[D] \subseteq L[U]$ because $L[U]$ satisfies that κ is measurable. Thus it suffices to prove that $U \cap L[D] \in L[D]$. Let $j = j_U$ be the natural embedding $j\colon V \to Ult_U(V)$, and let $\gamma = j(\kappa)$. Let $d(\alpha) = \alpha$ be the diagonal function and let δ be the ordinal represented in $Ult_U(V)$ by d; thus

$$(31.34) \qquad X \in U \quad \text{iff} \quad \delta \in j(X)$$

for all $X \subseteq \kappa$.

Since $L[j(D)] \models j(D)$ is a normal measure on γ, there exists α such that $\gamma = i_{0\alpha}(\kappa)$, $j(D) = i_{0\alpha}(D)$, and $L[j(D)] = Ult_D^{(\alpha)}(L[D])$. We shall show that for every $X \subseteq \kappa$, $X \in L[D]$,

$$(31.35) \qquad j(X) = i_{0\alpha}(X)$$

This, together with (31.34), gives

$$(31.36) \qquad U \cap L[D] = \{X \in L[D] : X \subseteq \kappa \text{ and } \delta \in i_{0\alpha}(X)\}$$

and therefore $U \cap L[D] \in L[D]$.

The proof of (31.35) again uses Lemma 31.13. We let A be a set of size κ^+ of ordinals greater than α such that $i_{0\alpha}(\xi) = j(\xi) = \xi$ for all $\xi \in A$, and let θ be a cardinal greater than all $\xi \in A$, such that $i_{0\lambda}(\theta) = j(\theta) = \theta$.

If $X \subseteq \kappa$ is in $L[D]$, then there is a Skolem term t such that

$$L_\theta[D] \models X = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_m, D]$$

for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\xi_1, \ldots, \xi_m \in A$. Since $i_{0\alpha}$ and j agree on $\kappa \cup A \cup \{\theta\}$, and $i_{0\alpha}(D) = j(D)$, it follows that $i_{0\alpha}(X) = j(X)$. ■

The proof of Lemma 31.17 gives additional information about κ-complete ultrafilters in $L[D]$. Let us assume that $V = L[D]$ and let U be a nonprincipal κ-complete ultrafilter over κ. By (31.36), we have

$$U = \{X \subseteq \kappa : \delta \in i_{0\alpha}(X)\} \tag{31.37}$$

where α is such that $j(\kappa) = i_{0\alpha}(\kappa)$, and $\delta < j(\kappa)$. Note that for any $\beta \geq \alpha$, we also have $U = \{X : \delta \in i_{0\beta}(X)\}$. Now a simple observation gives the following characterization of κ-complete ultrafilters over κ in $L[D]$:

Lemma 31.18. *Assume $V = L[D]$. Then if U is a nonprincipal κ-complete ultrafilter over κ, there exists $\delta < i_{0\omega}(\kappa)$ such that*

$$U = \{X \subseteq \kappa : \delta \in i_{0\omega}(X)\} \tag{31.38}$$

Proof. Let $j = j_U$ be the natural embedding of $V = L[D]$ in Ult_U. We have $j(\kappa) = i_{0\alpha}(\kappa)$ for some α. We shall show that α is a finite number; then the lemma follows by (31.37).

First we note that because $V = L[D] = L[U]$, we have $Ult_D^{(\alpha)} = Ult_U = L[i_{0\alpha}(D)] = L[j(U)]$. Now if $\alpha \geq \omega$, then in $Ult_D^{(\alpha)}$, $i_{0\omega}(\kappa)$ is an inaccessible cardinal (because it is measurable in $Ult_D^{(\omega)}$), while in Ult_U, $i_{0\omega}(\kappa)$ has cofinality ω (because it has cofinality ω in V and Ult_U contains all ω-sequences of ordinals). Hence $\alpha < \omega$. ■

Corollary. *If $V = L[D]$, there are exactly κ^+ nonprincipal κ-complete ultrafilters over κ.*

Proof. If κ is measurable, then it is easy to obtain 2^κ nonprincipal κ-complete ultrafilters over κ (because there are 2^κ subsets of κ of size κ such that $|X \cap Y| < \kappa$ for any two of them). By the lemma, if $V = L[D]$, there are at most $|i_{0\omega}(\kappa)| = \kappa^+$ of them. ■

Recall (Exercise 28.12) that a κ-complete ultrafilter over κ extends the closed unbounded filter just in case the diagonal function d is the least (mod U) nonconstant monotone function. If $V = L[D]$, then the characterization (31.38) shows that D is the only κ-complete ultrafilter over κ with this property; see Exercise 31.12. Let D_n be the measure on ${}^n\kappa$ defined from a normal measure D by (31.21). Let $\kappa^{(n)} = i_{0n}(\kappa)$ (where i_{0n}: $V \to Ult_D^{(n)}$).

Exercise 31.11. The ordinal $\kappa^{(n-1)}$ is represented in Ult_D by the function $d_n(\alpha_1, \ldots, \alpha_n) = \alpha_n$.

Exercise 31.12. Assume $V = L[D]$. If U is a κ-complete nonprincipal ultrafilter over κ and if $U \neq D$, then there is a monotone function f: $\kappa \to \kappa$ such that $\kappa \leq [f]_U < [d]_U$. (Hence U does not extend the closed unbounded filter.)

[U satisfies (31.38) for some δ; if $\delta = \kappa^{(n)}$ for some n, then $U = D$. Let n be such that

$\kappa^{(n-1)} < \delta < \kappa^{(n)}$; let $g\colon {}^n\kappa \to \kappa$ represent δ in Ult_{D_n}. Let $f(\xi) =$ least α such that $g(\alpha_1, \ldots, \alpha_{n-1}, \alpha) \geq \xi$ for some $\alpha_1 < \cdots < \alpha_{n-1} < \alpha$. The function f is monotone. To show that $[f]_U < [d]_U$, we argue as follows: For almost all (mod D_n) $\alpha_1, \ldots, \alpha_n$, $g(\vec{\alpha}) > \alpha_n$; hence for almost all $\alpha_1, \ldots, \alpha_n$, $f(g(\vec{\alpha})) < g(\vec{\alpha})$. Hence $(j_{D_n}(f))(\delta) < \delta$, and hence for almost all ξ (mod U), $f(\xi) < \xi$. Thus $[f]_U < [d]_U$.]

Some Generalizations

The uniqueness theorem for $L[D]$ can be generalized to the situation involving several measurable cardinals. If A_α, $\alpha < \theta$, is a sequence of sets, let us define the model

$$L\langle A_\alpha : \alpha < \theta\rangle \tag{31.39}$$

as the model $L[A]$ where

$$A = \{(\alpha, X) : X \in A_\alpha\}$$

Under this definition, $L\langle A_\alpha : \alpha < \theta\rangle = L[\langle B_\alpha : \alpha < \theta\rangle]$, where $B_\alpha = A_\alpha \cap L\langle A_\alpha : \alpha < \theta\rangle$ for all $\alpha < \theta$.

If κ_α, $\alpha < \theta$, is a sequence of measurable cardinals, and for each α, U_α is a κ_α-complete nonprincipal ultrafilter over κ_α, then in $L\langle U_\alpha : \alpha < \theta\rangle$, each $U_\alpha \cap L\langle U_\alpha : \alpha < \theta\rangle$ is again a κ_α-complete nonprincipal ultrafilter over κ_α.

The results of the present section can be generalized as follows: If $\langle\kappa_\alpha : \alpha < \theta\rangle$ is an increasing sequence of measurable cardinals, and if $\theta < \kappa_0$, then for any sequence $\langle D_\alpha : \alpha < \theta\rangle$ of normal measures on the κ_α, the model $M = L\langle D_\alpha : \alpha < \theta\rangle$ has the property that the κ_α are the only measurable cardinals and for each α, $D_\alpha \cap M$ is the only normal measure on κ_α.

The following exercise gives a canonical model with two normal measures, under the assumption of a normal measure concentrating on measurable cardinals. It can be proved that in M there are only two normal measures on κ.

Let D be a normal measure on κ such that $X = \{\alpha < \kappa : \alpha$ is measurable$\} \in D$. For each $\alpha < \kappa$, let U_α be a normal measure on α, and let U be the measure on κ in Ult_D represented by the function $\alpha \mapsto U_\alpha$; hence $Z \in U$ iff $\{\alpha : Z \cap \alpha \in U_\alpha\} \in D$. Let $M = L\langle U_\alpha, U, D : \alpha \in X\rangle$.

Exercise 31.13. In M, $U \cap M$ and $D \cap M$ are two distinct normal measures on κ.

[$Z \in U \cap M$ iff $\{\alpha : Z \cap \alpha \in U_\alpha \cap M\} \in D \cap M$. Hence $U \cap M$ is in the ultrapower $Ult_{D \cap M}(M)$; while $D \cap M$ is not.]

Indiscernibles for $L[D]$

If there exist two measurable cardinals, $\kappa < \lambda$, then it is possible to prove analogous theorems for the model $L[D]$ as we did in Section 30 for L under the assumption of one measurable cardinal. More specifically, one can prove existence of a closed unbounded set $I \subset \kappa$ and a closed unbounded class J of ordinals bigger than κ, such that $I \cup J$ contains all uncountable cardinals except κ, that every $X \in L[D]$ is definable in D from $I \cup J$, and that the elements of $I \cup J$ are indiscernibles for $L[D]$ in the following sense: The truth value of

$$L[D] \models \varphi[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m]$$

is independent of the choice of $\alpha_1 < \cdots < \alpha_n \in I$ and $\beta_1 < \cdots < \beta_m \in J$. In analogy with Silver indiscernibles, the above situation can be described by means of a certain set of formulas $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, which is usually called $0^\dagger$ (*zero-dagger*).

If $0^\dagger$ exists, then one can prove consistency of the theory ZFC + "there exists a measurable cardinal"; and hence one cannot prove the relative consistency of "$0^\dagger$ exists" with ZFC + "there exists a measurable cardinal."

I shall not give details of the theory of indiscernibles for $L[D]$. Instead, let me give a short argument showing that if there are two measurable cardinals, $\kappa < \lambda$, then there is a proper class of cardinals that are inaccessible in $L[D]$.

Let U be a normal measure on λ and let for each α, $i_{0\alpha}$ be the natural embedding of V in $Ult_U^{(\alpha)}$; let $i_{\alpha\beta}$: $Ult^{(\alpha)} \to Ult^{(\beta)}$. Let **K** be the class of all cardinals α such that $\operatorname{cf} \alpha > \lambda$ and $\gamma^\lambda < \alpha$ for all $\gamma < \alpha$. By Lemma 31.12, if $\alpha \in \mathbf{K}$, then $i_{0\alpha}(\kappa) = \alpha$ and $i_{0\alpha}(\beta) = \beta$ for all $\beta \in \mathbf{K}$ greater than α. Hence if $\alpha, \beta \in \mathbf{K}$, then $i_{\alpha\beta}(\alpha) = \beta$ and $i_{\alpha\beta}(\gamma) = \gamma$ for all $\gamma \in \mathbf{K}$ that are greater than β or less than α.

Now if D is a normal measure on κ, then because $\kappa < \lambda$, we have $i_{\alpha\beta}(D) = D$ for all $\alpha, \beta \in \mathbf{K}$. Thus each $i_{\alpha\beta}$ ($\alpha, \beta \in \mathbf{K}$), restricted to $L[D]$, is an elementary embedding of $L[D]$ in $L[D]$ such that $i_{\alpha\beta}(\alpha) = \beta$ and $i_{\alpha\beta}(\gamma)$ for all $\gamma \in \mathbf{K}$ below α or above β. Using these embeddings $i_{\alpha\beta}$ (as in the proof of Lemma 30.19), one shows that the elements of **K** are indiscernibles for the model $L[D]$.

Since some elements of **K** are regular cardinals, and some are limit cardinals, it follows that all elements of **K** are inaccessible cardinals in $L[D]$.

In the above argument, it was not necessary that κ be a measurable cardinal, only that κ be measurable in $L[D]$. Thus we have proved:

Lemma 31.19. *Let κ be a measurable cardinal, and assume that:*

(31.40) *For some $\gamma < \kappa$, there is $D \subseteq P(\gamma)$ such that $L[D] \models D$ is a normal measure on γ.*

Then there are arbitrarily large successor cardinals that are inaccessible in $L[D]$.

We have proved in Lemma 31.18 that if U is a nonprincipal κ-complete ultrafilter over κ, then $j_U(\kappa) < i_{0\omega}(\kappa)$, where $i_{0\omega}$ is the natural embedding of $L[D]$ in $Ult_D^{(\omega)}(L[D])$. We can now prove a stronger statement:

Lemma 31.20. *If there is a κ-complete nonprincipal ultrafilter U over κ such that $j_U(\kappa) \geq i_{0\omega}(\kappa)$, then* (31.40) *holds.*

Proof. Let us work in the model $M = Ult_U(V)$. The cardinal $j(\kappa)$ is measurable while $i_{0\omega}(\kappa)$ has cofinality ω, and so $i_{0\omega}(\kappa) < j(\kappa)$. Let F be the collection of all subsets X of $i_{0\omega}(\kappa)$ such that $X \supseteq \{i_{0n}(\kappa) : n \geq n_0\}$ for some n_0. Using Lemma 31.8, we proceed as in the proof of Lemma 31.14 to show that

$$L[F] \models F \cap L[F] \text{ is a normal measure on } i_{0\omega}(\kappa)$$

Thus (31.40) holds in M for $j(\kappa)$. Since j is an elementary embedding, (31.40) holds in V for κ. ■

Corollary. *If κ is a measurable cardinal and $2^\kappa > \kappa^+$, then* (31.40) *holds. Consequently, it is impossible to prove consistency of $2^\kappa > \kappa^+$ relative to* ZFC + "*κ is a measurable cardinal.*"

Proof. On the one hand, $|i_{0\omega}(\kappa)| = (\kappa^+)^{L[D]} \leq \kappa^+$; on the other hand, if U is any κ-complete nonprincipal ultrafilter over κ, we have $j_U(\kappa) > 2^\kappa > \kappa^+$. ■

We shall return to the problem of $2^\kappa > \kappa^+$ at a measurable cardinal in Section 36.

32. LARGE CARDINALS BELOW A MEASURABLE CARDINAL

Weakly Compact Cardinals

In Section 29 we defined weakly compact cardinals as the uncountable cardinals satisfying the partition property $\kappa \to (\kappa)^2$. We have shown that weakly compact cardinals are inaccessible, and in fact, exactly those inaccessible cardinals that have the tree property, that is, every tree of height κ whose levels have size less than κ has a branch of length κ. We have shown that every measurable cardinal is weakly compact and moreover if D is a normal measure on κ, then the set of all weakly compact cardinals below κ is in D.

We shall now investigate weakly compact cardinals in some detail. First we give a characterization of weakly compact cardinals which explains the name "weakly compact." This aspect of weakly compact cardinals has, as many other large cardinal properties, motivation in model theory.

We shall consider infinitary languages which are generalizations of the ordinary first order language. Let κ be an infinite cardinal number. The language $\mathscr{L}_{\kappa,\omega}$ consists of:

(a) κ variables;
(b) various relation, function, and constant symbols;
(c) logical connectives and the infinitary connectives $\bigvee_{\xi<\alpha} \varphi_\xi$, $\bigwedge_{\xi<\alpha} \varphi_\xi$ for $\alpha < \kappa$ (infinite disjunction and conjunction);
(d) quantifiers $\exists v$, $\forall v$.

The language $\mathscr{L}_{\kappa,\kappa}$ is like $\mathscr{L}_{\kappa,\omega}$ except that it also contains infinitary quantifiers:

(e) $\exists_{\xi<\alpha} v_\xi$, $\forall_{\xi<\alpha} v_\xi$ for $\alpha < \kappa$.

The interpretation of the infinitary symbols of $\mathscr{L}_{\kappa,\kappa}$ is the obvious generalization of the finitary case where $\bigvee_{\xi<n}\varphi_\xi$ is $\varphi_0\vee\cdots\vee\varphi_{n-1}$, $\exists_{\xi<n}v_\xi$ stands for $\exists v_0\cdots v_{n-1}$, etc.). The language $\mathscr{L}_{\omega,\omega}$ is just the language of the first order predicate calculus.

The finitary language $\mathscr{L}_{\omega,\omega}$ satisfies the *compactness theorem*: If Σ is a set of sentences such that every finite $S\subseteq\Sigma$ has a model, then Σ has a model. Let us say that the language $\mathscr{L}_{\kappa,\kappa}$ (or $\mathscr{L}_{\kappa,\omega}$) satisfies the *weak compactness theorem* if whenever Σ is a set of sentences of $\mathscr{L}_{\kappa,\kappa}$ ($\mathscr{L}_{\kappa,\omega}$) such that $|\Sigma|\le\kappa$ and that every $S\subseteq\Sigma$ with $|S|<\kappa$ has a model, then Σ has a model. Clearly, if $\mathscr{L}_{\kappa,\kappa}$ satisfies the weak compactness theorem, then so does $\mathscr{L}_{\kappa,\omega}$ because $\mathscr{L}_{\kappa,\omega}\subseteq\mathscr{L}_{\kappa,\kappa}$.

Lemma 32.1.

(a) *If κ is a weakly compact cardinal, then the language $\mathscr{L}_{\kappa,\kappa}$ satisfies the weak compactness theorem.*

(b) *If κ is an inaccessible cardinal and if $\mathscr{L}_{\kappa,\omega}$ satisfies the weak compactness theorem, then κ is weakly compact.*

Proof. (a) The proof of the weak compactness theorem for $\mathscr{L}_{\kappa,\kappa}$ is very much like the standard proof of the compactness theorem for $\mathscr{L}_{\omega,\omega}$. Let Σ be a set of sentences of $\mathscr{L}_{\kappa,\kappa}$ of size κ such that if $S\subset\Sigma$ and $|S|<\kappa$, then S has a model. Let us assume that the language $\mathscr{L}=\mathscr{L}_{\kappa,\kappa}$ has only the symbols that occur in Σ; thus $|\mathscr{L}|=\kappa$.

First we extend the language as follows: For each formula φ with free variables v_ξ, $\xi<\alpha$, we introduce new constant symbols c_ξ^φ, $\xi<\alpha$ (*Skolem constants*); let $\mathscr{L}^{(1)}$ be the extended language. Then we do the same for each formula φ of $\mathscr{L}^{(1)}$ and obtain $\mathscr{L}^{(2)}\supset\mathscr{L}^{(1)}$. We do this for each $n<\omega$, and then let $\mathscr{L}^*=\bigcup_{n=1}^{\infty}\mathscr{L}^{(n)}$. Since κ is inaccessible, it follows that $|\mathscr{L}^*|=\kappa$. $\mathscr{L}^*$ has the property that for each formula φ with free variables v_ξ, $\xi<\alpha$, there are in $\mathscr{L}^*$ constant symbols c_ξ^φ, $\xi<\alpha$ (which do not occur in φ).

For each $\varphi(v_\xi,\ldots)_{\xi<\alpha}$ let σ_φ be the sentence (a *Skolem sentence*)

$$\exists_{\xi<\alpha} v_\xi\, \varphi(v_\xi,\ldots)_{\xi<\alpha}\rightarrow\varphi(c_\xi^\varphi,\ldots)_{\xi<\alpha} \tag{32.1}$$

and let $\Sigma^*=\Sigma\cup\{\sigma_\varphi:\varphi \text{ a formula of } \mathscr{L}^*\}$.

Note that if $S\subset\Sigma^*$ and $|S|<\kappa$, then S has a model: Take a model for $S\cap\Sigma$ (for $\mathscr{L}$) and then expand it to a model for $\mathscr{L}^*$ by interpreting the Skolem constants so that each sentence (32.1) is true.

Let $\{\sigma_\alpha:\alpha<\kappa\}$ be an enumeration of all the sentences in $\mathscr{L}^*$. Let $(T,\subseteq)$ be the binary κ-tree consisting of all $t\colon\gamma\to\{0,1\}$, $\gamma<\kappa$, such that:

(32.2) there exists a model $\mathfrak{A}$ of $\Sigma\cap\{\sigma_\alpha:\alpha\in\operatorname{dom}(t)\}$ such that $\forall\alpha\in\operatorname{dom}(t)$

$$t(\alpha)=1 \qquad\text{iff}\qquad \mathfrak{A}\vDash\sigma_\alpha$$

Since κ has the tree property, there exists a branch B in T of length κ. Let

$$\Delta = \{\sigma_\alpha : t(\alpha) = 1 \text{ for some } t \in B\}$$

Clearly, $\Sigma^* \subset \Delta$. Let A_0 be the set of all constant terms of $\mathscr{L}^*$, and let $\approx$ be the equivalence relation over A_0 defined by

$$\tau_1 \approx \tau_2 \qquad \text{iff} \qquad (\tau_1 \approx \tau_2) \in \Delta$$

and let $A = A_0/\approx$.

We make A into a model $\mathfrak{A}$ for $\mathscr{L}^*$ as follows:

$$\mathfrak{A} \models P[[\tau_1], \ldots, [\tau_n]] \qquad \text{iff} \qquad P(\tau_1, \ldots, \tau_n) \in \Delta$$

and similarly for function and constant symbols. The proof is then completed by showing that $\mathfrak{A}$ is a model for Δ (and hence for Σ). The proof of

$$\text{(32.3)} \qquad \mathfrak{A} \models \sigma \qquad \text{iff} \qquad \sigma \in \Delta$$

is done by induction on the number of quantifier blocks in σ: If $\sigma = \exists_{\xi<\alpha} v_\xi\, \varphi(v_\xi, \ldots)$, then by induction hypothesis we have

$$\mathfrak{A} \models \sigma(c^\varphi_\xi, \ldots)_{\xi<\alpha} \qquad \text{iff} \qquad \sigma(c^\varphi_\xi, \ldots)_{\xi<\alpha} \in \Delta$$

and (32.3) follows.

(b) Let κ be inaccessible and assume that the language $\mathscr{L}_{\kappa,\omega}$ satisfies the weak compactness theorem. We shall show that κ has the tree property. Let $(T, <)$ be a tree of height κ such that each level of T has size $< \kappa$. Let us consider the $\mathscr{L}_{\kappa,\omega}$ language with one unary predicate B and constant symbols c_x, for all $x \in T$. Let Σ be the following set of sentences:

$$\text{(32.4)} \qquad \neg(B(c_x) \wedge B(c_y)) \qquad \text{for all } x, y \in T \text{ that are incomparable}$$

$$\bigvee_{x \in U_\alpha} B(c_x) \qquad \text{for all } \alpha < \kappa, \text{ where } U_\alpha \text{ is the } \alpha\text{th level of } T$$

(Σ says that B is branch in T of length κ). If $S \subset \Sigma$ and $|S| < \kappa$, then we get a model for S by taking a sufficiently large initial segment of T and some branch in this segment. By the weak compactness theorem for $\mathscr{L}_{\kappa,\omega}$, Σ has a model, which obviously yields a branch of length κ. ■

The weak compactness property for $\mathscr{L}_{\kappa,\kappa}$ alone does not imply that κ is inaccessible; but it does imply that κ is at least weakly inaccessible:

Exercise 32.1. If $\kappa = \lambda^+$ is a successor cardinal, then the weak compactness theorem for $\mathscr{L}_{\kappa,\omega}$ is false.

[Consider constants c_α, $\alpha \le \kappa$, a binary relation $<$ and a ternary relation R. Consider the sentences saying that (a) $<$ is a linear ordering; (b) $c_\alpha < c_\beta$; and (c) each f_x is a function, where $f_x(y) = z$ stands for $R(x, y, z)$. Let Σ consist of these sentences, the sentence $z < x \to \exists y\, R(x, y, z)$ (saying that $\operatorname{ran}(f_x) \supseteq \{z : z < x\}$), and the infinitary sentence $R(x, y, z) \to \bigvee_{\xi<\lambda} (y = c_\xi)$ (saying that $\operatorname{dom}(f_x) \subseteq \{c_\xi : \xi < \lambda\}$). Show that each $S \subset \Sigma$, $|S| \le \lambda$, has a model, but Σ does not.]

Exercise 32.2. If κ is a singular cardinal, then the weak compactness theorem for $\mathscr{L}_{\kappa,\omega}$ is false.

[Let $A \subset \kappa$ be a cofinal subset of size $< \kappa$. Consider constants c_α, $\alpha \le \kappa$, and a linear ordering $<$. There is Σ that says on the one hand that $\{c_\alpha : \alpha \in A\}$ is cofinal in the universe, and on the other hand that $c_\kappa >$ all c_α, $\alpha < \kappa$; and each $S \subset \Sigma$, $|S| < \kappa$, has a model.]

We have given three different characterizations of weak compactness of inaccessible cardinals: the partition property $\kappa \to (\kappa)^2$, the tree property, and the weak compactness of $\mathscr{L}_{\kappa,\kappa}$. Exercises 32.3 and 32.4 give yet another equivalence; Exercise 32.5 is just a formulation of what was proved in the course of proof of Lemma 29.6.

*Exercise 32.3.** If κ is weakly compact and if $(\mathscr{B}, \subseteq)$ is a κ-complete algebra of subsets of κ such that $|\mathscr{B}| = \kappa$, then every κ-complete filter F on $\mathscr{B}$ can be extended to a κ-complete ultrafilter on $\mathscr{B}$.

[Consider constants c_X for all $X \in \mathscr{B}$, and a unary predicate U. Let Σ be the following set of $L_{\kappa,\omega}$-sentences: $\neg U(c_\varnothing)$, $U(c_X) \vee U(c_{\kappa - X})$ for all $X \in \mathscr{B}$, $U(c_X) \to U(c_Y)$ for all $X \subseteq Y$ in $\mathscr{B}$, $U(c_X)$ for all $X \in F$, and $\bigwedge_{X \in \mathscr{A}} U(c_X) \to U(c_{\bigcap \mathscr{A}})$ for all $\mathscr{A} \subset \mathscr{B}$ such that $|\mathscr{A}| < \kappa$. Show that Σ has a model.]

Exercise 32.4. If κ is inaccessible and if every κ-complete filter on any κ-complete algebra $\mathscr{B}$ of subsets of κ such that $|\mathscr{B}| = \kappa$ can be extended to a κ-complete ultrafilter, then κ is weakly compact.

[Proceed as in the proof of Lemma 29.7.]

Exercise 32.5. If $(P, <)$ is a linearly ordered set of size κ, and κ is weakly compact, then there is a subset $W \subset P$ of size κ that is either well-ordered or conversely well-ordered by $<$.

The characterization of weak compactness from Exercise 32.3 can be used to obtain the following interesting result:

*Exercise 32.6.** If κ is weakly compact and if $|(\kappa^+)^L| = \kappa$, then $0^\#$ exists.

[Let $\mathscr{B}$ be the least κ-complete algebra of subsets of κ closed under inverses of constructible functions $f\colon \kappa \to \kappa$; we have $|\mathscr{B}| = \kappa$. Let U be a κ-complete ultrafilter on $\mathscr{B}$ containing all final segments $\{\alpha : \alpha_0 \le \alpha < \kappa\}$. $U \cap L$ is a nonprincipal L-ultrafilter, and $Ult_{U \cap L}(L)$ is well-founded. Thus there is a nontrivial elementary embedding of L in L.]

In Theorem 77 below, we prove equivalence of weak compactness with Π_1^1-indescribability, which will enable us to obtain further interesting results on weakly compact cardinals.

Let $n > 0$ be a natural number and let us consider the nth order predicate calculus. There are variables of orders $1, 2, \ldots, n$, and the quantifiers are applied to variables of all orders. An nth *order formula* contains, in addition to first order symbols and higher order quantifiers, predicates $X(z)$ where X and z are variables of order $k + 1$ and k respectively (for any $k < n$). Satisfaction for an nth order formula in a model $\mathfrak{A} = \langle A, P, \ldots, f, \ldots, c, \ldots\rangle$ is defined as follows: Variables of first order are interpreted as elements of the set A, variables of second order as elements of $P(A)$ (as subsets of A), etc.; variables of order n are interpreted as elements of $P^{n-1}(A)$. The predicate $X(z)$ is interpreted as $z \in X$. A Π^n_m *formula* is a formula of order $n + 1$ of the form

$$\underbrace{\forall X\ \exists Y \cdots}_{m \text{ quantifiers}} \psi \tag{32.5}$$

where X, Y, ... are $(n + 1)$th order variables and ψ is such that all quantified variables are of order at most n. Similarly, a Σ^n_m formula is as in (32.5), but with $\exists$ and $\forall$ interchanged.

We shall often exhibit a sentence σ and claim that it is Π^n_m (or Σ^n_m) although it is only *equivalent to* a Π^n_m (or Σ^n_m) sentence, in the following sense: We are considering a specific type of models in which σ is interpreted (e.g., the models $\langle V_\alpha, \in\rangle$) and there is a Π^n_m (or Σ^n_m) sentence $\tilde{\sigma}$ such that the equivalence $\sigma \leftrightarrow \tilde{\sigma}$ holds in all these models.

Note that every first order formula is equivalent to some Π^0_n formula (and also to some Σ^0_k formula).

A cardinal κ is Π^n_m*-indescribable* if whenever $U \subseteq V_\kappa$ and σ is a Π^n_m sentence such that $\langle V_\kappa, \in, U\rangle \vDash \sigma$, then for some $\alpha < \kappa$, $\langle V_\alpha, \in, U \cap V_\alpha\rangle \vDash \sigma$. Here U is considered a unary predicate.

Lemma 32.2. *Every measurable cardinal is* Π^2_1*-indescribable.*

Proof. Let κ be a measurable cardinal, let $U \subseteq V_\kappa$ and let σ be a Π^2_1 sentence of the (third order) language $\{\in, U\}$. Let us assume that $\langle V_\kappa, \in, U\rangle \vDash \sigma$.

We have $\sigma = \forall X\ \varphi(X)$ where X is a third order variable and $\varphi(X)$ contains only second and first order quantifiers. Thus

$$\forall X \subseteq V_{\kappa+1}\ \langle V_{\kappa+1}, \in, X, V_\kappa, U\rangle \vDash \tilde{\varphi} \tag{32.6}$$

where $\tilde{\varphi}$ is the (first order) sentence obtained from φ by replacing the first order quantifiers by the restricted quantifiers $\forall x \in V_\kappa$ and $\exists x \in V_\kappa$.

Now let D be a normal measure on κ and let $M = Ult_D(V)$. Since $(V_{\kappa+1})^M = V_{\kappa+1}$, we know that (32.6) holds also in M. Using the fact that V_κ is represented in the ultrapower by the function $\alpha \mapsto V_\alpha$, $V_{\kappa+1}$ by $\alpha \mapsto V_{\alpha+1}$, and U by $\alpha \mapsto U \cap V_\alpha$, we conclude that for almost all α,

$$\forall X \subseteq V_{\alpha+1}\ \langle V_{\alpha+1}, \in, X, V_\alpha, U \cap V_\alpha\rangle \vDash \tilde{\varphi} \tag{32.7}$$

Then, translating (32.7) back into the third order language, we obtain

$$\langle V_\alpha, \in, U \cap V_\alpha\rangle \vDash \sigma$$

for almost all, and hence for some, $\alpha < \kappa$. ■

Exercise 32.7. The least measurable cardinal is Σ^2_1-describable.

[Consider the sentence $\exists U$ (U is a κ-complete nonprincipal ultrafilter over κ).]

Lemma 32.3. *If κ is not inaccessible, then it is describable by a first order sentence, i.e., Π^0_m-describable for some m.*

Proof. Let κ be a singular cardinal, and let f be a function with $\operatorname{dom}(f) = \lambda < \kappa$ and $\operatorname{ran}(f)$ cofinal in κ. Let $U_1 = f$ and $U_2 = \{\lambda\}$, and let σ be the first order sentence saying that U_2 is nonempty and that the unique element of U_2 is the domain of U_1. Clearly, κ is describable in the sense that $\langle V_\kappa, \in, U_1, U_2\rangle \vDash \sigma$ and there is no $\alpha < \kappa$ such that $\langle V_\alpha, \in, U_1 \cap V_\alpha, U_2 \cap V_\alpha\rangle \vDash \sigma$. It is routine to find a single $U \subseteq V_\kappa$ and an $(\in, U)$-sentence $\tilde{\sigma}$ attesting to describability of κ.

If $\kappa \leq 2^\lambda$ for some $\lambda < \kappa$, there is a function f that maps $P(\lambda)$ onto κ. We let $U_1 = f$ and $U_2 = \{P(\lambda)\}$; then κ is described by the same sentence as above.

Finally, $\kappa = \omega$ is describable as follows: $\langle V_\kappa, \in\rangle \vDash \forall x\ \exists y\ x \in y$. ■

The converse of Lemma 32.3 is also true:

Exercise 32.8. Every inaccessible cardinal is Π^0_m-indescribable, for all m.

[Let $U \subseteq V_\kappa$. The model $\langle V_\kappa, \in, U\rangle$ has a countable elementary submodel M_0. Let $\alpha_0 < \kappa$ be such that $M_0 \subseteq V_{\alpha_0}$. For each n, let M_{n+1} be an elementary submodel of $\langle V_\kappa, \in, U\rangle$ such that $V_{\alpha_n} \subseteq M_{n+1}$, and let $M_{n+1} \subseteq V_{\alpha_{n+1}}$. Let $\alpha = \lim_{n\to\omega} \alpha_n$; then V_α is an elementary submodel of $\langle V_\kappa, \in, U\rangle$.]

Theorem 77 (Hanf-Scott). *A cardinal κ is Π^1_1-indescribable if and only if it is weakly compact.*

Corollary. *Every weakly compact cardinal κ is a Mahlo cardinal, and the set of Mahlo cardinals below κ is stationary.*

Proof of corollary. Let $C \subseteq \kappa$ be a closed unbounded set. Since κ is inaccessible, $\langle V_\kappa, \in, C\rangle$ satisfies the following Π^1_1 sentence

$$\neg\exists F(F \text{ is a function from some } \lambda < \kappa \text{ cofinally into } \kappa)$$
$$\text{and } C \text{ is unbounded in } \kappa.$$

By Π^1_1-indescribability, there exists a regular $\alpha < \kappa$ such that $C \cap \alpha$ is unbounded in α; hence $\alpha \in C$. Thus κ is Mahlo.

Being Mahlo is also expressible by a Π^1_1 sentence:

$$\forall X(\text{if } X \text{ is closed unbounded, then } \exists \text{ regular } \alpha \text{ in } X)$$

and so the same argument as above shows that there is a stationary set of Mahlo cardinals below κ. ■

The proof that a weakly compact cardinal is Π^1_1-indescribable uses the following property of weakly compact cardinals.

Lemma 32.4. *If κ is a weakly compact cardinal, then for every $U \subseteq V_\kappa$, the model $\langle V_\kappa, \in, U\rangle$ has a transitive elementary extension $\langle M, \in, U'\rangle$ such that $\kappa \in M$.*

[It will follow from the proof of Theorem 77 that the property in Lemma 32.4 is yet another characterization of weakly compact cardinals.]

Proof. Let Σ be the set of all $L_{\kappa,\kappa}$ sentences true in the model $\langle V_\kappa, \in, U, x\rangle_{x \in V_\kappa}$ plus the sentences

$$c \text{ is an ordinal}$$

$$c > \alpha \qquad (\text{all } \alpha < \kappa)$$

Clearly $|\Sigma| = \kappa$, and if $S \subset \Sigma$ is such that $|S| < \kappa$, then S has a model (namely V_κ, where the constant c can be interpreted as some ordinal greater than all the α's mentioned in S).

Hence Σ has a model $\mathfrak{A} = \langle A, E, U^{\mathfrak{A}}, x^{\mathfrak{A}}\rangle_{x \in V_\kappa}$; we may assume that $A \supseteq V_\kappa$, $E \cap (V_\kappa \times V_\kappa) = \in$, $U^{\mathfrak{A}} \cap V_\kappa = U$ and $x^{\mathfrak{A}} = x$ for all $x \in V_\kappa$. Moreover, $V_\kappa \prec \langle A, E, U^{\mathfrak{A}}\rangle$ because $\mathfrak{A}$ satisfies all formulas true in V_κ of all $x \in V_\kappa$. If we show that the model $\langle A, E\rangle$ is well-founded, then the lemma follows. Here we make use of the expressive power of the infinitary language $L_{\kappa,\kappa}$: We consider the sentence

$$\neg \exists v_0\, \exists v_1 \cdots \exists v_n \cdots \bigwedge_{n \in \omega} (v_{n+1} \in v_n) \tag{32.8}$$

The sentence (32.8) holds in a model $\mathfrak{A} = \langle A, E\rangle$ if and only if $\mathfrak{A}$ is well founded. Since Σ contains the sentence (32.8), every model of Σ is well founded. ■

Exercise 32.9. If κ is weakly compact and if $A \subseteq \kappa$ is not constructible, then for some $\alpha < \kappa$, $A \cap \alpha$ is not constructible.

[Let $A \subseteq \kappa$ be such that $A \cap \alpha \in L$ for all $\alpha < \kappa$. By Lemma 32.4 there is transitive $\langle M, \in, A'\rangle \succ \langle V_\kappa, \in, A\rangle$ such that $\kappa \in M$. Consider the sentence $\forall\alpha\, \exists x$($x$ is constructible and $x = A \cap \alpha$).]

Proof of Theorem 77. First we show that every Π^1_1-indescribable cardinal is weakly compact. If κ is Π^1_1-indescribable, then by Lemma 32.3, κ is inaccessible, and it suffices to show that κ has the tree property. In fact, by the proof of Lemma 29.6 (or by the proof of Lemma 32.1), it suffices to show that every binary κ-tree has a branch of length κ.

Let $(T, \subseteq)$ be a binary κ-tree, consisting of sequences $t: \gamma \to \{0, 1\}$, $\gamma < \kappa$. Clearly, for every $\alpha < \kappa$, the model $\langle V_\alpha, \in, T \cap V_\alpha\rangle$ satisfies the Σ^1_1 sentence

$$\exists B(B \subseteq T \text{ and } B \text{ is a branch of unbounded length}) \tag{32.9}$$

Namely, let $B = \{t \mid \xi : \xi < \alpha\}$ where t is any $t \in T$ with domain α. Since κ is Π^1_1-indescribable, the sentence (32.9) holds in $\langle V_\kappa, \in, T\rangle$ and hence T has a branch of length κ.

To show that a weakly compact cardinal is Π^1_1-indescribable, we use Lemma 32.4. Let κ be weakly compact, let $U \subseteq V_\kappa$ and let σ be a Π^1_1 sentence true in $\langle V_\kappa, \in, U\rangle$. We have $\sigma = \forall X\, \varphi(X)$ where X is a second order variable and φ has only first order quantifiers.

Let $\langle M, \in, U'\rangle$ be a transitive elementary extension of $\langle V_\kappa, \in, U\rangle$ such that $\kappa \in M$. Since

$$(\forall X \subseteq V_\kappa)\,\langle V_\kappa, \in, U\rangle \vDash \varphi(X)$$

and $V_\kappa^M = V_\kappa$, we have

$$\langle M, \in, U'\rangle \vDash [(\forall X \subseteq V_\kappa)\,\langle V_\kappa, \in, U' \cap \kappa\rangle \vDash \varphi(X)]$$

Therefore,

$$\langle M, \in, U'\rangle \vDash [\exists\alpha(\forall X \subseteq V_\alpha)\,\langle V_\alpha, \in, U' \cap \alpha\rangle \vDash \varphi(X)]$$

and so

$$\langle V_\kappa, \in, U\rangle \vDash [\exists\alpha(\forall X \subseteq V_\alpha)\,\langle V_\alpha, \in, U \cap \alpha\rangle \vDash \varphi(X)]$$

Hence for some $\alpha < \kappa$,

$$\langle V_\alpha, \in, U\rangle \vDash \sigma \quad \blacksquare$$

Π_1^1-indescribability can be used to obtain various results on weakly compact cardinals. For instance:

Exercise 32.10. If κ is weakly compact and if $S \subseteq \kappa$ is stationary, then there is a regular uncountable $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ. (Compare this with (24.17).)

["κ is regular" is expressible by a Π_1^1 sentence in $\langle V_\kappa, \in\rangle$ and so is "κ is uncountable." "S is stationary" is Π_1^1 in $\langle V_\kappa, \in, S\rangle$: For every C if C is closed unbounded, then $S \cap C \neq \emptyset$.]

We recall that a complete Boolean algebra B is countably generated if there is a countable set $A \subset B$ such that B is the least complete subalgebra containing A.

Exercise 32.11. If κ is weakly compact, then there is no countably generated complete Boolean algebra B such that $|B| = \kappa$.

[Assume B is such. Note that $\operatorname{sat}(B) = \kappa$. We may assume that $B = \langle \kappa, +, \cdot, -\rangle$; let $A \subset \kappa$ be a countable set of generators. Let U_1 be the set of all pairs (u, x) such that $u \in \kappa$, $x \subset \kappa$, $|x| < \kappa$, and $u = \sum x$, let $U_2 = \{A\}$. Let σ be the conjunction of these sentences: (a) B is a Boolean algebra and $B \supseteq A$ (first order), (b) $\forall x\ \exists u$ (if $x \subseteq \kappa$, then $u = \sum x$) (first order), and (c) $\forall X$[if $X \subset \kappa$ and X is a partition of B, then $\exists x(x = X)$] (here x is a first order variable; the sentence (c) is Π_1^1). Since $\langle V_\kappa, \in, U_1, U_2\rangle$ satisfies σ, there is $\alpha < \kappa$ such that $\langle V_\alpha, \in, U_1 \cap V_\alpha, U_2 \cap V_\alpha\rangle \vDash \sigma$. Then $\langle \alpha, +, \cdot, -\rangle$ is a complete subalgebra of B containing A.]

A similar argument can be used to prove that if κ is a weakly compact cardinal, then there is no simple complete Boolean algebra of size κ.

Unlike measurable (or Ramsey) cardinals, weakly compact cardinals do not contradict the axiom of constructibility (assuming that their existence is consistent with ZFC). In fact, we shall show that every weakly compact cardinal is weakly compact in L.

Lemma 32.5. *Let κ be a regular cardinal and assume that $L \vDash \kappa$ is inaccessible. If κ has the tree property, then for every $\alpha < \kappa^+$, there exists an elementary embedding j: $L_\alpha \to L_\beta$ (for some β) such that $j(\kappa) > \kappa$ and κ is the least ordinal moved by j.*

[It can be proved that the assumption that κ is inaccessible in L is redundant and follows from the other assumptions.]

Proof. Let $\alpha < \kappa^+$. We may assume that $\langle L_\alpha, \in\rangle$ is a model of ZF$^-$ (by taking a larger α' that has this property). Let $\langle X_\xi : \xi < \kappa\rangle$ be an enumeration of $P(\kappa) \cap L_\alpha$.

For each t: $\gamma \to \{0, 1\}$, $\gamma < \kappa$, let

$$X_t = \bigcap_{\xi<\gamma} Y_\xi$$

where $Y_\xi = X_\xi$ if $t(\xi) = 1$, and $Y_\xi = \kappa - X_\xi$ if $t(\xi) = 0$. Let

(32.10) $$T = \{t \in L : t \in {}^\gamma\{0, 1\} \text{ for some } \gamma < \kappa \text{ and } |X_t| = \kappa\}$$

$(T, \subseteq)$ is a binary tree, and since for every $\gamma < \kappa$, $|2^\gamma|^L < \kappa$, the tree T has height κ and every level of T has size $< \kappa$.

By the tree property, T has a branch B of length κ; let F: $\kappa \to \{0, 1\}$ be the function $F = \bigcup \{t : t \in B\}$. Let U be the filter on $P(\kappa) \cap L_\alpha$ generated by $\{X_\xi : F(\xi) = 1\} \cup \{\kappa - X_\xi : F(\xi) = 0\}$. Clearly, U is a nonprincipal L_α-ultrafilter, is L_α-κ-complete, and moreover, the intersection of less than κ elements of U is nonempty.

Now we consider the ultrapower $Ult_U(L_\alpha)$; that is, we use functions f: $\kappa \to L_\alpha$ such that $f \in L_\alpha$. The ultrapower is well-founded and thus isomorphic to some L_β. The natural embedding j_U: $L_\alpha \to Ult_U(L_\alpha)$ has the required properties. ■

Lemma 32.6. *Let κ be an inaccessible cardinal in L and assume that for every $\alpha < (\kappa^+)^L$, there exists an elementary embedding j: $L_\alpha \to L_\beta$ (for some β) such that κ is the least ordinal moved by j. Then κ is weakly compact in L.*

Corollary 1. *If κ is weakly compact, then κ is weakly compact in L.*

Proof. This is an immediate consequence of Lemma 32.5 and 32.6. ■

Corollary 1 can be proved directly, using Lemma 32.4:

Exercise 32.12. If κ is weakly compact, then κ has the tree property in L (and hence $L \vDash \kappa$ is weakly compact).

[In L, let $T = (\kappa, <_T)$ be a tree of height κ such that each level of T has size less than κ. T has a branch B (in the universe). By Exercise 32.9, B is constructible.]

Corollary 2. *If there is no Aronszajn ω_2-tree, then $\aleph_2$ is weakly compact in L.*

Proof. We have shown in Section 29 that if $\aleph_2$ has the tree property then $\aleph_2$ is inaccessible in L. Now, the statement follows from Lemmas 32.5 and 32.6. ■

Corollary 3. *Every Silver indiscernible is a weakly compact cardinal in L.*

Proof. It suffices to show that i_ω, the ωth Silver indiscernible, is weakly compact in L. Let j be an order-preserving map from the class of Silver indiscernibles into itself such that $j(i_n) = i_n$ for all $n < \omega$, and $j(i_\omega) > i_\omega$, and let us extend j to an elementary embedding of L into L. By remarkability, i_ω is the least ordinal moved. By Lemma 32.6, i_ω is weakly compact in L. ■

Proof of Lemma 32.6. We shall prove that in L every binary tree of height κ has a branch of length κ. In L let us consider a tree $(T, \subseteq)$ of height κ whose elements on level $\gamma < \kappa$ are functions $t: \gamma \to \{0, 1\}$. If $t \in T$, then $t|\gamma \in T$ for each $\gamma \in \mathrm{dom}(t)$. Since κ is inaccessible in L, we have $t \in L_\kappa$ for all $t \in T$ and so $T \subset L_\kappa$; let $\alpha < (\kappa^+)^L$ be such that $T \in L_\alpha$, that L_α is a model of ZF^- and that $L_\alpha \vDash (T, \subseteq)$ is a tree of height κ.

Let j be an elementary embedding of L_α into some L_β such that κ is the least ordinal moved. Then $j(x) = x$ for all $x \in L_\kappa$, and $L_\beta \vDash j(T)$ is a tree of height greater than κ. Thus let $f \in j(T)$ be such that $\mathrm{dom}(f) = \kappa$ (an element of the κth level of $j(T)$). We shall show that $f|\gamma \in T$ for all $\gamma < \kappa$, and hence $B = \{f|\gamma : \gamma < \kappa\}$ is a branch in T of length κ (and clearly B is constructible).

If $\gamma < \kappa$, let $t = f|\gamma$; clearly $t \in j(T)$. Now because $t \in L_\kappa$ we have $j(t) = t$; hence $j(t) \in j(T)$ and it follows that $t \in T$. ■

We remark that if κ is weakly compact, then although κ is weakly compact in L, it is not necessarily true that κ is weakly compact in any transitive model M of ZFC containing all ordinals (a counterexample is known).

One way to prove that a measurable cardinal κ is weakly compact is to use a normal measure D on κ to prove $\kappa \to (\kappa)^2$; to show that every partition $f: [\kappa]^2 \to \{0, 1\}$ has a homogeneous set H of D-measure one. The homogeneous set H has not only size κ, but because $H \in D$, H is stationary.

Let us call an uncountable cardinal κ *ineffable* if every partition $f: [\kappa]^2 \to \{0, 1\}$ has a stationary homogeneous set H. Every measurable cardinal is ineffable, and every ineffable cardinal is weakly compact.

The following characterization of ineffable cardinals is useful:

Lemma 32.7. *A cardinal κ is ineffable if and only if it has the following property: If $\langle A_\alpha : \alpha < \kappa\rangle$ is such that $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, then there is $A \subseteq \kappa$ such that $\{\alpha \in \kappa : A_\alpha = A \cap \alpha\}$ is stationary in κ.*

Proof. (a) Let κ be ineffable, and let A_α, $\alpha < \kappa$, be such that $A_\alpha \subseteq \alpha$ for all α. Let $\prec$ be the lexicographic ordering of subsets of κ, i.e., $X \prec Y$ whenever the least element of the symmetric difference $X \triangle Y$ lies in Y. Let $f: [\kappa]^2 \to \{0, 1\}$ be as follows:

$$f(\alpha, \beta) = \begin{cases} 1 & \text{if } A_\alpha \preceq A_\beta \\ 0 & \text{if } A_\alpha \succ A_\beta \end{cases} \qquad (\alpha < \beta)$$

By ineffability, f has a stationary homogeneous set H. Let us consider the case when $A_\alpha \precsim A_\beta$ for all $\alpha < \beta$ in H; the other case is handled similarly.

We show, by induction on $\xi < \kappa$, that for every ξ there is $\eta(\xi)$ such that $\xi \in A_\alpha$ for all $\alpha \geq \eta(\xi)$ in H or $\xi \notin A_\alpha$ for all $\alpha \geq \eta(\xi)$ in H. If this is true for all $\nu < \xi$, then for all $\alpha \geq \sup\{\eta(\nu) : \nu < \xi\}$ in H, $A_\alpha \cap \xi$ is the same; then (because $A_\alpha \precsim A_\beta$ for all $\alpha < \beta$ in H) once ξ is in A_α for one such α, then ξ is in A_α for all greater $\alpha \in H$. Hence we find $\eta(\xi)$ that has the claimed property.

Now let C be the closed unbounded set

$$\{\alpha : (\forall \xi < \alpha)\, \eta(\xi) < \alpha\}$$

and let S be the stationary set $H \cap C$. If $\alpha < \beta$ are in S, then $A_\alpha = A_\beta \cap \alpha$, and we let $A = \bigcup \{A_\alpha : \alpha \in S\}$.

(b) Assume that the condition holds and let f: $[\kappa]^2 \to \{0, 1\}$ be a partition of $[\kappa]^2$. Let us define sets $A_\alpha \subseteq \alpha$, $\alpha < \kappa$, as follows:

$$A_\alpha = \{\xi < \alpha : f(\xi, \alpha) = 1\}$$

Let $A \subseteq \kappa$ be such that the set $S = \{\alpha : A_\alpha = A \cap \alpha\}$ is stationary. Either $S \cap A$ or $S - A$ is a stationary set; let us consider the case when $S \cap A$ is stationary, and let $H = S \cap A$. The other case is handled similarly.

If $\alpha < \beta$ are in H, then because $\beta \in S$ and $\alpha \in A$, we have $\alpha \in A_\beta$ and therefore $f(\alpha, \beta) = 1$. Thus H is homogeneous for f. ■

Corollary 1. *If κ is ineffable, then κ is ineffable in L.*

Proof. Let $\langle A_\alpha : \alpha < \kappa \rangle$ be a constructible sequence such that $A_\alpha \subseteq \alpha$ for all α. Since κ is ineffable, there is $A \subseteq \kappa$ such that $S = \{\alpha : A_\alpha = A \cap \alpha\}$ is stationary. For each $\alpha < \kappa$, $A \cap \alpha$ is constructible, and so, by Exercise 32.9, A is constructible. Hence S is also constructible; and because S is stationary, clearly $L \vDash S$ is stationary. Thus κ is ineffable in L. ■

Corollary 2. *Every Silver indiscernible is ineffable in L.*

Proof. It suffices to prove it for one Silver indiscernible. Let κ be an uncountable cardinal (or $\kappa = i_\omega$) and let j: $L \to L$ be an elementary embedding such that κ is the least ordinal moved. Let $\langle A_\alpha : \alpha < \kappa \rangle \in L$ be such that $A_\alpha \subseteq \alpha$ for all α. For each $\alpha < \kappa$, we have $j(A_\alpha) = A_\alpha$, and so $j(\langle A_\alpha : \alpha < \kappa \rangle) = \langle A_\alpha : \alpha < j(\kappa) \rangle$ for some A_α, $\kappa \leq \alpha < j(\kappa)$, such that $A_\alpha \subseteq \alpha$ for all $\alpha < j(\kappa)$. Let $A = A_\kappa$; we shall prove that $S = \{\alpha < \kappa : A_\alpha = A \cap \alpha\}$ is stationary in L.

Let $C \in L$ be a closed unbounded subset of κ. Since $C = j(C) \cap \kappa$ and $j(C)$ is closed unbounded in $j(\kappa)$, we have $\kappa \in j(C)$. We also have $A = j(A) \cap \kappa$ and hence $\kappa \in j(S) = \{\alpha < j(\kappa) : A_\alpha = j(A) \cap \alpha\}$. Thus $j(C) \cap j(S) \neq \emptyset$ and hence $C \cap S$ is nonempty. Therefore S is stationary in L. ■

As for the relative strength of ineffable and Ramsey cardinals: Ramsey cardinals imply that $0^\#$ exists, while ineffable cardinals are consistent with

$V = L$. One can show that if Ramsey cardinals exist, then the least ineffable cardinal is smaller than the least Ramsey cardinal. However, it is not true that every Ramsey cardinal is ineffable: Using Lemma 32.7, one can show that ineffable cardinals are Π^1_2-indescribable; but the least Ramsey cardinal is Π^1_2-describable.

Partition Cardinals

If X is a set of ordinals and $n < \omega$, $[X]^n$ denotes the set of all increasing n-sequences in X; $[X]^{<\omega}$ is the set $\bigcup_{n=1}^{\infty} [X]^n$.

In (29.7) we have introduced the partition property

$$\kappa \to (\alpha)^{<\omega}_m$$

where κ is an infinite cardinal number, m is a cardinal number ≥ 2, and α is a limit ordinal: Every function $f\colon [\kappa]^{<\omega} \to m$ has a homogeneous set $H \subseteq \kappa$ of order type α; that is, for every $n < \omega$, f is constant on $[H]^n$.

Let α be a limit ordinal number and let us define

$$\eta_\alpha = \text{the least } \kappa \text{ such that } \kappa \to (\alpha)^{<\omega} \tag{32.11}$$

Note that α is Ramsey just in case $\eta_\alpha = \alpha$.

Theorem 78.

(a) *Each η_α is an inaccessible cardinal, and every model $\mathfrak{A} \supseteq \eta_\alpha$ whose language has less than η_α symbols has a set of indiscernibles of order type α.*
(b) *If $\alpha < \beta$, then $\eta_\alpha < \eta_\beta$.*
(c) *If η_ω exists, then there exists a weakly compact cardinal $\kappa < \eta_\omega$.*
(d) *If η_{ω_1} exists, then $0^\#$ exists.*
(e) *If $\kappa \to (\omega)^{<\omega}$, then $L \models \kappa \to (\omega)^{<\omega}$.*

Our first step toward the proof of Theorem 78 is the following easy lemma:

Lemma 32.8. *If $\kappa \to (\alpha)^{<\omega}$, then $\kappa \to (\alpha)^{<\omega}_{2^{\aleph_0}}$.*

Note that by Lemma 29.8, every model $\mathfrak{A} \supseteq \kappa$ of a countable language has α indiscernibles.

Proof. Let f be a partition, $f\colon [\kappa]^{<\omega} \to {}^\omega\{0, 1\}$. For each $n < \omega$, let $f_n = f \restriction [\kappa]^n$, and for each $k < \omega$, let $f_{n,k}\colon [\kappa]^n \to \{0, 1\}$ be as follows:

$$f_{n,k}(\alpha_1, \ldots, \alpha_n) = h(k), \qquad \text{where} \quad h = f_n(\alpha_1, \ldots, \alpha_n)$$

Let π be a one-to-one correspondence between ω and $\omega \times \omega$ such that if $\pi(m) = (n, k)$, then $m \geq n$; for each m, let $g_m\colon [\kappa]^m \to \{0, 1\}$ be the partition defined by

$$g_m(\alpha_1, \ldots, \alpha_m) = f_{n,k}(\alpha_1, \ldots, \alpha_n)$$

where $(n, k) = \pi(m)$.

By the assumption, there exists $H \subseteq \kappa$ of order type α which is homogeneous for all g_m. We claim that H is homogeneous for f. If not, then $f_n(\alpha_1, \ldots, \alpha_n) \neq f_n(\beta_1, \ldots, \beta_n)$ for some α's and β's in H. Then for some k, $f_{nk}(\alpha_1, \ldots, \alpha_n) \neq f_{nk}(\beta_1, \ldots, \beta_n)$, contrary to the assumption that H is homogeneous for g_m, where $\pi(m) = (n, k)$. ■

The following lemma is analogous to (but more difficult to prove than) Lemma 29.9 (which is the special case for $\eta_\alpha = \alpha$):

Lemma 32.9. *For every* $\kappa < \eta_\alpha$, $\eta_\alpha \to (\alpha)_\kappa^{<\omega}$.

An immediate consequence of this lemma is that η_α is a strong limit cardinal: If $\kappa < \eta_\alpha$, then $2^\kappa \not\to (\alpha)_\kappa^2$ by Lemma 29.2, and so $2^\kappa < \eta_\alpha$.

Proof. Let $\kappa < \eta_\alpha$, and let $f\colon [\eta_\alpha]^{<\omega} \to \kappa$. We wish to find a homogeneous set for f, of order type α. Since $\kappa < \eta_\alpha$, there exists $g\colon [\kappa]^{<\omega} \to \{0, 1\}$ that has no homogeneous set of order-type α. For each n, let $f_n = f \,|\, [\eta_\alpha]^n$ and $g_n = g \,|\, [\kappa]^n$, and let $\mathfrak{A}$ be the model

$$\mathfrak{A} = \langle V_{\eta_\alpha}, \in, f_n, g_n \rangle_{n=0,1,\ldots}$$

By Lemma 32.8, and by Lemma 29.8, the model $\mathfrak{A}$ has a set of indiscernibles H of order-type α. We shall show that H is homogeneous for f. It suffices to show that for each n, the formula

$$f_n(\alpha_1, \ldots, \alpha_n) = f_n(\beta_1, \ldots, \beta_n) \tag{32.12}$$

holds in $\mathfrak{A}$ for any increasing sequence $\alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots < \beta_n$ of indiscernibles: Then if $\alpha_1 < \cdots < \alpha_n$ and $\alpha'_1 < \cdots < \alpha'_n$ are arbitrary in H, we choose $\beta_1 < \cdots < \beta_n$ in H such that $\alpha_n < \beta_1$ and $\alpha'_n < \beta_1$, and $f_n(\alpha_1, \ldots, \alpha_n) = f_n(\alpha'_1, \ldots, \alpha'_n)$ follows from (32.12).

Thus let us assume that the negation of (32.12) holds for any $\alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots < \beta_n$ in H. Let u_ξ, $\xi < \alpha$, be increasing n-sequences in H such that the last element of u_ξ is less than the first element of u_η whenever $\xi < \eta$. Let

$$\gamma_\xi = f(u_\xi)$$

for all $\xi < \alpha$, and let $G = \{\gamma_\xi : \xi < \alpha\}$. By indiscernibility, and since $\gamma_0 > \gamma_1 > \cdots > \gamma_\xi > \cdots$ is impossible, we have $\gamma_0 < \gamma_1 < \cdots < \gamma_\xi < \cdots$.

We shall reach a contradiction by showing that G is homogeneous for g. For each k, consider the formula

$$g(f(u_{\xi_1}), \ldots, f(u_{\xi_k})) = g(f(u_{\nu_1}), \ldots, f(u_{\nu_k})) \tag{32.13}$$

By indiscernibility, either (32.13) or its negation holds for all increasing sequences $\xi_1 < \cdots < \xi_k < \nu_1 < \cdots < \nu_k$. The inequality cannot hold because g takes only two values, 0 and 1, and three sequences $\langle \xi_1, \ldots, \xi_k \rangle$ would give three different values. Thus (32.13) holds, and the same argument as earlier in this proof shows that g is constant on $[G]^k$. ■

Proof of Theorem 78a. We know already that η_α is a strong limit cardinal. It suffices to prove that η_α is regular since the second part of the assertion follows from Lemma 29.8.

Let us assume that η_α is singular and that $\kappa = \mathrm{cf}(\eta_\alpha)$; let $\eta_\alpha = \lim_{\nu \to \kappa} \lambda_\nu$. For each $\nu < \kappa$, let $f^\nu \colon [\lambda_\nu]^{<\omega} \to \{0, 1\}$ be such that f^ν has no homogeneous set of order type α. For each n, let $f^\nu_n = f^\nu \mid [\lambda_\nu]^n$; let $\mathfrak{A}$ be the model

$$\mathfrak{A} = \langle V_{\eta_\alpha}, \in, \lambda_\nu, f^\nu_n \rangle_{\nu < \kappa,\, n = 0, 1, \ldots}$$

Since η_α is a strong limit and $\kappa < \eta_\alpha$, the model $\mathfrak{A}$ has a set of indiscernibles H of order type α.

Let ν be such that λ_ν is greater than the least element of H. Then by indiscernibility, *all* elements of H are smaller than λ_ν. Since the function f^ν takes only two values, it follows that for each n, it is the equality

$$f^\nu_n(\alpha_1, \ldots, \alpha_n) = f^\nu_n(\beta_1, \ldots, \beta_n)$$

that holds for all increasing sequences $\alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots < \beta_n$ in H, and not its negation. Hence H is homogeneous for f^ν, contrary to the assumption on f^ν. ■

Proof of Theorem 78b. Let $\alpha < \beta$ be limit ordinals, and let us assume that $\eta_\alpha = \eta_\beta$. For each $\xi < \eta_\alpha$, there exists a function $f_\xi \colon [\xi]^{<\omega} \to \{0, 1\}$ that has no homogeneous subset of ξ of order-type α. Let us define $g \colon [\eta_\beta]^{<\omega} \to \{0, 1\}$ by

$$g(\xi_1, \ldots, \xi_n) = f_{\xi_n}(\xi_1, \ldots, \xi_{n-1})$$

Now if H is homogeneous for g, then for each $\xi \in H$, $H \cap \xi$ is homogeneous for f_ξ. Hence the order-type of each $H \cap \xi$ is less than α, and therefore the order-type of H is at most α, which is less than β. A contradiction. ■

Before giving the proof of (c), let me formulate a lemma that is an abstract version of something we proved earlier:

Lemma 32.10. *Let M and N be transitive models of* ZFC *and let $j \colon M \to N$ be a nontrivial elementary embedding; let κ be the least ordinal moved. If $P^M(\kappa) = P^N(\kappa)$, then κ is an ineffable cardinal in M.*

Proof. [Note that $P^M(\kappa) \subseteq P^N(\kappa)$ holds always, because if $X \subseteq \kappa$ is in M, then $X = j(X) \cap \kappa$ and hence $X \in N$.]

The proof is virtually given in Corollary 2 of Lemma 32.7. Let $\langle A_\alpha : \alpha < \kappa \rangle \in M$ be such that $A_\alpha \subseteq \alpha$ for all α. We have $j(A_\alpha) = A_\alpha$ for all $\alpha < \kappa$, and hence $j(\langle A_\alpha : \alpha < \kappa \rangle) = \langle A_\alpha : \alpha < j(\kappa) \rangle$ (for some A_α, $\kappa \le \alpha < j(\kappa)$). The set A_κ is in M and witnesses ineffability of κ in M. ■

Proof of Theorem 78c. Let h_φ, $\varphi \in$ *Form*, be Skolem functions for the language $\{\in\}$ of set theory, and let us consider the model

$$\mathfrak{A} = \langle V_{\eta_\omega}, \in, h^{\mathfrak{A}}_\varphi \rangle_{\varphi \in Form}$$

where for each φ, $h_\varphi^{\mathfrak{A}}$ is a Skolem function for φ in $\langle V_{\eta_\omega}, \in\rangle$. The model $\mathfrak{A}$ has a set of indiscernibles I of order type ω. Let B be the closure of I under the Skolem functions $h_\varphi^{\mathfrak{A}}$.

Let us consider some nontrivial order-preserving mapping of H into H. Using the Skolem functions, we extend this mapping (in the unique way) to a nontrivial elementary embedding of B into B. Let M be the transitive set isomorphic to B and let j: $M \to M$ be the corresponding nontrivial elementary embedding.

Since η_ω is inaccessible, V_{η_ω} is a model of ZFC and thus M is a transitive model of ZFC. Using Lemma 32.10, we conclude that $M \vDash$ there exists an ineffable cardinal; and therefore $V_{\eta_\omega} \vDash$ there exists an ineffable cardinal. Hence there is a weakly compact (even ineffable) cardinal below η_ω. ■

Exercise 32.13. If M is a transitive model of ZFC and if j: $M \to M$ is a nontrivial elementary embedding, then the least ordinal κ moved by j is Π_m^n-indescribable in M, for all n and m.

[If $U \subseteq V_\kappa^M$, then $U = j(U) \cap V_\kappa^M$. If σ is a Π_m^n sentence and $M \vDash [\langle V_\kappa, \in, U\rangle \vDash \sigma]$, then $M \vDash [(\exists\alpha < j(\kappa))\langle V_\alpha, \in, j(U) \cap V_\alpha\rangle \vDash \sigma]$.]

Thus the proof of (c) shows also that if η_ω exists, then there is $\kappa < \eta_\omega$ such that κ is Π_m^n-indescribable for all n and m. Another consequence of Exercise 32.13 is that every Silver indiscernible is Π_m^n-indescribable in L.

Exercise 32.14. The cardinal η_ω is not weakly compact.

[η_ω is Π_1^1-describable.]

Proof of Theorem 89d. If $\kappa \to (\omega_1)^{<\omega}$, then the model $\langle L_\kappa, \in\rangle$ has an uncountable set of indiscernibles and $0^\#$ exists by Theorem 73. ■

Finally, (e) follows from this lemma:

Lemma 32.11. *Let f be a constructible partition, f: $[\kappa]^{<\omega} \to \{0, 1\}$. If there is an infinite homogeneous set for f, then there is a constructible one.*

Corollary 1. Theorem 78e.

Corollary 2. *If $0^\#$ exists, then $L \vDash$ there is κ such that $\kappa \to (\omega)^{<\omega}$. (And $(\eta_\omega)^L$, being a definable set in L, is countable and in fact smaller than the least Silver indiscernible.)*

Proof. Let κ be any uncountable cardinal. If f: $[\kappa]^{<\omega} \to \{0, 1\}$ is in L, then there is a Skolem term t and Silver indiscernibles $\alpha_1 < \cdots < \alpha_n < \kappa \leq \beta_1 < \cdots < \beta_m$ such that $f = t(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$. I leave it to the reader to verify that the set H of all Silver indiscernibles between α_n and κ is homogeneous for f. Hence $L \vDash \kappa \to (\omega)^{<\omega}$. ■

Proof of Lemma 32.11. Let T be the set of all finite increasing sequences $t = \langle\alpha_0, \ldots, \alpha_{n-1}\rangle$ in κ such that for every $k \leq n$, f is constant on

$[\{\alpha_0, \ldots, \alpha_{n-1}\}]^k$, and let us consider the tree $(T, \supseteq)$; clearly, T is constructible. We note that an infinite homogeneous set for f exists just in case $(T, \supseteq)$ is not well-founded. However, being well-founded is an absolute property for models of ZFC; and so if the tree is not well-founded, then it is not well-founded in L, and the lemma follows. ■

We conclude the discussion on partition cardinals with the following observation:

Exercise 32.15. If κ is a Ramsey cardinal, then κ is the κth weakly compact cardinal.

[Given $\alpha < \kappa$, find a weakly compact cardinal λ such that $\alpha \leq \lambda < \kappa$, as follows. Consider a Skolem expansion $\mathfrak{A}$ of the model $\langle V_\kappa, \in, \xi \rangle_{\xi < \alpha}$ and proceed as in the proof of Theorem 78c to show that $V_\kappa \vDash \exists$ ineffable cardinal $\lambda \geq \alpha$.]

A Few Remarks on Jónsson Cardinals

Jónsson cardinals were defined in Section 29; a cardinal κ is Jónsson if every model (of countable language) of size κ has a proper elementary submodel of size κ. Here is what is known about Jónsson cardinals:

No $\aleph_n$, $n < \omega$, can be a Jónsson cardinal (for $\aleph_0$, see Exercise 29.12). If GCH holds, then no successor cardinal is Jónsson. The least Jónsson cardinal is either weakly inaccessible or has cofinality ω.

Every Ramsey cardinal is a Jónsson cardinal (by Lemma 29.10). Conversely, the method of Section 31 has been used to show that in $L[U]$, every Jónsson cardinal is Ramsey. In Chapter 6, we shall present a model that has a Jónsson cardinal of cofinality ω.

It is not known whether $\aleph_\omega$ can be a Jónsson cardinal; however, if it is, then $\aleph_\omega$ is measurable in some transitive model M.

We shall prove only one, rather easy fact about Jónsson cardinals, namely that they imply that $0^\#$ exists:

Lemma 32.12. *Let j: $L_\alpha \to L_\beta$ be an elementary embedding and let γ be the least ordinal moved by j. If $\gamma < |\alpha|$, then $0^\#$ exists.*

Corollary 1. *If there is a Jónsson cardinal, then $0^\#$ exists.*

Proof. Let κ be a Jónsson cardinal and let us consider the model $\langle L_\kappa, \in \rangle$. Let A be an elementary submodel, of size κ, such that $A \neq L_\kappa$. Let π be the transitive collapse of A; clearly, $\pi[A] = L_\kappa$. Thus $j = \pi^{-1}$ is a nontrivial elementary embedding of L_κ in L_κ. Since κ is a cardinal, Lemma 32.12 applies and hence $0^\#$ exists. ■

Let us recall that Chang's conjecture is the statement that every model of type $(\aleph_2, \aleph_1)$ has an elementary submodel of type $(\aleph_1, \aleph_0)$.

Corollary 2. *Chang's conjecture implies that $0^\#$ exists.*

Proof. Consider the model $\langle L_{\omega_2}, \omega_1, \in \rangle$, and let $\mathfrak{A} = \langle A, \omega_1 \cap A, \in \rangle$ be its elementary submodel such that $|A| = \aleph_1$ and $|\omega_1 \cap A| = \aleph_0$. Let π be the transitive collapse of A; clearly, $\pi[A] = L_\alpha$ for some α such that $\omega_1 \leq \alpha < \omega_2$. Also, $\pi[\omega_1 \cap A]$ is a countable ordinal, and hence $\pi(\omega_1) < \omega_1$. Then $j = \pi^{-1}$ is an elementary embedding of L_α in L_{ω_2}, and the least ordinal moved is countable. By Lemma 32.12, $0^\#$ exists. ■

Proof of Lemma 32.12. We define an L-ultrafilter D over γ as follows

$$(32.14) \qquad X \in D \quad \text{iff} \quad \gamma \in j(X) \qquad (X \subseteq \gamma, \quad X \in L)$$

Note that $j(X)$ is defined for all constructible $X \subseteq \gamma$ because every such X is in L_α.

Let us consider the ultrapower $Ult_D(L)$. If the ultrapower is well-founded, then we are done because then the natural embedding $j_D\colon L \to Ult_D(L)$ is a nontrivial elementary embedding of L in L and the lemma follows by Theorem 73. Thus we shall prove that $Ult_D(L)$ is well-founded.

Let us assume that $f_0, f_1, \ldots, f_n, \ldots$ is a counterexample to well-foundedness of the ultrapower. Each f_n is a constructible function on γ and $\{\xi < \gamma : f_{n+1}(\xi) \in f_n(\xi)\} \in D$ for all $n < \omega$. Let θ be a limit ordinal such that $f_n \in L_\theta$ for all n and let S be an elementary submodel of $\langle L_\theta, \in\rangle$ such that $|S| = |\gamma|$, $\gamma \subseteq S$, and $f_n \in S$ for all n. Let π be the transitive collapse of S, $\pi[S] = L_\eta$, and let $g_n = \pi(f_n)$, for all n.

Since $\pi(\xi) = \xi$ for all $\xi < \gamma$, we see that for each $\xi < \gamma$ and all n, $g_{n+1}(\xi) \in g_n(\xi)$ just in case $f_{n+1}(\xi) \in f_n(\xi)$, and hence $g_0, g_1, \ldots, g_n, \ldots$ is also a counterexample to well-foundedness of the ultrapower. However, since each g_n is in L and $|\theta| = |\gamma| < |\alpha|$, we have $g_n \in L_\alpha$ for all n. Thus $j(g_n)$ is defined for all n, and we have, by (32.14),

$$\{\xi < \gamma : g_{n+1}(\xi) \in g_n(\xi)\} \in D \qquad \text{iff} \qquad (j(g_{n+1}))(\gamma) \in (j(g_n))(\gamma)$$

Now we reached a contradiction because $(jg_0)(\gamma) \ni (jg_1)(\gamma) \ni \cdots$ would be a descending sequence. ■

CHAPTER 6

OTHER LARGE CARDINALS

33. COMPACT CARDINALS

In Section 32 we studied weakly compact cardinals, which are inaccessible cardinals satisfying the weak compactness theorem for the infinitary language $\mathscr{L}_{\kappa,\omega}$. If we remove the restriction on the size of sets of sentences in the model theoretic characterization of weakly compact cardinals, we obtain a considerably stronger notion of large cardinals. This notion, (*strong*) *compactness*, turns out to be much stronger than measurability.

Compact cardinals can be characterized in several different ways. Let me use, as a definition, the property that is a natural generalization of the ultrafilter theorem:

An uncountable regular cardinal κ is *compact* (or, *strongly compact*) if for any set S, every κ-complete filter over S can be extended to a κ-complete ultrafilter over S. Obviously, every compact cardinal κ is measurable, for any ultrafilter over κ that extends the filter $\{X : |\kappa - X| < \kappa\}$ is nonprincipal.

Let us say that the language $\mathscr{L}_{\kappa,\omega}$ (or $\mathscr{L}_{\kappa,\kappa}$) satisfies the *compactness theorem* if whenever Σ is a set of sentences of $\mathscr{L}_{\kappa,\omega}$ ($\mathscr{L}_{\kappa,\kappa}$) such that every $S \subseteq \Sigma$ with $S < \kappa$ has a model, then Σ has a model.

Let A be a set of size at least κ. By $P_\kappa(A)$, we denote the set

$$P_\kappa(A) = \{P \subset A : |P| < \kappa\} \tag{33.1}$$

For each $P \in P_\kappa(A)$, let

$$\hat{P} = \{Q \in P_\kappa(A) : P \subseteq Q\} \tag{33.2}$$

and let us consider the filter over $P_\kappa(A)$ generated by the sets $\hat{P}$ for all $P \in P_\kappa(A)$; that is, the filter

$$\{X \subseteq P_\kappa(A) : X \supseteq \hat{P} \text{ for some } P \in P_\kappa(A)\} \tag{33.3}$$

If κ is a regular cardinal, then the filter (33.3) is κ-complete. We call U a *fine measure* on $P_\kappa(A)$ if U is a κ-complete ultrafilter over $P_\kappa(A)$ that extends the filter (33.3); i.e., if $\hat{P} \in U$ for all $P \in P_\kappa(A)$.

Lemma 33.1. *The following are equivalent, for any regular cardinal* κ:

(i) *For any set* S*, every* κ*-complete filter over* S *can be extended to a* κ*-complete ultrafilter over* S.
(ii) *For any* A *such that* $|A| \geq \kappa$*, there exists a fine measure on* $P_\kappa(A)$.
(iii) *The language* $\mathscr{L}_{\kappa,\omega}$ *satisfies the compactness theorem.*

Proof. (i) → (ii) is clear.

(ii) → (iii): Let Σ be a set of sentences of $\mathscr{L}_{\kappa,\omega}$ and assume that every $S \subseteq \Sigma$ of size less than κ has a model, say $\mathfrak{A}_S$. Let U be a fine measure on $P_\kappa(\Sigma)$, and let us consider the ultraproduct $\mathfrak{A} = Ult_U\langle \mathfrak{A}_S : S \in P_\kappa(\Sigma)\rangle$. It is routine to verify that the fundamental theorem on ultraproducts holds for the language $\mathscr{L}_{\kappa,\omega}$ provided the ultrafilter is κ-complete; in order to prove the induction step for the infinitary connective $\bigwedge_{\xi<\alpha} \varphi_\xi$, one uses the κ-completeness of U. Thus we have, for any sentence σ of $\mathscr{L}_{\kappa,\omega}$,

$$\mathfrak{A} \models \sigma \quad \text{iff} \quad \{S : \mathfrak{A}_S \models \sigma\} \in U \tag{33.4}$$

Now if $\sigma \in \Sigma$, then $\{\sigma\}\hat{} \in U$ and since $\mathfrak{A}_S \models \sigma$ whenever $S \ni \sigma$, (33.4) implies that σ holds in $\mathfrak{A}$. Hence $\mathfrak{A}$ is a model of Σ.

(iii) → (i): Let S be a set and let F be a κ-complete filter over S. Let us consider the $\mathscr{L}_{\kappa,\omega}$-language which has a unary predicate symbol X for each $X \subseteq S$, and a constant symbol c. Let Σ be the set of $\mathscr{L}_{\kappa,\omega}$ sentences consisting of:

(a) all sentences true in $\langle S, X\rangle_{X\subseteq S}$,
(b) $X(c)$ for all $X \in F$.

Every set of less than κ sentences in Σ has a model: Take S as the universe, interpret each X as X and let c be some element of S that lies in every X whose name is mentioned in the given set of sentences; since F is κ-complete, such c exists.

Hence Σ has a model $\mathfrak{A} = \langle A, X^{\mathfrak{A}}, c\rangle_{X\subseteq S}$. Let us define $U \subseteq P(S)$ as follows:

$$X \in U \quad \text{iff} \quad \mathfrak{A} \models X(c) \tag{33.5}$$

It is easy to verify that U is a κ-complete ultrafilter and that $U \supseteq F$: For instance, U is κ-complete because if $\alpha < \kappa$ and $X = \bigcap_{\xi<\alpha} X_\xi$, then $\mathfrak{A}$ satisfies the sentence $\bigwedge_{\xi<\alpha} X_\xi(c) \to X(c)$. ■

The proof of (ii) → (iii) gives a stronger result:

Exercise 33.1. If κ is compact, then $\mathscr{L}_{\kappa,\kappa}$ satisfies the compactness theorem.

[Prove the fundamental theorem on ultraproducts modulo a κ-complete ultrafilter for $\mathscr{L}_{\kappa,\kappa}$.]

Every compact cardinal is measurable, but not every measurable cardinal is necessarily compact (although it is consistent that there is exactly one measurable cardinal which is also compact). We shall show that existence of compact cardinals is a much stronger assumption than existence of measurable cardinals. We start with the following theorem:

Theorem 79 (Vopěnka–Hrbáček). *If there exists a compact cardinal, then there is no set A such that $V = L[A]$.*

Proof. Let us assume that $V = L[A]$ for some set A. Since there is a set of ordinals A' such that $L[A] = L[A']$, we may as well assume that A is a set of ordinals. Let κ be a compact cardinal, and let $\lambda \geq \kappa$ be a cardinal such that $A \subseteq \lambda$. Since κ is compact, there exists a κ-complete ultrafilter U over λ^+ such that $|X| = \lambda^+$ for every $X \in U$ (let U extend the filter $\{X: |\lambda^+ - X| \leq \lambda\}$).

Since U is κ-complete, the ultrapower $Ult_U(V)$ is well-founded, and thus can be identified with a transitive model M. As usual, if f is a function on λ^+, then $[f]$ denotes the element of M represented by f. Let $j = j_U$ be the natural elementary embedding of V into M given by U.

Let us now consider another version of ultrapower. Let us consider only those functions on λ^+ that assume at most λ values. For these functions, we still define $f =^* g \pmod U$ and $f \in^* g \pmod U$ in the usual way, and therefore obtain a model of the language of set theory, which we denote $Ult_U^-(V)$. The fundamental theorem holds for this version of ultrapower too: If $f, \ldots$ are functions on λ^+ with $|\operatorname{ran} f| \leq \lambda$, then

$$Ult^- \models \varphi(f, \ldots) \quad \text{iff} \quad \{\alpha : \varphi(f(\alpha), \ldots)\} \in U \tag{33.6}$$

(Check the induction step for $\exists$.) Hence Ult^- is a model of ZFC, elementarily equivalent to V. Also, since U is κ-complete, Ult^- is well-founded and thus is isomorphic to a transitive model N. Every element of N is represented by a function f on λ^+ such that $|\operatorname{ran} f| \leq \lambda$. We denote by $[f]^-$ the element of N represented by f. We also define an elementary embedding $i: V \to N$ by $i(x) = [c_x]^-$ where c_x is the constant function on λ^+ with value x.

For every function f on λ^+ with $|\operatorname{ran} f| \leq \lambda$, we let

$$k([f]^-) = [f] \tag{33.7}$$

It is easy to see that the definition (33.7) of $k([f]^-)$ does not depend on the choice of f representing $[f]^-$ in N, and that k is an elementary embedding of N in M. In fact, we have the commutative diagram in Fig. 33.1.

If $\gamma < \lambda^+$, then every function from λ^+ into γ has at most λ values, and hence $[f]^- = [f]$ for all $f: \lambda^+ \to \gamma$. If $f: \lambda^+ \to \lambda^+$ has at most λ values, then $f: \lambda^+ \to \gamma$ for some $\gamma < \lambda^+$; it follows that $i(\lambda^+) = \lim_{\gamma \to \lambda^+} i(\gamma)$, and we have $k(\xi) = \xi$ for all $\xi < i(\lambda^+)$.

Similarly, $i(A) = j(A)$, and we have $M = L[j(A)] = L[i(A)] = N$.

Now we reach a contradiction by observing that $j(\lambda^+) > i(\lambda^+)$: Since the diagonal function $d(\alpha) = \alpha$ represents in M an ordinal greater than each $j(\gamma)$, $\gamma < \lambda^+$, we have $j(\lambda^+) > \lim_{\gamma \to \lambda^+} j(\gamma)$. While N thinks that $i(\lambda^+)$ is the successor of $i(\lambda)$, M thinks that $j(\lambda^+)$ is the successor of $j(\lambda)$ (and $j(\lambda) = i(\lambda)$). Thus $M \neq N$, a contradiction. ∎

$$\begin{tikzcd}
 & V \arrow[dl, "i"'] \arrow[dr, "j"] & \\
N \arrow[rr, "k"'] & & M
\end{tikzcd}$$

FIGURE 33.1

The proof of Theorem 79 provides this additional information:

Exercise 33.2. If κ is compact, $\lambda \geq \kappa$, and $A \subseteq \lambda$, then λ^+ is an ineffable cardinal in $L[A]$.

[Let U be as above, let $M = Ult_U(L[A])$, $N = Ult_{\bar{U}}(L[A])$, let j: $L[A] \to M$, i: $L[A] \to N$, and let k: $N \to M$ be as above. Again, $M = N$, and $i(\lambda^+)$ is the least ordinal moved. By Lemma 32.10, N thinks that $i(\lambda^+)$ is ineffable; hence λ^+ is ineffable in $L[A]$.]

We shall now state a theorem which shows that, consistencywise, strong compactness is much stronger than measurability.

Theorem 80 (Kunen). *If there exists a compact cardinal, then for every ordinal number θ there exists a transitive model M of* ZFC *with at least θ measurable cardinals* (i.e., *the set of measurable cardinals in M has order type at least θ*).

I shall not give a detailed proof of this theorem. I shall give the proof only for the case $\theta = 2$, which is fairly representative of the method, and then make some remarks on the general version of the theorem.

The key lemma, on which the proof of the theorem is based, is the following:

Lemma 33.2. *Let κ be a compact cardinal. For every $\delta < (2^\kappa)^+$, there exists a κ-complete ultrafilter U over κ such that $j_U(\kappa) > \delta$.*

[We know that $j_U(\kappa) < (2^\kappa)^+$.]

The proof of Lemma 33.2 uses the following combinatorial result:

Lemma 33.33 *Let κ be an inaccessible cardinal. There exists a family $\mathscr{G}$ of functions g: $\kappa \to \kappa$ such that $|\mathscr{G}| = 2^\kappa$, and whenever $\mathscr{H} \subset \mathscr{G}$ is a subfamily of size $<\kappa$, and $\{\beta_g$: $g \in \mathscr{H}\}$ is any collection of ordinals $<\kappa$, then there exists α such that $g(\alpha) = \beta_g$ for all $g \in \mathscr{H}$.*

Proof. Let $\mathscr{A}$ be a family of almost disjoint subsets of κ (i.e., $|A| = \kappa$ for each $A \in \mathscr{A}$ and $|A \cap B| < \kappa$ for any distinct $A, B \in \mathscr{A}$), such that $|\mathscr{A}| = 2^\kappa$. For each $A \in \mathscr{A}$, let f_A be a mapping of A onto κ such that for each $\beta < \kappa$, the set $\{a \in A : f_A(a) = \beta\}$ has size κ. Let s_α, $\alpha < \kappa$, enumerate all subsets $s \subset \kappa$ of size $< \kappa$.

For each $A \in \mathscr{A}$, let g_A: $\kappa \to \kappa$ be defined as follows:

$$\text{if} \quad s_\alpha \cap A = \{x\}, \qquad \text{then} \quad g_A(\alpha) = f_A(x)$$

$$g_A(\alpha) = 0 \quad \text{otherwise}$$

Let $\mathscr{G} = \{g_A : A \in \mathscr{A}\}$.

If $A \neq B \in \mathscr{A}$, then it is easy to find s_α such that $g_A(\alpha) \neq 0$ and $g_B(\alpha) = 0$; hence $|\mathscr{G}| = 2^\kappa$. If $\mathscr{H} \subset \mathscr{A}$ has size $< \kappa$ and if $\{\beta_A: A \in \mathscr{H}\}$ are given, then for each $A \in \mathscr{H}$ we choose $x_A \in A$ such that $x_A \notin B$ for any other $B \in \mathscr{H}$ and that $f_A(x_A) = \beta_A$. Then if α is such that $s_\alpha = \{x_A: A \in \mathscr{H}\}$, we have $g_A(\alpha) = \beta_A$ for every $A \in \mathscr{H}$. ■

Proof of Lemma 33.2. Let $\delta < (2^\kappa)^+$. Let $\mathscr{G}$ be a family of functions $g: \kappa \to \kappa$ of size $|\delta|$ with the property stated in Lemma 33.3; let us enumerate $\mathscr{G} = \{g_\alpha : \alpha \leq \delta\}$.

For any $\alpha < \beta \leq \delta$, let $X_{\alpha\beta} = \{\xi : g_\alpha(\xi) < g_\beta(\xi)\}$. Using the property of $\mathscr{G}$ stated in Lemma 33.3, we can see that any collection of less than κ of the $X_{\alpha\beta}$ has a nonempty intersection and hence $F = \{X : X \supseteq X_{\alpha\beta} \text{ for some } \alpha < \beta \leq \delta\}$ is a κ-complete filter over κ. Since κ is compact, there exists a κ-complete ultrafilter U extending F. It is clear that if $\alpha < \beta \leq \delta$, then $g_\alpha < g_\beta \bmod U$, and hence $j_U(\kappa) > \delta$. ■

Before I start the proof of the theorem, let us go back to iterated ultrapowers. Let D be a normal measure on κ, and let $i_{0\alpha}$ denote, for each α, the elementary embedding $i_{0\alpha}: V \to Ult^{(\alpha)}$; let $\kappa^{(\alpha)} = i_{0\alpha}(\kappa)$ and $D^{(\alpha)} = i_{0\alpha}(D)$.

First recall (31.18): If λ is a limit ordinal, then $X \in Ult^{(\lambda)}$ belongs to $D^{(\lambda)}$ just in case $X \supseteq \{\kappa^{(\gamma)} : \alpha \leq \gamma < \lambda\}$ for some $\alpha < \lambda$. Let

(33.8) $\quad K = \{\nu : \nu \text{ is a strong limit cardinal, } \nu > 2^\kappa, \text{ and cf } \nu > \kappa\}$

By Lemma 31.12, if $\nu \in K$, then $\kappa^{(\nu)} = \nu$, and $i_{0\alpha}(\nu) = \nu$ for all $\alpha < \nu$. Thus if $\gamma_0 < \gamma_1 < \cdots < \gamma_n < \cdots$ are elements of the class K, and if $\lambda = \lim_{n\to\infty} \gamma_n$, then $\kappa^{(\lambda)} = \lambda$, and $X \in Ult^{(\lambda)}$ belongs to $D^{(\lambda)}$ just in case $X \supseteq \{\gamma_n : n_0 \leq n\}$ for some n_0.

If A is a set of ordinals of order type ω, $A = \{\gamma_n\}_{n=0}^\infty$, we define a filter $F(A)$ over $\lambda = \sup A$ as follows

(33.9) $\quad X \in F(A) \quad \text{iff} \quad \exists n_0 \forall n \geq n_0\, (\gamma_n \in X)$

The above discussion leads us to this: If $A \subset K$ has order type ω, and if $\lambda = \sup A$, then for every $X \in Ult^{(\lambda)}$

(33.10) $\quad X \in D^{(\lambda)} \quad \text{iff} \quad X \in F(A)$

In other words,

(33.11) $\quad D^{(\lambda)} = F(A) \cap Ult^{(\lambda)}$

Hence $F(A) \cap Ult^{(\lambda)} \in Ult^{(\lambda)}$; and so by Lemma 15.4, $L[F(A)] = L[D^{(\lambda)}]$. Thus $F(A) \cap L[F(A)] = D^{(\lambda)} \cap L[D^{(\lambda)}]$, and we have

(33.12) $\quad L[F(A)] \models F(A) \cap L[F(A)]$ is a normal measure on λ.

The only assumption needed to derive (33.12) is that κ is measurable and A is a subset of the class K. We shall now use Lemma 33.2 and a similar construction to obtain a model with two measurable cardinals.

Suppose that $A = \{\gamma_n\}_{n=0}^{\infty}$ is as above, and that $A' = \{\gamma'_n\}_{n=0}^{\infty}$ is another subset of K of order type ω, such that $\gamma'_0 > \lambda = \sup A$; let $\lambda' = \sup A'$. Let $F = F(A)$ and $F' = F(A')$. Our intention is to choose A and A' such that the model $L[F, F']$ has two measurable cardinals, namely λ and λ', and that $F \cap L[F, F']$ and $F' \cap L[F, F']$ are normal measures on λ and λ', respectively.

The argument leading to (33.12) can again be used to show that $F \cap L[F, F']$ is a normal measure on λ in $L[F, F']$. This is because we have again

$$D^{(\lambda)} = F \cap Ult^{(\lambda)}$$

moreover, $i_{0\lambda}(\gamma'_n) = \gamma'_n$ for each n, and hence $i_{0\lambda}(A') = A'$ and we have

$$i_{0\lambda}(F') = F' \cap Ult^{(\lambda)} \tag{33.13}$$

Therefore

$$L[F, F'] = L[D^{(\lambda)}, F'] = L[D^{(\lambda)}, i_{0\lambda}(F')]$$

and

$$F \cap L[F, F'] = D^{(\lambda)} \cap L[D^{(\lambda)}, i_{0\lambda}(F')] \tag{33.14}$$

which gives

$$L[F, F'] \vDash F \cap L[F, F'] \text{ is a normal measure on } \lambda. \tag{33.15}$$

In order to find A, A' so that F' also gives a normal measure in $L[F, F']$, let us make the following observation: Let us think for a moment that $A \subset \kappa$ and $A' \subset K$. Then $i_{0\lambda'}(A) = A$ and $D^{(\lambda')} = F' \cap Ult^{(\lambda')}$, and the same argument as above shows that

$$L[F, F'] \vDash F' \cap L[F, F'] \text{ is a normal measure on } \lambda'. \tag{33.16}$$

We shall use this observation below.

Let us define the following classes of cardinals (compare with (30.34)):

$$\begin{aligned} K_0 &= K \\ K_{\alpha+1} &= \{\nu \in K_\alpha : |K_\alpha \cap \nu| = \nu\} \\ K_\gamma &= \bigcap_{\alpha<\gamma} K_\alpha \qquad (\gamma \text{ a limit}) \end{aligned} \tag{33.17}$$

Each K_α is nonempty; in fact each K_α is unbounded and δ-closed for all δ of cofinality $> \kappa$.

Now we let

$$\begin{aligned} \gamma_n &= \text{the least element of } K_n \\ A &= \{\gamma_n\}_{n=0}^{\infty}, \qquad \lambda = \lim_{n\to\infty} \gamma_n \end{aligned} \tag{33.18}$$

and let $A' = \{\gamma'_n\}_{n=0}^{\infty}$ be a subset of $K_{\omega+1}$.

Let us consider the model $L[A, A']$, and let for each $n \leq \omega$

$$\begin{aligned}(33.19)\quad M_n &= \text{the Skolem hull of } K_n \text{ in } L[A, A'] \\ &= \text{the class of all } x \in L[A, A'] \text{ such that} \\ &\quad L[A, A'] \vDash x = t[v_1, \ldots, v_k, \gamma_0, \ldots, \gamma_k, \gamma'_0, \ldots, \gamma'_k, A, A'] \\ &\quad \text{where } t \text{ is a Skolem term and } v_1, \ldots, v_k \in K_n.\end{aligned}$$

(Let us not worry about the problem whether (33.19) is expressible in the language of set theory; it can be shown that it is, similarly as in the case of ordinal definable sets. Alternatively, we can consider the model $L_\theta[A, A']$ where θ is some large enough cardinal in $K_{\omega+1}$.)

Each M_n is an elementary submodel of $L[A, A']$; let π_n be the transitive collapse of M_n; then $\pi_n[M_n] = L[\pi_n(A), \pi_n(A')]$ and $j_n = \pi_n^{-1}$ is an elementary embedding

$$(33.20)\qquad j_n\colon L[\pi_n(A), \pi_n(A')] \to L[A, A']$$

Lemma 33.4. *For each* $n < \omega$, $\pi_n(\gamma_n) < (2^\kappa)^+$.

Proof. By induction on n. First let $n = 0$. Let $\alpha < \gamma_0$ be in M_0. Then $\alpha = t(v_1, \ldots, v_k, A, A')$ for some Skolem term t and some $v_1, \ldots, v_k \in K_0$. Let $i_{0\alpha}$ be the natural embedding into $Ult_U^{(\alpha)}$ for some U over κ. Since γ_0 is the least element of K_0, we have $\alpha < v$ for all $v \in K_0$ and hence $i_{0\alpha}(v) = v$ for all $v \in K_0$. Hence also $i_{0\alpha}(A) = A$ and $i_{0\alpha}(A') = A'$ and it follows that $i_{0\alpha}(\alpha) = \alpha$. Now $i_{0\alpha}(\alpha) = \alpha$ is possible only if $\alpha < \kappa$. Hence each $\alpha < \gamma_0$ in M_0 is less than κ and therefore $\pi_0(\gamma_0) \leq \kappa$.

Now let us assume that $\pi_n(\gamma_n) < (2^\kappa)^+$ and let us show that $\pi_{n+1}(\gamma_{n+1}) < (2^\kappa)^+$. By Lemma 33.2 there exists U such that $j_U(\kappa) > \pi_n(\gamma_n)$. We shall show that $\pi_n(\alpha) < j_U(\kappa)$ for all $\alpha < \gamma_{n+1}$ in M_{n+1}; since $\pi_{n+1}(\alpha) \leq \pi_n(\alpha)$ it follows that $\pi_{n+1}(\gamma_{n+1}) = \sup\{\pi_{n+1}(\alpha)\colon \alpha < \gamma_{n+1} \text{ and } \alpha \in M_{n+1}\} \leq j_U(\kappa) < (2^\kappa)^+$.

First notice that it follows from the definition of K_{n+1} in (33.17) that $\pi_n(v) = v$ for all $v \in K_{n+1}$. Note also that $\gamma_m \in K_{n+1}$ for all $m \geq n + 1$, and $A' \subset K_{n+1}$.

Let $\alpha < \gamma_{n+1}$ be in M_{n+1}. Then (in $L[A, A']$),

$$\alpha = t(\gamma_0, \ldots, \gamma_n, v_1, \ldots, v_k, A, A')$$

where t is some Skolem term and $v_1, \ldots, v_k \in K_{n+1}$. Hence (in $L[\pi_n(A), \pi_n(A')]$)

$$\pi_n(\alpha) = t(\pi_n(\gamma_0), \ldots, \pi_n(\gamma_n), v_1, \ldots, v_k, \pi_n(A), A')$$

Now we argue inside the model $Ult_U(V)$ (which contains both $\pi_n(A)$ and A'): Consider the αth iterated ultrapower (modulo some measure on $j_U(\kappa)$). Since $\pi_n(\gamma_0), \ldots, \pi_n(\gamma_n)$ are all less than $j_U(\kappa)$, we have $i_{0\alpha}(\pi_n(\gamma_i)) = \pi_n(\gamma_i)$ for all $i = 0, \ldots, n$. We also have $i_{0\alpha}(v) = v$ for each $v \in K_{n+1}$ (because $\alpha < v$ for each

$\nu \in K_{n+1}$ and $K_{n+1} \subset K$). It follows that $i_{0\alpha}(\pi_n(\alpha)) = \pi_n(\alpha)$. Now (because $\pi_n(\alpha) \leq \alpha$) this is only possible if $\pi_n(\alpha) < j_U(\kappa)$. ■

We can now complete the proof of Theorem 80 for $\theta = 2$. Let us consider the model M_ω, the Skolem hull in $L[A, A']$ of K_ω. Let π_ω be the transitive collapse of M_ω and $B = \pi_\omega(A)$. Since $A' \subset K_{\omega+1}$, we have $\pi_\omega(A') = A'$, and $j_\omega = \pi_\omega^{-1}$ is an elementary embedding

$$j_\omega \colon L[B, A'] \to L[A, A']$$

By Lemma 33.4, $\pi_\omega(\gamma_n) \leq \pi_n(\gamma_n) < (2^\kappa)^+$ for all n, and hence $\pi_\omega(\lambda) < (2^\kappa)^+$. Let U be a κ-complete ultrafilter over κ such that $j_U(\kappa) > \pi_\omega(\lambda)$ (by Lemma 33.2).

In Ult_U, B is a subset of $j_U(\kappa)$ and A' is a subset of the class K. Thus we can apply (33.16) and get

$$Ult_U \vDash (L[F(B), F(A')] \vDash F(A') \cap L[F(B), F(A')] \text{ is a normal measure on } \lambda')$$

Hence $L[B, A'] \vDash$

$$(L[F(B), F(A')] \vDash F(A') \cap L[F(B), F(A')] \text{ is a normal measure on } \lambda')$$

and applying j_ω, we get

$L[A, A'] \vDash$

$$(L[F(A), F(A')] \vDash F(A') \cap L[F(A), F(A')] \text{ is a normal measure on } \lambda')$$

Therefore $F' \cap L[F, F']$ is (in $L[F, F']$) a normal measure on λ'.

This completes the proof of the theorem for $\theta = 2$. The proof of the general case is completely analogous for $\theta < \kappa$. Instead of A, A', one uses a sequence $\langle A_\xi \colon \xi < \theta \rangle$, where $\gamma_n^{(\xi)}$, the nth element of A_ξ, is defined as the least element of $K_{\omega\cdot\xi+n}$. One uses the models $M_{\omega\cdot\xi+n}$ to prove an analog of Lemma 33.4: $\pi_{\omega\cdot\xi+n}(\gamma_n^{(\xi)}) < (2^\kappa)^+$. The assumption $\theta < \kappa$ is important because when one uses the ultrapower Ult_U, one needs to know that $\langle A_\xi : \xi < \theta \rangle \in Ult_U$.

Once the theorem is proved for all $\theta < \kappa$, then the general version is derived as follows: If θ is arbitrary, let us consider $Ult^{(\alpha)}(V)$ for some $\alpha > \theta$. Then the model $Ult^{(\alpha)}(V)$ has a strongly compact cardinal greater than θ and therefore has a transitive model with θ measurable cardinals. ■

The following theorem is a further evidence of the impact of large cardinals on cardinal arithmetic.

Theorem 81. (Solovay). *If there exists a compact cardinal, then the singular cardinals hypothesis holds above the compact cardinal. That is, if κ is strongly compact and $\lambda > \kappa$ is a singular cardinal, then $2^{\operatorname{cf}\lambda} < \lambda$ implies $\lambda^{\operatorname{cf}\lambda} = \lambda^+$. (Consequently, if $\lambda > \kappa$ is a singular strong limit cardinal, then $2^\lambda = \lambda^+$.)*

We shall prove the theorem in a sequence of lemmas. An ultrafilter over a cardinal λ is *uniform* if every set in the ultrafilter has size λ.

Lemma 33.5. *If κ is a compact cardinal and $\lambda > \kappa$ is a regular cardinal, then there exists a κ-complete uniform ultrafilter D over λ with the property that almost all* (mod D) *ordinals $\alpha < \lambda$ have cofinality less than κ.*

Proof. Let U be a fine measure on $P_\kappa(\lambda)$. Since U is fine, every $\alpha < \lambda$ belongs to almost all (mod U) $P \in P_\kappa(\lambda)$. Let us consider the ultrapower $Ult_U(V)$ and let f be the least ordinal function in Ult_U greater than all the constant functions c_γ, $\gamma < \lambda$:

$$[f] = \lim_{\gamma \to \lambda} j_U(\gamma) \tag{33.21}$$

We note first that $f(P) < \lambda$ for almost all P: Let g: $P_\kappa(\lambda) \to \lambda$ be the function $g(P) = \sup P$. If $\gamma < \lambda$, then $\gamma \leq g(P)$ for almost all P and hence $j(\gamma) \leq [g]$; thus $[f] \leq [g] < j(\lambda)$.

Let D be the ultrafilter over λ defined as follows:

$$X \in D \quad \text{iff} \quad f_{-1}(X) \in U \qquad (X \subseteq \lambda) \tag{33.22}$$

It is clear that D is κ-complete, and since f is greater than the constant functions, D is nonprincipal. For the same reason, the diagonal function $d(\alpha) = \alpha$ is greater (in Ult_D) than all the constant functions c_γ, $\gamma < \lambda$, and since λ is regular, D is uniform. In order to show that almost all (mod D) $\alpha < \lambda$ have cofinality $< \kappa$, it suffices, by (33.22), to show that $\operatorname{cf}(f(P)) < \kappa$ for almost all P (mod U).

That will follow immediately once we show that for almost all P (mod U),

$$f(P) = \sup\{\alpha \in P : \alpha < f(P)\} \tag{33.23}$$

We clearly have $\geq$ in (33.23). To prove $\leq$, consider the function $h(P) = \sup\{\alpha \in P : \alpha < f(P)\}$. For each $\gamma < \lambda$, γ is in almost every P and hence $\gamma \leq h(P)$ almost everywhere. Thus $[h] \geq j_U(\gamma)$ for all $\gamma < \lambda$ and so $f(P) \leq h(P)$ almost everywhere. ■

Lemma 33.6. *If κ is compact and $\lambda > \kappa$ is a regular cardinal, then there exists a κ-complete nonprincipal ultrafilter D over λ, and a collection $\{M_\alpha : \alpha < \lambda\}$ such that*

(33.24) (a) $|M_\alpha| < \kappa$ *for all* $\alpha < \lambda$,

(b) *for every $\gamma < \lambda$, γ belongs to M_α for almost all α* (mod D)

[An ultrafilter D that has a family $\{M_\alpha : \alpha < \lambda\}$ with property (33.24) is called (κ, λ)-*regular*.]

Proof. Let D be the ultrafilter over λ from Lemma 33.5. For almost all α (mod D), there exists $A_\alpha \subseteq \alpha$ of size less than κ and cofinal in α. If $\operatorname{cf} \alpha \geq \kappa$, let $A_\alpha = \emptyset$. Let A be set of ordinals represented in $Ult_D(V)$ by the function $\langle A_\alpha : \alpha < \lambda \rangle$. The set A is cofinal in the ordinal represented by the diagonal function d; and since $[d] > j_D(\gamma)$ for all $\gamma < \lambda$, it follows that for each $\eta < \lambda$ there is $\eta' > \eta$ such that $A \cap \{\xi : j_D(\eta) \leq \xi < j_D(\eta')\}$ is nonempty.

We construct a sequence $\langle \eta_\gamma : \gamma < \lambda \rangle$ of ordinals $< \lambda$ as follows: let $\eta_0 = 0$ and $\eta_\gamma = \lim_{\delta \to \gamma} \eta_\delta$ if γ is a limit; let $\eta_{\gamma+1}$ be some ordinal such that there exists $\xi \in A$ such that $j_D(\eta_\gamma) \leq \xi < j_D(\eta_{\gamma+1})$.

In other words, if we denote I_γ the interval $\{\xi : \eta_\gamma \leq \xi < \eta_{\gamma+1}\}$, then for every γ, the interval I_γ has nonempty intersection with almost every A_α. Thus if we let

$$M_\alpha = \{\gamma < \lambda : I_\gamma \cap A_\alpha \neq \emptyset\}$$

for each $\alpha < \lambda$, then $\{M_\alpha : \alpha < \lambda\}$ has property (33.24b). To see that M_α has property (a) as well, notice that $|A_\alpha| < \kappa$ for all α and that since the I_γ are mutually disjoint, each A_α intersects less than κ of them. ■

Lemma 33.7. *If κ is compact and $\lambda > \kappa$ is a regular cardinal, then there exists a collection $\{M_\alpha : \alpha < \lambda\} \subseteq P_\kappa(\lambda)$ such that*

$$P_\kappa(\lambda) = \bigcup_{\alpha < \lambda} P(M_\alpha) \tag{33.25}$$

Consequently,

$$\lambda^{<\kappa} = \lambda$$

Proof. Let $\{M_\alpha : \alpha < \lambda\}$ be as in Lemma 33.6. If P is a subset of λ of size less than κ, then by (33.24b) and by κ-completeness of D, $P \subseteq M_\alpha$ for almost all α. Hence $P \in P(M_\alpha)$ for some $\alpha < \lambda$. This proves (33.25); since κ is inaccessible, it follows that $|P_\kappa(\lambda)| = \lambda$. ■

Proof of Theorem 81. Let κ be a compact cardinal. If $\lambda > \kappa$ is an arbitrary cardinal, then we have, by Lemma 33.7,

$$\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+$$

In particular, we have $\lambda^{\aleph_0} \leq \lambda^+$ for every $\lambda > \kappa$. By Theorem 23 (or rather by Lemma 8.3), this implies that the singular cardinals hypothesis holds for every $\lambda > \kappa$. ■

Supercompact Cardinals

We proved in Lemma 33.1 that a compact cardinal κ is characterized by the property that every $P_\kappa(A)$ has a fine measure. If we require the fine measure to satisfy a normality condition, then we obtain a stronger notion—a *supercompact* cardinal. Ultrapowers modulo normal measures on $P_\kappa(A)$ induce elementary embeddings that can be used to derive strong consequences of supercompact cardinals. For instance, Theorems 79 and 80 become almost trivial if existence of a strongly compact cardinal is replaced by existence of a supercompact cardinal. It is consistent to assume that a compact cardinal is not supercompact, or that every compact cardinal is supercompact, but it is not known whether supercompact cardinals are consistent relative to compact cardinals.

A fine measure U on $P_\kappa(A)$ is *normal* if whenever $f\colon P_\kappa(A) \to A$ is such that $f(P) \in P$ for almost all P, then f is constant on a set in U. A cardinal κ is *supercompact* if for every A such that $|A| \geq \kappa$, there exists a normal measure on $P_\kappa(A)$.

Exercise 33.3. If κ is measurable, then there is a normal measure on $P_\kappa(\kappa)$.

Normality of filters over $P_\kappa(A)$ is a generalization of normality of filters over κ. Many results on normality generalize from κ to $P_\kappa(A)$ (but not the existence of a normal measure on a measurable cardinal). For instance, let us call $C \subseteq P_\kappa(\lambda)$ *closed unbounded* if (a) for every P in $P_\kappa(\lambda)$ there is $Q \in C$ such that $P \subseteq Q$, and (b) if $D \subseteq C$ is a nonempty directed (that is, $\forall P_1, P_2 \in D$ $\exists Q \in D$ such that $Q \supseteq P_1 \cup P_2$) set of size $<\kappa$, then $\bigcup D \in C$. [It is not difficult to show that (b) can be replaced by: closed under unions of chains of length $< \kappa$.] The *closed unbounded filter* over $P_\kappa(A)$ is the collection of all subsets of $P_\kappa(A)$ that have a closed unbounded subset. *Stationary* sets are defined as sets that intersect every closed unbounded set.

Exercise 33.4.

(a) Each $\hat{P} = \{Q : P \subseteq Q\}$ is closed unbounded.
(b) The closed unbounded filter is κ-complete.
(c) If $f(P) \in P$ on a stationary set, then f is constant on a stationary set.

[For (c), prove that the closed unbounded filter is closed under the diagonal intersections $\triangle_{a \in A} C_a = \{P : P \in C_a \text{ for all } a \in P\}$.]

Let $\lambda \geq \kappa$ be a cardinal and let us consider the ultrapower $Ult_U(V)$ by a normal measure U over $P_\kappa(\lambda)$; let $j = j_U$ be the corresponding elementary embedding. Clearly, a set $X \subseteq P_\kappa(\lambda)$ belongs to U just in case $[d] \in j(X)$, where d, the "diagonal function" is the function $d(P) = P$.

Lemma 33.8. *If U is a normal measure on $P_\kappa(\lambda)$, then $[d] = \{j(\gamma) : \gamma < \lambda\}$ and hence for every $X \subseteq P_\kappa(\lambda)$,*

$$X \in U \quad \text{iff} \quad \{j(\gamma) : \gamma < \lambda\} \in j(X) \tag{33.26}$$

Proof. On the one hand, if $\gamma < \lambda$, then $\gamma \in P$ for almost all P and hence $j(\gamma) \in [d]$. On the other hand, if $[f] \in [d]$, then $f(P) \in P$ for almost all P and by normality, there is $\gamma < \lambda$ such that $[f] = j(\gamma)$. ■

Let $G = \{j(\gamma) : \gamma < \lambda\}$. It follows from (33.26) that if f and g are functions on $P_\kappa(\lambda)$, then

$$[f] = [g] \quad \text{iff} \quad (jf)(G) = (jg)(G)$$

and

$$[f] \in [g] \qquad \text{iff} \qquad (jf)(G) \in (jg)(G)$$

Consequently,

$$[f] = (jf)(G) \tag{33.27}$$

for every function f on $P_\kappa(\lambda)$.

For each $P \in P_\kappa(\lambda)$, let us denote

$$\kappa_P = P \cap \kappa$$

$$\lambda_P = \text{the order type of } P$$

Note that if we let $G = \{j(\gamma) : \gamma < \lambda\}$, then the order type of G is λ and hence by (33.27), λ is represented in the ultrapower by the function $P \mapsto \lambda_P$. Also, since $\lambda_P < \kappa$ for all P, we have $j(\kappa) > \lambda$. By κ-completeness of U, we have $j(\gamma) = \gamma$ for all $\gamma < \kappa$; and since κ is moved by j, it follows that $G \cap j(\kappa) = \kappa$ and therefore κ is represented by the function $P \mapsto \kappa_P$.

This gives the following characterization of supercompact cardinals:

Lemma 33.9. *Let $\lambda \geq \kappa$. A normal measure on $P_\kappa(\lambda)$ exists if and only if there is an elementary embedding j: $V \to M$ such that*

(33.28) (i) $j(\gamma) = \gamma$ *for all* $\gamma < \kappa$;
(ii) $j(\kappa) > \lambda$;
(iii) ${}^\lambda M \subseteq M$; i.e., *every sequence $\langle a_\alpha : \alpha < \lambda \rangle$ of elements of M is a member of M.*

Proof. (a) Let U be a normal measure on $P_\kappa(\lambda)$. We let $M = Ult_U(V)$ and let j be the natural embedding j: $V \to Ult$. We have already proved (i) and (ii). To prove (iii), it suffices to show that whenever $\langle a_\alpha : \alpha < \lambda \rangle$ is such that $a_\alpha \in M$ for all $\alpha < \lambda$, then the set $\{a_\alpha : \alpha < \lambda\}$ belongs to M. Let f_α, $\alpha < \lambda$, be functions representing elements of M: $[f_\alpha] \in M$. We consider the function f on $P_\kappa(\lambda)$ defined as follows: $f(P) = \{f_\alpha(P) : \alpha \in P\}$; I claim that $[f] = \{a_\alpha : \alpha < \lambda\}$.

On the one hand, if $\alpha < \lambda$, then $\alpha \in P$ for almost all P and hence $[f_\alpha] \in [f]$. On the other hand, if $[g] \in [f]$, then for almost all P, $g(P) = f_\alpha(P)$ for some $\alpha \in P$. By normality, there exists $\gamma < \lambda$ such that $g(P) = f_\gamma(P)$ for almost all P, and hence $[g] = a_\gamma$.

(b) Let j: $V \to M$ be an elementary embedding that satisfies (i), (ii), and (iii). By (iii), the set $\{j(\gamma) : \gamma < \lambda\}$ belongs to M and so the following defines an ultrafilter over $P_\kappa(\lambda)$:

$$X \in U \qquad \text{iff} \qquad \{j(\gamma) : \gamma < \lambda\} \in j(X) \tag{33.29}$$

A standard argument shows that U is a κ-complete ultrafilter. U is a fine measure because for every $\alpha \in \lambda$, $\{P : \alpha \in P\}$ is in U. Finally, U is normal: If

$f(P) \in P$ for almost all P, then $(jf)(G) \in G$ where $G = \{j(\gamma) : \gamma < \lambda\}$. Hence $(jf)(G) = j(\gamma)$ for some $\gamma < \lambda$, and so $f(P) = \gamma$ for almost all P. ■

Exercise 33.5. If U is a normal measure on $P_\kappa(\lambda)$, then every closed unbounded subset of $P_\kappa(\lambda)$ is in U.

[If $C \subseteq P_\kappa(\lambda)$ is closed unbounded, then $D = \{j(P) : P \in C\}$ is a directed subset of $j(C)$ and $|D| = \lambda^{<\kappa} < j(\kappa)$. Hence $\bigcup D \in j(C)$, and since $\bigcup D = \{j(\gamma) \colon \gamma < \lambda\}$, we have $C \in U$.]

We have seen several examples how large cardinals restrict the behavior of the continuum function (e.g., if κ is measurable and $2^\kappa > \kappa^+$, then $2^\alpha > \alpha^+$ for cofinally many $\alpha < \kappa$). This is more so for supercompact cardinals:

Exercise 33.6. If $\lambda \geq \kappa$ and if U is a normal measure on $P_\kappa(\lambda)$, then the ultraproduct $Ult_U\langle(V_{\lambda_P}, \in) \colon P \in P_\kappa(\lambda)\rangle$ is isomorphic to $(V_\lambda, \in)$.

As a typical application, if κ is supercompact and if there is a regular cardinal α such that $2^\alpha = \alpha^+$, then there is a regular cardinal $\alpha < \kappa$ such that $2^\alpha = \alpha^+$.

The following two results illustrate the technique of elementary embeddings.

Lemma 33.10. *If κ is supercompact, then there exists a normal measure D on κ such that almost every $\alpha < \kappa$* (mod D) *is measurable.*

Proof. Let $\lambda = 2^\kappa$ and let $j \colon V \to M$ be an elementary embedding with properties (33.28). Let D be defined by $D = \{X : \kappa \in j(X)\}$, and let $j_D \colon V \to Ult_D$ be the corresponding elementary embedding. Let $k \colon Ult_D \to M$ be the elementary embedding defined in Lemma 28.6:

$$k([f]_D) = (jf)(\kappa)$$

Note that $k(\kappa) = \kappa$.

Now, $P(\kappa) \subseteq M$ and every subset of M of size λ is in M; hence every $U \subseteq P(\kappa)$ is in M and it follows that in M, κ is a measurable cardinal. Since k is elementary and $k(\kappa) = \kappa$, we have $Ult_D \vDash \kappa$ is a measurable cardinal, and the lemma follows. ■

Corollary. *If κ is supercompact, then κ is the* κth *measurable cardinal.*

Incidentally, it is consistent that there exists a compact cardinal with no measurable cardinals below it.

Lemma 33.11. *If κ is supercompact, then for every $\mathcal{W} \subseteq P(\kappa)$ there exists a normal measure D on κ such that $\mathcal{W} \in Ult_D(V)$.*

Proof. Assume on the contrary that $\exists \mathcal{W} \subseteq P(\kappa) \Phi(\kappa, \mathcal{W})$, where Φ stands for:

(33.30) For every normal measure U on κ, $\mathcal{W} \notin Ult_U$.

Let $\lambda = 2^\kappa$, let $j: V \to M$ be an elementary embedding with the properties (33.28), let $D = \{X : \kappa \in j(X)\}$, $j_D: V \to Ult_D$, and let $k: Ult_D \to M$ be as in Lemma 28.6.

By (33.28), every subset of $P(\kappa)$ is in M, and hence $\mathscr{W} \in M$; we have

$$M \vDash \exists \mathscr{W} \subseteq P(\kappa) \Phi(\kappa, \mathscr{W})$$

and by elementarity of k (since $k(\kappa) = \kappa$),

$$Ult_D \vDash \exists \mathscr{W} \subseteq P(\kappa) \Phi(\kappa, \mathscr{W})$$

Let $\mathscr{W}' \in Ult_D$ be such that $Ult_D \vDash \Phi(\kappa, \mathscr{W}')$. Since $k(\mathscr{W}') = \mathscr{W}'$, we have (applying k)

$$M \vDash \Phi(\kappa, \mathscr{W}')$$

and because every $U \subseteq P(\kappa)$ is in M, it follows that $\Phi(\kappa, \mathscr{W}')$ holds (in V). In other words, $\mathscr{W}'$ satisfies (33.30), contrary to the fact that $\mathscr{W}' \in Ult_D$. ∎

Corollary. *If κ is supercompact, then there are 2^{2^κ} normal measures on κ.*

Proof. If D is a normal measure on κ and $\mathscr{W} \subseteq P(\kappa)$ is in Ult_D, then $\mathscr{W}$ is represented by a function f on κ such that $f(\alpha) \subseteq P(\alpha)$ for all $\alpha < \kappa$. Since the number of such functions is 2^κ, it follows that Ult_D contains only 2^κ subsets of $P(\kappa)$.

However, by the lemma, each $\mathscr{W} \subseteq P(\kappa)$ is contained in some ultrapower Ult_D where D is a normal measure on κ, and therefore there must exist 2^{2^κ} normal measures on κ. ∎

It is not known whether the same conclusion follows if κ is only compact.

The following lemma indicates that a strongly compact cardinal is not necessarily supercompact.

Lemma 33.12. *Let κ be a measurable cardinal such that there are κ compact cardinals below κ. Then κ is compact.*

Proof. Let F be a nonprincipal κ-complete ultrafilter over κ such that $C \in F$ where $C = \{\alpha < \kappa : \alpha \text{ is compact}\}$. Let A be such that $|A| \geq \kappa$; we shall show that there is a fine measure on $P_\kappa(A)$.

For each $\alpha \in C$, let U_α be a fine measure on $P_\alpha(A)$, and let us define $U \subseteq P_\kappa(A)$ as follows:

$$X \in U \quad \text{iff} \quad \{\alpha \in C : X \cap P_\alpha(A) \in U_\alpha\} \in F$$

It is easy to verify that U is a fine measure on $P_\kappa(A)$. ∎

Corollary. *If there exists a measurable cardinal that is a limit of compact cardinals, then the least such cardinal is compact but not supercompact.*

Proof. Let κ be the least measurable limit of compact cardinals. By the lemma, κ is compact. Let us assume that κ is supercompact. Let $\lambda = 2^\kappa$ and let $j\colon V \to M$ be an elementary embedding such that κ is the least ordinal moved, and that ${}^\lambda M \subseteq M$. If $\alpha < \kappa$ is compact, then $M \vDash j(\alpha)$ is compact, but $j(\alpha) = \alpha$ and therefore $M \vDash \kappa$ is a limit of compact cardinals. Since every $U \subseteq P(\kappa)$ is in M, κ is measurable in M and hence $M \vDash \kappa$ is a measurable limit of compact cardinals. This is a contradiction because $M \vDash j(\kappa)$ is the least measurable limit of compact cardinals. ■

The assumption of the corollary holds if there are extendible cardinals (see below). There is also a consistency proof that uses forcing showing that not every compact cardinal is supercompact. (And another consistency proof gives a model that has exactly one compact cardinal and the cardinal is supercompact.)

Let κ be a supercompact cardinal. Let $\lambda \geq \kappa$, and let D be a normal measure on $P_\kappa(\lambda)$. Let $G = \{j_D(\alpha) : \alpha < \lambda\}$, where j_D is the elementary embedding of V in $Ult_D(V)$. Recall that κ, λ are represented by κ_P, λ_P where $\kappa_P = P \cap \kappa$ and $\lambda_P =$ the order-type of P. If λ is a cardinal, note that $\lambda_P = |P|$ almost everywhere.

In Section 28 I stated without proof the following combinatorial result (see the remark following Lemma 28.8): If X is an infinite set, then there exists $F\colon {}^\omega X \to X$ such that whenever $Y \subseteq X$ is such that $|Y| = |X|$ then $F[{}^\omega Y] = X$. Let us call F an E–H function on X (Erdös–Hajnal).

Exercise 33.7. Let λ be a cardinal, $\lambda \geq \kappa$, and let D be a normal measure on $P_\kappa(\lambda)$. There is $X \in D$ such that for any $P, Q \in X$, if $P \subset Q$, then $\lambda_P < \lambda_Q$.

[In Ult_D, consider an E–H function F on $G = j_D[\lambda]$. For each $P \in P_\kappa(\lambda)$, let F_P be the restriction of F to ${}^\omega P$. F is represented in Ult by $\langle F_P : P \in P_\kappa(\lambda)\rangle$, and there is $X \in D$ such that F_P is an E–H function on P for all $P \in X$. If $P, Q \in X$ such that $P \subseteq Q$ and $|P| = |Q|$, then $Q = F_Q[{}^\omega Q] = F_P[{}^\omega Q] = P$.]

Exercise 33.8. Let λ be a regular cardinal $\geq \kappa$, let D be a normal measure on $P_\kappa(\lambda)$. There is $X \in D$ such that the function $P \mapsto \sup(P)$ is one-to-one on X.

[Let F be, in Ult_D, an E–H function on G, and let F_P be as above. There is $X \in D$ such that for each $P \in X$, F_P is an E–H function on P and P has the property that if $s \subseteq P$ is countable, then $\sup(s) \in P$ (because G has this property). Now if $P, Q \in X$ and $\sup P = \sup Q = \gamma$, show that $P \cap Q$ is cofinal in γ. Hence $|P \cap Q| = |P|$ and as in Exercise 33.7 it follows that $P \cap Q = P$. Similarly, $P \cap Q = Q$, and hence $P = Q$.]

Note that Exercise 33.8 gives an alternative proof of Theorem 81 if κ is supercompact: We have found $X \in D$ such that $|X| \leq \lambda$, and since $P_\kappa(\lambda) = \bigcup_{Q \in X} P(Q)$, we obtain $\lambda^{<\kappa} = \lambda$.

Let κ be supercompact and let $\lambda \geq \kappa$ be a cardinal. A normal measure D on $P_\kappa(\lambda)$ is *strongly normal* if there exists $X \in D$ such that for every function f on X, if for each $P \in X$, $f(P)$ is in X and $f(P) \subset P$ (and $f(P) \neq P$), then f is constant on some $Y \in D$.

Exercise 33.9. The following are equivalent:

- (i) D is strongly normal.
- (ii) There is $X \in D$ such that if $\{Z_P : P \in X\} \subseteq D$, then $\triangle_{P \in X} Z_P \in D$, where $\triangle_{P \in X} Z_P = \{Q : Q \in Z_P$ for each $P \subset Q$ such that $P \in X\}$.
- (iii) D has this partition property: If $F: [P_\kappa(\lambda)]^2 \to \{0, 1\}$ is a partition, then there is $X \in D$ such that F is constant on $\{\{P, Q\} \in [X]^2 : P \subset Q$ or $Q \subset P\}$.
- (iv) There is $X \in D$ such that if $P, Q \in X$ and $P \subset Q$, then $\lambda_P < \kappa_Q$.

[(i) → (ii): Let $X \in D$ be a witness to strong normality. Prove by contradiction that D is closed under $\triangle_{P \in X} Z_P$.

(ii) → (iii): Let $F: [P_\kappa(\lambda)]^2 \to \{0, 1\}$; for each P, let $F_P: \hat{P} \to \{0, 1\}$ be $F_P(Q) = F(P, Q)$. For each P there is $Z_P \subseteq \hat{P}$, $Z_P \in D$, such that F_P is constant on Z_P. Let $X \in D$ be as in (ii) and such that the constant value of F_P is the same for all $P \in X$. Then $X \cap \triangle_{P \in X} Z_P$ is homogeneous for F in the sense of (iii).

(iii) → (iv): Note that if $X \in D$, then there exist $P, Q \in X$ such that $P \subset Q$ and $\lambda_P < \kappa_Q$.

(iv) → (i): Let $X \in D$ be as in (iv) and let $f: X \to X$ be such that $f(P) \subset P$ for all P. In Ult_D, if $P \in j_D(X)$ and $P \subset G$ (where $G = j[\lambda]$), then $|P| < G \cap \kappa = \kappa$ and hence $P = j(Q)$ for some $Q \in P_\kappa(\lambda)$. Hence $(jf)(G) = j(Q)$ for some Q and so $f(P) = Q$ for almost all P.]

Exercise 33.10. If $\lambda = \kappa^+$, then every normal measure on $P_\kappa(\lambda)$ is strongly normal.

[There is $X \in D$ such that for all $P \in X$, κ_P is inaccessible, $\lambda_P = \kappa_P^+$, and if $P, Q \in X$, $P \subset Q$, then $\lambda_P < \lambda_Q$. Hence if $P, Q \in X$ and $P \subset Q$, then $\kappa_P = \kappa_Q$ is impossible and so $\kappa_P < \kappa_Q$ and consequently $\lambda_P = \kappa_P^+ < \kappa_Q$.]

It has been proved that if κ is supercompact, then every $P_\kappa(\lambda)$ has a strongly normal measure; however, not every normal measure is necessarily strongly normal:

Exercise 33.11. If $\lambda > \kappa$ is measurable, then there is a normal measure U on $P_\kappa(\lambda)$ that is not strongly normal.

[Let $j: V \to M$ be elementary, κ least moved, $j(\kappa) > \lambda$, and ${}^\lambda M \subseteq M$. Let D be a normal measure on λ. Let us define a normal measure U on $P_\kappa(\lambda)$ as follows: $X \in U$ iff $\{\alpha < \lambda : j[\alpha] \in j(X)\} \in D$. If $X \in U$, then there are $\alpha < \beta$ such that $j[\alpha]$ and $j[\beta]$ are in $j(X)$; hence $Ult_U \vDash \exists P, Q \in j(X)$ such that P is an initial segment of Q. Thus $\exists P, Q \in X$ such that P is an initial segment of Q, and so $\lambda_P \geq \kappa_Q$.]

Beyond Supercompactness

I shall conclude this section with several large cardinal notions that are even stronger than supercompactness and in fact are bordering on inconsistency.

A cardinal κ is *extendible* if for every $\alpha > \kappa$ there is β and an elementary embedding $j: V_\alpha \to V_\beta$ such that κ is the least ordinal moved by j.

We shall show that extensibility is stronger than supercompactness. If there is a normal measure on $P_\kappa(\lambda)$, let us say that κ is *λ-supercompact*, and recall the equivalence (33.28).

Lemma 33.13. *Let $\lambda \geq \kappa$ be a regular cardinal and let κ be λ-supercompact. Let $\alpha < \kappa$. If α is γ-supercompact for all $\gamma < \kappa$, then α is λ-supercompact.*

Proof. Let U be a normal measure on $P_\kappa(\lambda)$, and let us consider j_U: $V \to Ult_U$. Since $j(\alpha) = \alpha$, we have $Ult \vDash$ (α is γ-supercompact for all $\gamma < j(\kappa)$); in particular, $Ult \vDash \alpha$ is λ-supercompact. Hence there is D such that $Ult \vDash D$ is a normal measure on $P_\alpha(\lambda)$. Now, $|P_\alpha(\lambda)| = \lambda$ and ${}^\lambda Ult \subseteq Ult$, and hence every subset of $P_\alpha(\lambda)$ is in Ult. It follows that D is a normal measure on $P_\alpha(\lambda)$. ■

Lemma 33.14.

(a) *If κ is extendible, then κ is supercompact.*
(b) *If κ is extendible, then there is a normal measure D on κ such that $\{\alpha < \kappa\colon \alpha$ is supercompact$\} \in D$.*

Proof. (a) Let $\alpha > \kappa$ be a limit cardinal with the property that if $V_\alpha \vDash$ (κ is λ-supercompact for all λ), then κ is supercompact. (Such an α exists by the reflection principle.) Thus it suffices to show that κ is λ-supercompact for all regular $\lambda < \alpha$.

Let j: $V_\alpha \to V_\beta$ be such that κ is the least ordinal moved. Consider the sequence $\kappa_0 = \kappa$, $\kappa_1 = j(\kappa), \ldots, \kappa_{n+1} = j(\kappa_n), \ldots$, as long as $j(\kappa_n)$ is defined. First we note that as in Theorem 68 (see Exercise 28.9) either there is n such that $\kappa_n < \alpha \leq j(\kappa_n)$ or $\alpha = \lim_{n\to\infty} \kappa_n$. Therefore, it is sufficient to prove, by induction on n, that κ is λ-supercompact for each regular $\lambda < \kappa_n$ (if $\lambda < \alpha$).

Clearly, κ is λ-supercompact for each $\lambda < \kappa_1$. Thus let $n \geq 1$ and let us assume that κ is λ-supercompact for all $\lambda < \kappa_n$. Applying j, we get: $V_\beta \vDash$ ($j(\kappa)$ is λ-supercompact for all regular $\lambda < \kappa_{n+1}$). Now we also have $V_\beta \vDash$ (κ is γ-supercompact for all $\gamma < j(\kappa)$), and we can apply Lemma 33.13 (in V_β) to conclude that $V_\beta \vDash$ (κ is λ-supercompact for all regular $\lambda < \kappa_{n+1}$). This completes the induction step.

(b) Let α be some limit ordinal greater than κ and let j: $V_\alpha \to V_\beta$ be such that κ is the least moved. Let $D = \{X \subseteq \kappa : \kappa \in j(X)\}$. By (a), κ is supercompact, and so $V_\beta \vDash$ (κ is γ-supercompact for all $\gamma < j(\kappa)$). Hence $A \in D$, where $A = \{\alpha < \kappa : \alpha$ is γ-supercompact for all $\gamma < \kappa\}$. By Lemma 33.13, every $\alpha \in A$ is supercompact. ■

Let me consider now the following axiom schema called *Vopěnka's principle* (VP): Let **C** be a proper class of models of the same language. Then there exist two members $\mathfrak{A}$, $\mathfrak{B}$ of the class **C** such that $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$.

Lemma 33.15. *If Vopěnka's principle holds, then there exists an extendible cardinal.*

Proof. Let **A** be the class of all limit ordinals α such that $\operatorname{cf} \alpha = \omega$ and that for every $\kappa < \alpha$, if $V_\alpha \vDash$ (κ is extendible), then κ is extendible. Using the reflection principle, we see that **A** is a proper class. Let **C** consist of the models $\langle V_{\alpha+1}, \in\rangle$, for $\alpha \in \mathbf{A}$.

By Vopěnka's principle, there exist $\alpha, \beta \in \mathbf{A}$ and an elementary embedding $j\colon V_{\alpha+1} \to V_{\beta+1}$. Since $j(\alpha) = \beta$, j moves some ordinal; the least ordinal moved is clearly measurable and so it is not α (which has cofinnality ω). Let κ be the least ordinal moved.

Now $V_\alpha \models$ (κ is extendible) because for every $\gamma < \alpha$, $j \mid V_\gamma$ is a witness to it. By the definition of **A**, κ is extendible. ■

A similar argument shows that Vopěnka's principle implies existence of arbitrarily large extendible cardinals.

Let us call a cardinal κ *huge* if there is an elementary embedding $j\colon V \to M$ such that κ is the least ordinal moved and that ${}^{j(\kappa)}M \subseteq M$.

Lemma 33.16. *If κ is a huge cardinal, then Vopěnka's principle is consistent; in fact $\langle V_\kappa, \in\rangle$ is a model of* VP.

Proof. We shall show that if C is a set of models and rank $(C) = \kappa$, then there exist two members $\mathfrak{A}, \mathfrak{B} \in C$ and an elementary embedding $h\colon \mathfrak{A} \to \mathfrak{B}$.

Let $j\colon V \to M$ such that κ is the least ordinal moved and that ${}^{j(\kappa)}M \subseteq M$. Since $\operatorname{rank}(C) = \kappa$, there exists $\mathfrak{A}_0 \in j(C)$ such that $\mathfrak{A}_0 \notin C$. It follows that $j(\mathfrak{A}_0) \neq \mathfrak{A}_0$.

Let $e_0 = j \mid \mathfrak{A}_0$; it is easy to see that e_0 is an elementary embedding of $\mathfrak{A}_0$ in $j(\mathfrak{A}_0)$, and since $|\mathfrak{A}_0| < j(\kappa)$, we have $e_0 \in M$. Hence

$M \models$ there exists $\mathfrak{A} \in j(C)$, $\mathfrak{A} \neq j(\mathfrak{A}_0)$, and there exists an elementary $e\colon \mathfrak{A} \to j(\mathfrak{A}_0)$;

and so there exists $\mathfrak{A} \in C$, $\mathfrak{A} \neq \mathfrak{A}_0$, and there exists an elementary $e\colon \mathfrak{A} \to \mathfrak{A}_0$. Let $\mathfrak{A}$, e be such; clearly,

$$M \models e \text{ is an elementary embedding of } \mathfrak{A} \text{ in } \mathfrak{A}_0$$

and because $\operatorname{rank}(\mathfrak{A}) < \kappa$, we have $\mathfrak{A} = j(\mathfrak{A})$, and hence $\mathfrak{A} \in j(C)$, and so

$M \models$ there exist distinct $\mathfrak{A}, \mathfrak{B} \in j(C)$ and there exists an elementary $h\colon \mathfrak{A} \to \mathfrak{B}$

It follows that there exist distinct $\mathfrak{A}, \mathfrak{B} \in C$, and an elementary embedding $h\colon \mathfrak{A} \to \mathfrak{B}$. ■

Exercise 33.12. A cardinal κ is huge, with $j\colon V \to M$ and $j(\kappa) = \lambda$, if and only if there is a normal κ-complete ultrafilter U over $[\lambda]^\kappa = \{X \subseteq \lambda : |X| = \kappa\}$.

[$X \in U$ iff $j[\lambda] \in j(X)$.]

Exercise 33.13. Let κ be the least huge cardinal and let μ be the least supercompact cardinal. Then $\kappa < \mu$.

[If $\kappa = \mu$ then by Lemmas 33.16, 33.15, 33.14, and 33.13 we get: $V_\mu \models VP$,

$V_\mu \vDash$ ($\exists$ supercompact α), there is a supercompact $\alpha < \mu$, a contradiction. If $\kappa > \mu$, let $j\colon V \to M$ with $\lambda = j(\kappa)$ and ${}^\lambda M \subseteq M$. Since μ is supercompact, let $i\colon V \to N$ be such that $i(\mu) > \lambda$ and $V_{\lambda+2} \subseteq N$. If U is a normal measure on $[\lambda]^\kappa$, then $U \in N$, and hence $N \vDash$ ($\exists$ huge cardinal below $i(\mu)$). Thus there exists a huge cardinal below μ, a contradiction.]

With the definition of huge cardinals, we are approaching the brink of inconsistency. By the proof of Theorem 68 (see Exercise 28.9), if $j\colon V \to M$ is elementary and κ is the least ordinal moved, and if $\lambda = \lim_{n\to\infty} \{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$, then $P(\lambda) \not\subseteq M$.

34. REAL-VALUED MEASURABLE CARDINALS

In this section we prove the following theorems:

Theorem 82 (Solovay).

(a) *If κ is a real-valued measurable cardinal, then there is a transitive model of set theory in which κ is measurable.*

(b) *If κ is a measurable cardinal, then there exists a generic extension in which $\kappa = 2^{\aleph_0}$ and κ is real-valued measurable.*

Theorem 83 (Prikry). *If $2^{\aleph_0}$ is real-valued measurable, then $2^\lambda = 2^{\aleph_0}$ for all infinite $\lambda < 2^{\aleph_0}$.*

We defined real-valued measurable cardinals in Section 27. If μ is a κ-additive real-valued measure on κ, then the ideal I_μ of all sets of measure 0 is a σ-saturated κ-complete ideal over κ. We have proved that if an uncountable cardinal κ carries a σ-saturated κ-complete ideal, then κ is weakly inaccessible.

We shall prove (a) in Theorem 82 and Theorem 83 for this generalization of real-valued measurability, namely under the assumption that κ is uncountable and carries a σ-saturated κ-complete ideal. Thus let κ be an uncountable cardinal and let I be a σ-saturated κ-complete ideal over κ. Let us adopt the following terminology, motivated by real-valued measurability: We say that $X \subseteq \kappa$ has *measure zero* if $X \in I$, and that X has *positive measure* if $X \notin I$. We also say that X has *measure one* if $\kappa - X$ has measure zero.

Let us call $A \subseteq \kappa$ an *atom* if A has positive measure and is not the union of two disjoint sets of positive measure. I is *atomless* if it has no atoms. What we proved in Lemma 27.5b can be formulated as follows: If I is atomless, then $\kappa \le 2^{\aleph_0}$. It follows that if $2^{\aleph_0} < \kappa$, then every set X of positive measure contains an atom $A \subseteq X$, and hence there exists an at most countable disjoint collection W of atoms such that $\kappa = \bigcup \{A\colon A \in W\}$.

We start with the following analog of Lemma 28.11. We recall that a κ-complete ideal over κ is normal if every function $f\colon S \to \kappa$ regressive on a set $S \subseteq \kappa$ of positive measure is constant on some $T \subseteq S$ of positive measure. A real-valued measure μ is *normal* if I_μ is normal.

Lemma 34.1.

(a) *If I is a σ-saturated κ-complete ideal over an uncountable cardinal κ, then there exists a function $f\colon \kappa \to \kappa$ such that*

$$J = f_*(I) = \{X \subseteq \kappa : f_{-1}(X) \in I\}$$

is a normal σ-saturated κ-complete ideal over κ.

(b) *If μ is a κ-additive real-valued measure on κ, then there exists a function $f\colon \kappa \to \kappa$ such that $v = f_*(\mu)$ defined by*

$$v(X) = \mu(f_{-1}(X)) \qquad (X \subseteq \kappa)$$

is a normal κ-additive real-valued measure on κ.

Proof. We shall prove (a) and leave the completely analogous proof of (b) to the reader. Let us say that a function $g\colon S \to \kappa$ is *unbounded* on the set S of positive measure if there is no $\gamma < \kappa$ and no $T \subseteq S$ of positive measure such that $g(\alpha) < \gamma$ for all $\gamma \in T$. Let us consider the family $\mathscr{F}$ of all functions g into κ defined on a set of positive measure and unbounded on its domain. Let us define $g < h$ if $\operatorname{dom}(g) \subseteq \operatorname{dom}(h)$ and if $g(\alpha) < h(\alpha)$ everywhere on $\operatorname{dom}(g)$. Let us also define $g \leq h$ if $\operatorname{dom}(g) \subseteq \operatorname{dom}(h)$ and if $g(\alpha) \leq h(\alpha)$ everywhere on $\operatorname{dom}(g)$. (Note that $g \leq h$ does not mean $g < h$ or $g = h$.) Let us call $g \in \mathscr{F}$ *minimal* if there is no $h \in \mathscr{F}$ such that $h < g$.

We shall first show that there exists a minimal $g \in \mathscr{F}$. Otherwise, for every $g \in \mathscr{F}$ there is $h \in \mathscr{F}$ such that $h < g$. Thus let $g \in \mathscr{F}$ be arbitrary. Let W be a maximal collection of elements of $\mathscr{F}$ such that $h < g$ for each $h \in W$, and that $\operatorname{dom}(h_1) \cap \operatorname{dom}(h_2) = \emptyset$ whenever h_1 and h_2 are distinct elements of W. Since I is σ-saturated, W is at most countable and by our assumption, the set $\operatorname{dom}(g) - \bigcup \{\operatorname{dom}(h) : h \in W\}$ has measure zero. Thus if we let $f = \bigcup \{h : h \in W\}$, we have $\operatorname{dom}(g) - \operatorname{dom}(f) \in I$, and $f < g$. Since g was arbitrary, we can construct a countable sequence $g_0 > g_1 > \cdots > g_n > \cdots$ such that $\operatorname{dom}(g_n) - \operatorname{dom}(g_{n+1}) \in I$ for each n. It follows that $\bigcap_{n=0}^{\infty} \operatorname{dom}(g_n)$ has positive measure and we get a contradiction since for any $\alpha \in \bigcap_{n=0}^{\infty} \operatorname{dom}(g_n)$ we would have $g_0(\alpha) > g_1(\alpha) > \cdots$.

The same argument shows that for every $h \in \mathscr{F}$ there exists a minimal $g \in \mathscr{F}$ such that $g \leq h$. Thus if W is a maximal family of minimal functions $g \in \mathscr{F}$ such that $\operatorname{dom}(g_1) \cap \operatorname{dom}(g_2) = \emptyset$ whenever g_1 and g_2 are distinct elements of W, W is at most countable and $\bigcup\{\operatorname{dom}(g) : g \in W\}$ has measure one. Thus if we let $f = \bigcup \{g : g \in W\}$, then $\operatorname{dom}(f)$ has measure one and f is a *least* unbounded function: On the one hand, if $\gamma < \kappa$, then there is no $S \subseteq \kappa$ of positive measure such that $f(\alpha) < \gamma$ everywhere on S; on the other hand, if S is a set of positive measure and g is a function on S such that $g(\alpha) < f(\alpha)$ everywhere on S, then g is constant on some $T \subseteq S$ of positive measure. We can clearly assume that $\operatorname{dom}(f) = \kappa$.

Let $f\colon \kappa \to \kappa$ be a least unbounded function; we shall show that $J = f_*(I)$ is a normal σ-saturated κ-complete ideal over κ. It is obvious that J is a κ-complete ideal. For every $\gamma \in \kappa$, $f_{-1}(\{\gamma\})$ has measure zero and hence $\{\gamma\} \in J$. If $X \notin J$, then $f_{-1}(X) \notin I$, and if $X \cap Y = \emptyset$, then $f_{-1}(X) \cap f_{-1}(Y) = \emptyset$, and hence J is σ-saturated because I is σ-saturated.

To show that J is normal, let $S \notin I$, and let $g(\alpha) < \alpha$ for all $\alpha \in S$. Then $g(f(\xi)) < f(\xi)$ for all $\xi \in f_{-1}(S)$ and since f is a least unbounded function, $g(f(\xi))$ is constant on some $X \subseteq f_{-1}(S)$ of positive I-measure. Hence g is constant on $f[X]$, and $f[X] \notin J$. ■

Exercise 34.1. Show that if g is a minimal function in $\mathscr{F}$, then $g_*(I)$ is a normal σ-saturated κ-complete ideal over κ.

Lemma 34.2. *Let I be a normal σ-saturated κ-complete ideal over κ. If S is a set of positive measure and $f\colon S \to \kappa$ is regressive on S, then f is bounded almost everywhere on S; that is, there exists $\gamma < \kappa$ such that $\{\alpha \in S : f(\alpha) \geq \gamma\} \in I$.*

Proof. For every $X \subseteq S$ of positive measure there exists $Y \subseteq X$ of positive measure such that f is constant on Y. Thus let W be a maximal disjoint family of sets $X \subseteq S$ of positive measure such that f is constant on X. Let $T = \bigcup \{X : X \in W\}$. The family W is at most countable and hence there is γ such that $f(\alpha) < \gamma$ for all $\alpha \in T$. Clearly, $S - T$ has measure 0. ■

Corollary. *If κ is real-valued measurable (or if κ carries a σ-saturated κ-complete ideal), then κ is a weakly Mahlo cardinal.*

Proof. Let I be a normal σ-saturated κ-complete ideal over κ. Since I is normal, every closed unbounded set has I-measure one (see Lemma 7.7). Because κ is weakly inaccessible, it suffices to show that the set of all regular cardinals $\alpha < \kappa$ has measure one.

Let us assume that the set S of all limit ordinals $\alpha < \kappa$ such that $\operatorname{cf} \alpha < \alpha$ has positive measure. Considering the regressive function $\alpha \mapsto \operatorname{cf}(\alpha)$, we find a set T of positive measure and some $\lambda < \kappa$ such that $\operatorname{cf} \alpha = \lambda$ for all $\alpha \in T$. For each $\alpha \in T$, let $\langle \alpha_\nu : \nu < \lambda \rangle$ be an increasing λ-sequence with the limit α.

For each $\nu < \lambda$, the function $\alpha \mapsto \alpha_\nu$ is regressive on T and so, by Lemma 34.2 there is γ_ν such that $\alpha_\nu < \gamma_\nu$ for almost all $\alpha \in T$. Let $\gamma = \sup\{\gamma_\nu : \nu < \lambda\}$. Since $\lambda < \kappa$, we conclude, by κ-completeness of I, that for almost all $\alpha \in T$, $\alpha_\nu < \gamma$ for all $\nu < \lambda$. But this means that for almost all $\alpha \in T$, $\alpha = \lim_\nu \alpha_\nu$ is $\leq \gamma$. This is a contradiction since T is unbounded. ■

Since every closed unbounded set has measure one (if I is a normal σ-saturated κ-complete ideal over κ), every set of positive measure is stationary. It can even be proved that if S has positive measure, then $S \cap \alpha$ is stationary in α for almost all α. Then it follows that κ is the κth weakly Mahlo cardinal, κth cardinal which is a limit of weakly Mahlo cardinals, etc. I shall return to this subject in the next section.

Exercise 34.2. If κ carries a σ-saturated κ-complete ideal, then κ has the tree property.

[As in Lemma 29.7, let T' be the set of all $x \in T$ such that the set of all successors of x has positive measure. T' is a tree of height κ and each level of T' is at most countable; show that T' has a κ-branch.]

We shall now show that every real-valued measurable cardinal is a Rowbottom cardinal; we shall show that the statement of Lemma 29.10 for a measurable cardinal holds under the weaker assumption that κ carries a σ-saturated κ-complete ideal.

Lemma 34.3. *Let I be a normal σ-saturated κ-complete ideal over κ, and let λ be an infinite cardinal less than κ. Let $\mathfrak{A} = \langle A, \ldots\rangle$ be a model of a language $\mathscr{L}$ such that $|\mathscr{L}| \leq \lambda$, and let $A \supseteq \kappa$. If $P \subset A$ is such that $|P| < \kappa$, then $\mathfrak{A}$ has an elementary submodel $\mathfrak{B} = \langle B, \ldots\rangle$ such that $B \cap \kappa$ has measure one and $|P \cap B| \leq \lambda$. Moreover, if $X \subset A$ is of size at most λ, then we can find $\mathfrak{B}$ such that $X \subset B$.*

As in the proof of Lemma 29.10 (or Theorem 71), the result follows from this combinatorial lemma:

Lemma 34.4. *Let I be as in Lemma* 34.3, *let $\gamma < \kappa$ and let $f\colon [\kappa]^{<\omega} \to \gamma$ be a partition. Then there exists $H \subseteq \kappa$ of measure one such that the image of $[H]^{<\omega}$ under f is at most countable.*

Proof. We proceed as in the proof of Theorem 70. It suffices to show that for each $n = 1, 2, \ldots$ there is H_n of measure one such that $f[[H]^n]$ is at most countable; then we take $H = \bigcap_{n=1}^{\infty} H_n$.

We prove, by induction on n, that for every partition of $[\kappa]^n$ into less than κ pieces there is $H \subseteq \kappa$ of measure one such that $f[[H]^n]$ is at most countable. For $n = 1$, let $f\colon \kappa \to \gamma$ and $\gamma < \kappa$; let W be a maximal pairwise disjoint family of subsets $X \subseteq \kappa$ such that X has positive measure and f is constant on X. Let $H = \bigcup \{X : X \in W\}$. Since $|W| \leq \aleph_0$, we have $|f[H]| \leq \aleph_0$, and since $\gamma < \kappa$ and I is κ-complete, we clearly have $\kappa - H \in I$.

Let us assume that the assertion is true for n and let us prove that it holds also for $n + 1$. Let $f\colon [\kappa]^{n+1} \to \gamma$ where $\gamma < \kappa$. For each $\alpha < \kappa$, we define f_α on $[\kappa - \{\alpha\}]^n$ by $f_\alpha(x) = f(\{\alpha\} \cup x)$. By the induction hypothesis, there exists for each $\alpha < \kappa$ a set X_α of measure one such that $f_\alpha[[X_\alpha]^n]$ is at most countable; let A_α be the image of $[X_\alpha]^n$ under f_α. Let X be the diagonal intersection

$$X = \{\alpha < \kappa : \alpha \in \bigcap_{\xi < \alpha} X_\xi\}$$

The set X has measure one since I is normal; also if $\alpha < \alpha_1 < \cdots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\alpha]^n$ and so $f(\{\alpha, \alpha_1, \ldots, \alpha_n\}) = f_\alpha(\{\alpha_1, \ldots, \alpha_n\}) \in A_\alpha$.

For each $\alpha \in X$, let $A_\alpha = \{a_{\alpha,n} : n < \omega\}$. For each n, consider the function $g_n\colon X \to \gamma$ defined by $g_n(\alpha) = a_{\alpha,n}$. There exists a set $H_n \subseteq X$ of measure one such that $g_n[H_n]$ is at most countable. Thus let $H = \bigcap_{n=0}^{\infty} H_n$; the set H has measure one, and moreover $\bigcup \{A_\alpha : \alpha \in H\} = \bigcup_{n=0}^{\infty} g_n[H]$ is at most countable. It follows that $f[[H]^{n+1}]$ is at most countable. ■

We can now proceed as in Theorem 75 and prove that if I is a normal σ-saturated κ-complete ideal over κ and $V = L[I]$, then GCH holds. In fact, if D denotes the filter dual to I, that is, the filter of all sets of I-measure one, then the proof of Theorem 75 goes through verbatim in the present context.

Now we recall the results of Section 27: If κ carries a σ-saturated κ-complete ideal then either $\kappa \leq 2^{\aleph_0}$ or κ is measurable. Thus we conclude: If I is a normal σ-saturated κ-complete ideal over κ and $V = L[I]$, then κ is measurable.

Proof of Theorem 82a. Let κ be real-valued measurable. Then there is a normal κ-additive measure μ on κ by Lemma 34.1. Let I be the ideal of sets of measure zero, and let us consider the model $L[I]$. Let $J = I \cap L[I]$. We have $L[J] = L[I]$, and it is easy to verify that in $L[I]$, J is a normal σ-saturated κ-complete ideal over κ. Thus in $L[I]$, κ is a measurable cardinal. ■

Let me now prove the generalization of Theorem 82a for σ-saturated ideals. Since I am not able to show directly that if I is countably saturated, then $I \cap L[I]$ is countably saturated in $L[I]$, let me consider a somewhat more general situation. Let ν be a regular uncountable cardinal less than κ, and let us consider ν-saturated κ-complete ideals over κ. (Hence σ-saturated is the special case when $\nu = \aleph_1$.)

Lemmas 34.1 and 34.2 hold again; in Lemma 34.4, we have to replace "at most countable" by "of size less than ν." Lemma 34.3 holds for all $\lambda \geq \nu$ and the analog of Theorem 75 is: If $V = L[I]$ and I is normal, then $2^{<\nu} = \nu$ and $2^\lambda = \lambda^+$ for all $\lambda \geq \nu$.

Lemma 27.5 can also be generalized, and we get: If I is atomless, then $\kappa \leq 2^{<\nu}$. Hence if $V = L[I]$, every set of positive measure contains a subset that is an atom, and therefore κ is the union of a disjoint family W of atoms such that $|W| < \nu$.

Lemma 34.5. *Let $\nu < \kappa$ be a regular uncountable cardinal, and let I be a normal ν-saturated κ-complete ideal over κ; let F be the dual filter. Then in $L[F]$, $F \cap L[F]$ is a normal measure on κ (and $L[F]$ is the model $L[D]$ of Section* 31).

Proof. To start, it is easy to verify that $L[F] = L[I]$, and that in $L[F]$, $I \cap L[I]$ is a normal ν-saturated κ-complete ideal over κ. Thus we may assume that $V = L[F]$; we want to show that F is an ultrafilter.

We know that κ is the union of a disjoint family W of atoms. (What we want to show is that W has only one element.).For $A \in W$, let

$$F_A = \{X \subseteq \kappa : X \cap A \text{ has positive measure}\}$$

Since A is an atom, F_A is a filter, and it is easy to see that F_A is in fact a normal measure on κ. Hence $F_A \cap L[F_A]$ is the unique normal measure D in $L[F_A]$, and $L[F_A]$ is *the* model $L[D]$.

We shall now show that $F \cap L[D] = D$. Let $X \in L[D]$ be a subset of κ. If $X \in F$, then $X \in F_A$ for all $A \in W$ and hence $X \in F_A \cap L[F_A] = D$. If $X \notin F$, then there is $A \in W$ such that $X \notin F_A$ and hence $X \notin F_A \cap L[F_A] = D$. It follows that $F \cap L[D] = D$ and so $F \cap L[D] \in L[D]$.

Consequently, $L[F] = L[D]$; since we assumed that $V = L[F]$ and because $F \cap L[D] = D$, we have $F = D$. ■

A Model in Which $2^{\aleph_0}$ is Real-Valued Measurable

Let κ be a measurable cardinal, and let $\lambda \geq \kappa$ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. We shall construct a generic extension in which $2^{\aleph_0} = \lambda$ and κ is real-valued measurable.

Let F be a σ-complete field of sets and let μ be a measure on F (cf. Section 27). Let $I \subseteq F$ be the ideal of sets of measure 0 and let us consider the Boolean algebra $B = F/I$. That is, the members of B are equivalence classes $[X]$ where $X \in F$ and where $X \equiv Y$ just in case $\mu(X \bigtriangleup Y) = 0$.

Since both F and I are countably complete, it follows easily that B is countably complete and $\sum_{n=0}^{\infty} [X_n] = [\bigcup_{n=0}^{\infty} X_n]$. Since μ is a measure, I is countably saturated and so B satisfies the countable chain condition. Now a Boolean algebra that is both σ-complete and σ-saturated is complete, and so B is a complete Boolean algebra.

For $[X] \in B$, let us define $m([X]) = \mu(X)$. Clearly, the definition of m does not depend on the particular choice of X, and furthermore, m has the following properties:

(34.1) (i) m is a real-valued function on B;
(ii) $m(0) = 0$, $m(a) > 0$ if $a \neq 0$, and $m(1) = 1$;
(iii) if $a \leq b$, then $m(a) \leq m(b)$;
(iv) if a_n, $n = 0, 1, \ldots$, are pairwise disjoint, then

$$m\left(\sum_{n=0}^{\infty} a_n\right) = \sum_{n=0}^{\infty} m(a_n)$$

A Boolean algebra B with a measure m (satisfying (34.1)) is called a *measure algebra*; a set S with a field of sets $\mathscr{F}$ and a measure μ on $\mathscr{F}$ is called a *measure space*.

We need from measure theory the following basic fact about *products* of measure spaces. Let I be a set (of indices), and for each $i \in I$ let $(S_i, \mathscr{F}_i, \mu_i)$ be a measure space. Let us consider the Cartesian product $S = \prod_{i \in I} S_i$, and let us consider the following σ-complete field of subsets of S:

Let E be a finite subset of I, and for each $i \in E$, let $Z_i \in \mathscr{F}_i$. Let $Z \subseteq S$ be as follows: If $t \in \prod_i S_i$, then

$$t \in Z \quad \text{iff} \quad t(i) \in Z_i \text{ for all } i \in E \tag{34.2}$$

Let $\mathscr{F}$ be the least σ-complete field of subsets of S such that $\mathscr{F}$ contains every $Z \subseteq S$ of the form (34.2), for any finite $E \subseteq I$ and any $Z_i \in \mathscr{F}_i$, $i \in E$.

There exists a unique measure μ on $\mathscr{F}$ (the *product measure*) such that for every Z of the form (34.2), $\mu(Z)$ is the product of $\mu_i(Z_i)$, $i \in E$. (In case of the product $S = S_1 \times S_2$, the measure of a "rectangle" $Z_1 \times Z_2$ is equal to $\mu_1(Z_1) \cdot \mu_2(Z_2)$.)

We shall use the following simple example of a product measure space. Let I be an infinite set, and for each $i \in I$ let us consider the space $\{0, 1\}$ of two elements. We give measure $\frac{1}{2}$ to both $\{0\}$ and $\{1\}$:

$$(34.3) \qquad S_i = \{0, 1\}, \qquad \mathscr{F}_i = P(S_i)$$

$$\mu_i(\{0\}) = \mu_i(\{1\}) = \tfrac{1}{2}, \qquad \mu(\varnothing) = 0, \qquad \mu(\{0, 1\}) = 1$$

(The measures μ_i do not satisfy the condition (iii) in 27.1.) Let $S = \prod_{i \in I} S_i$, and let μ be the product measure on $\mathscr{F}$, the least σ-complete field of subsets of S containing the sets $\{t \in {}^I\{0, 1\} : t(i) = 0\}$, for all $i \in I$.

Let $\mathfrak{M}$ be a transitive model of ZFC (the ground model). In $\mathfrak{M}$, let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$. Let $(S, \mathscr{F}, \mu)$ be the product measure space ${}^I\{0, 1\}$ defined above, where $I = \lambda \times \omega$. Let B be the corresponding measure algebra $\mathscr{F}$/the ideal of sets of measure 0.

Let G be an $\mathfrak{M}$-generic ultrafilter on B. Since B satisfies the countable chain condition, the generic extension $\mathfrak{M}[G]$ preserves cardinals. We shall show that in $\mathfrak{M}[G]$, $2^{\aleph_0} = \lambda$.

On the one hand, an easy computation gives $|\mathscr{F}| = \lambda$ (because $\lambda^{\aleph_0} = \lambda$) and since B satisfies the c.c.c., we get $|B| = \lambda$. Therefore

$$(2^{\aleph_0})^{\mathfrak{M}[G]} \leq (|B|^{\aleph_0})^{\mathfrak{M}} = \lambda$$

and we have $(2^{\aleph_0})^{\mathfrak{M}[G]} \leq \lambda$.

On the other hand, we shall exhibit λ distinct subsets of ω in $\mathfrak{M}[G]$. For each $\alpha < \lambda$ and each $n < \omega$, let $u_{\alpha, n} = [U_{\alpha, n}]$, where $U_{\alpha, n} \in \mathscr{F}$ is as follows:

$$(34.4) \qquad U_{\alpha, n} = \{t \in {}^{\lambda \times \omega}\{0, 1\} : t(\alpha, n) = 1\}$$

For $\alpha < \lambda$, let $\boldsymbol{x}_\alpha$ be the B-valued subset of ω such that

$$(34.5) \qquad \|\check{n} \in \boldsymbol{x}_\alpha\| = u_{\alpha n} \qquad (n < \omega)$$

Let x_α be the G-interpretation of $\boldsymbol{x}_\alpha$.

We shall show that $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$, and in fact that $\|\boldsymbol{x}_\alpha = \boldsymbol{x}_\beta\| = 0$. Let k be any natural number. Then

$$\|\boldsymbol{x}_\alpha \cap \check{k} = \boldsymbol{x}_\beta \cap \check{k}\| = [N_{\alpha, \beta, k}]$$

where

$$N_{\alpha, \beta, k} = \{t : t(\alpha, n) = t(\beta, n) \text{ for all } n < k\}$$

It is easy to verify that for each k,

$$\mu(N_{\alpha\beta k}) = 1/2^k$$

But $\|x_\alpha = x_\beta\| = \prod_{k=0}^{\infty} [N_{\alpha\beta k}] = [\bigcap_{k=0}^{\infty} N_{\alpha\beta k}] = 0$ since $\mu(\bigcap_{k=0}^{\infty} N_{\alpha\beta k}) = 0$. This completes the proof that $2^{\aleph_0} = \lambda$ in $\mathfrak{M}[G]$.

Exercise 34.3. No x_α is in the ground model. If $a \subseteq \omega$, then $\|x_\alpha = \check{a}\| = 0$.
[Show that for each k, $\|x_\alpha \cap \check{k} = \check{a} \cap \check{k}\| = [D_{\alpha, k}]$ where $\mu(D_{\alpha, k}) = 1/2^k$.]

Now let us assume that κ is a measurable cardinal in the ground model, and let $\lambda \geq \kappa$ be such that $\lambda^{\aleph_0} = \lambda$. We construct a generic extension $\mathfrak{M}[G]$ of $\mathfrak{M}$, using the measure algebra described above. In $\mathfrak{M}[G]$, we have $2^{\aleph_0} = \lambda$, and all we have to do is to show that κ is real-valued measurable in $\mathfrak{M}[G]$. This follows from this general lemma:

Lemma 34.6. *Let κ be a measurable cardinal in the ground model $\mathfrak{M}$, let B be (in $\mathfrak{M}$) a measure algebra, and let G be an $\mathfrak{M}$-generic ultrafilter on B. Then in $\mathfrak{M}[G]$, there exists a nontrivial κ-additive measure on κ.*

Proof. Let U be a κ-complete nonprincipal ultrafilter over κ. Let B be a complete Boolean algebra and let m be a measure on B. We shall define a B-valued name μ and show that if G is a generic ultrafilter, then the G-interpretation of μ is a nontrivial κ-additive measure on κ.

Let a be a nonzero element of B, and let $A \in \mathfrak{M}^B$ be a B-valued name such that $a \Vdash A \subseteq \check{\kappa}$. For each $\alpha < \kappa$, we let

$$f_a(A, \alpha) = \frac{m(a \cdot \|\check{\alpha} \in A\|)}{m(a)} \tag{34.6}$$

Since U is κ-complete, there is a unique real number r such that $f_a(A, \alpha) = r$ for almost all α (mod U). Thus let

(34.7)

$\mu_a(A) =$ the unique r such that $f_a(A, \alpha) = r$ almost everywhere(mod U)

Note that if $a \Vdash A = A'$, then $\mu_a(A) = \mu_a(A')$. Also, if $a \Vdash A_1 \subseteq A_2$, then $\mu_a(A_1) \leq \mu_a(A_2)$. If $X \subseteq \kappa$ is in $\mathfrak{M}$, then $f_a(\check{X}, \alpha) = 1$ for all $\alpha \in X$ and $= 0$ for all $\alpha \notin X$. Hence $\mu_a(\check{X}) = 1$ if $X \in U$ and $\mu_a(\check{X}) = 0$ if $X \notin U$.

Let $\gamma < \kappa$ and let A_ξ, $\xi < \gamma$, be such that $a \Vdash A_\xi \subseteq \check{\kappa}$ for all $\xi < \gamma$, and that $a \Vdash A_\xi \cap A_\eta = \emptyset$ whenever $\xi \neq \eta$. Let A be such that $a \Vdash A = \bigcup_{\xi<\gamma} A_\xi$. Then $f_a(A, \alpha) = \sum_{\xi<\gamma} f_a(A_\xi, \alpha)$ for all $\alpha < \kappa$, and hence (because U is κ-complete),

$$\mu_a(A) = \sum_{\xi<\gamma} \mu_a(A_\xi) \tag{34.8}$$

Let r be a real number, $0 \leq r \leq 1$, and let $\{a_n\}_{n=0}^{\infty}$ be a partition of $a \in B$. If $\mu_{a_n}(A) < r$ for all n, then for almost all α, $m(a_n \cdot \|\check{\alpha} \in A\|) < r \cdot m(a_n)$, and it follows that for almost all α, $m(a \cdot \|\check{\alpha} \in A\|) < r \cdot m(a)$; hence $\mu_a(A) < r$.

As a consequence, we obtain:

(34.9) If for every nonzero $b \leq a$ there is a nonzero $c \leq b$ such that $\mu_c(A) < r$, then $\mu_a(A) < r$.

(And a similar statement holds when $<$ is replaced by $\leq$, $>$ or $\geq$.)

Now if $b \Vdash A \subseteq \check{\kappa}$, we define

$$\mu_b^*(A) = \inf_{a \leq b} \mu_a(A) \tag{34.10}$$

Again, if $b \Vdash A_1 \subseteq A_2$, then $\mu_b^*(A_1) \leq \mu_b^*(A_2)$, and if $X \in \mathfrak{M}$, then $\mu_b^*(\check{X}) = 1$ if $X \in U$ and $= 0$ if $X \in U$. However, μ_b^* is not additive and we only have (using (34.8)), for $\gamma < \kappa$:

$$\mu_b^*(A) \geq \sum_{\xi < \gamma} \mu_b^*(A_\xi) \tag{34.11}$$

under the assumption that $b \Vdash A_\xi \cap A_\eta = \emptyset$ whenever $\xi \neq \eta$, and that $b \Vdash A = \bigcup_{\xi < \gamma} A_\xi$.

Note that if $b_1 \leq b_2$, then $\mu_{b_1}(A) \geq \mu_{b_2}(A)$.

Now we are ready to define $\boldsymbol{\mu}$. Let G be a generic ultrafilter; in $\mathfrak{M}[G]$, we define μ: $P(\kappa) \to [0, 1]$ as follows:

$$\mu(A) = \sup_{b \in G} \mu_b^*(A) \tag{34.12}$$

where A is a name for A. Let $\boldsymbol{\mu}$ be the canonical name for μ (defined in $\mathfrak{M}^B$ by (34.12) using the canonical $\boldsymbol{G}$).

It is clear that μ does not depend on the name A for A, that $A_1 \subseteq A_2$ implies $\mu(A_1) \leq \mu(A_2)$ and that if $X \in \mathfrak{M}$, then $\mu(X) = 1$ if $X \in U$ and $\mu(X) = 0$ if $X \notin U$. It remains to show that μ is κ-additive.

I shall now explain the meaning of μ_b^*. Let r be a real number (in $\mathfrak{M}$) such that $0 \leq r \leq 1$. I claim that

$$\mu_b^*(A) \geq r \qquad \text{iff} \qquad b \Vdash \boldsymbol{\mu}(A) \geq \check{r} \tag{34.13}$$

If $\mu_b^*(A) \geq r$, then for every generic G such that $b \in G$, $\mu(A) \geq r$, and hence $b \Vdash \boldsymbol{\mu}(A) \geq \check{r}$. Thus assume that $b \Vdash \boldsymbol{\mu}(A) \geq \check{r}$. Then

$$b \Vdash \forall q < \check{r}\ \exists d \in \boldsymbol{G}\ \mu_d^*(A) \geq q$$

that is,

$$\forall q < r\ \forall c \leq b\ \exists d \leq c\ \mu_d^*(A) \geq q \tag{34.14}$$

Let $q < r$; I claim that $\mu_b^*(A) \geq q$. If $a \leq b$, then $\forall c \leq a\ \exists d \leq c$ such that $\mu_d^*(A) \geq q$ and hence (by a variant of (34.9)), $\mu_a(A) \geq q$. Thus $\mu_b^*(A) \geq q$. Since this holds for any $q < r$, we have $\mu_b^*(A) \geq r$.

Next we show that μ is finitely additive. Let A, A_1, and A_2 be such that every condition forces that A is the disjoint union of A_1 and A_2. If r_1 and r_2 are real numbers and if $b \Vdash (\boldsymbol{\mu}(A_1) \geq \check{r}_1$ and $\boldsymbol{\mu}(A_2) \geq \check{r}_2)$, then by (34.13) and (34.11), $b \Vdash \boldsymbol{\mu}(A) \geq \check{r}_1 + \check{r}_2$; hence $\mu(A) \geq \mu(A_1) + \mu(A_2)$. Conversely, let us assume that $\mu(A) > \mu(A_1) + \mu(A_2)$. There are reals r_1, $r_2 \in \mathfrak{M}$ and $b \in G$ such that

$$b \Vdash \boldsymbol{\mu}(A_1) < \check{r}_1, \boldsymbol{\mu}(A_2) < \check{r}_2, \text{and } \boldsymbol{\mu}(A) \geq \check{r}_1 + \check{r}_2$$

Since $b \Vdash \boldsymbol{\mu}(A_1) < \check{r}_1$, there is for each $c \leq b$ some $d \leq c$ such that $\mu_d(A_1) < r_1$;

hence by (34.9), $\mu_b(A_1) < r_1$. Similarly, $\mu_b(A_2) < r_2$, and so $\mu_b^*(A) \le \mu_b(A) < r_1 + r_2$. This is a contradiction.

Now when we know that μ is finitely additive, it suffices to show that $\mu(\bigcup_{\xi<\gamma} A_\xi) \le \sum_{\xi<\gamma} \mu(A_\xi)$ for any family $\{A_\xi : \xi < \gamma\}$ of less than κ subsets of κ. Thus let $\gamma < \kappa$ and let A_ξ, $\xi < \gamma$, and A be such that $\|A = \bigcup_{\xi<\gamma} A_\xi\| = 1$, and let us assume that $\mu(A) > \sum_{\xi<\gamma} \mu(A_\xi)$. Then there is $r \in \mathfrak{M}$ and $b \in G$ such that

$$b \Vdash \sum_{\xi<\gamma} \boldsymbol{\mu}(A_\xi) < \check{r} \text{ and } \boldsymbol{\mu}(A) > \check{r}$$

Let $E \subseteq \gamma$ be an arbitrary finite set, let $A_E = \bigcup_{\xi \in E} A_\xi$. Since $\|\boldsymbol{\mu}(A_E) \le \sum_{\xi \in E} \boldsymbol{\mu}(A_\xi)\| = 1$, we have $b \Vdash \boldsymbol{\mu}(A_E) < \check{r}$. By (34.9), we get $\mu_b(A_E) < r$.

Since $\mu_b(A_E) < r$ for all finite $E \subseteq \gamma$, it follows from (34.8) that $\mu_b(A) \le r$. Hence $\mu_b^*(A) \le r$, a contradiction.

This completes the proof that in $\mathfrak{M}[G]$ μ is a nontrivial κ-additive measure on κ. ■

Another Model in Which $2^{\aleph_0}$ Carries a σ-Saturated Ideal

Let κ be a measurable cardinal, and let $\lambda \ge \kappa$ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. We shall construct a generic extension that satisfies $2^{\aleph_0} = \lambda$ and such that there is a σ-saturated κ-complete ideal over κ.

Let P be the notion of forcing that adjoins λ Cohen reals; i.e., a condition is a finite 0–1 function p with $\operatorname{dom}(p) \subset \lambda$. If G is a generic filter on P, then $\mathfrak{M}[G] \vDash 2^{\aleph_0} = \lambda$, and all cardinals are preserved in $\mathfrak{M}[G]$ because P satisfies the countable chain condition. That κ carries in $\mathfrak{M}[G]$ a σ-saturated ideal follows from this lemma:

Lemma 34.7. *Let κ be a measurable cardinal (in $\mathfrak{M}$) and let I be a nonprincipal κ-complete prime ideal over κ. Let P be a notion of forcing that satisfies the countable chain condition. Then in $\mathfrak{M}[G]$, the ideal J generated by I is a σ-saturated κ-complete ideal over κ.*

Proof. Let J be the ideal in $\mathfrak{M}[G]$ defined as follows:

$$X \in J \qquad \text{iff} \qquad X \subseteq Y \text{ for some } Y \in I$$

First we show that J is κ-complete. Let $\mathscr{X} = \{X_\xi : \xi < \gamma\}$ be a family of less than κ elements of J; let $\dot{\mathscr{X}}$ be a name for $\mathscr{X}$ and let $p_0 \in G$ be such that $p_0 \Vdash \forall \xi < \check{\gamma}\ \boldsymbol{X}_\xi \in \boldsymbol{J}$.

For each $\xi < \gamma$ and each $p \le p_0$, there exist $q \le p$ and some $Y \in I$ such that $q \Vdash \boldsymbol{X}_\xi \subseteq \check{Y}$. Let W_ξ be a maximal incompatible family of $q \le p_0$ for which there is $Y_{\xi,q}$ such that $q \Vdash \boldsymbol{X}_\xi \subseteq \check{Y}_{\xi,q}$. Since P satisfies the countable chain condition, each W_ξ is countable, and hence $Y = \bigcup \{Y_{\xi,q} : \xi < \gamma \text{ and } q \in W_\xi\}$ belongs to I. Now it is easy to verify that

$$p_0 \Vdash \bigcup_{\xi<\gamma} \boldsymbol{X}_\xi \subseteq \check{Y}$$

and hence $\bigcup \mathscr{X} \in J$.

To prove that J is countably saturated, let us assume that $\mathscr{X} = \{X_\xi : \xi < \omega_1\}$ is a family of pairwise disjoint sets of positive J-measure. Let $\mathscr{X}$ be a name for $\mathscr{X}$ and let $p \in G$ be such that $p \Vdash X_\xi \notin J$, for each $\xi < \omega_1$, and $p \Vdash \mathscr{X}_\xi \cap \mathscr{X}_\eta = \emptyset$ for any $\xi \neq \eta$.

For each $\xi < \omega_1$, let $Y_\xi = \{\alpha < \kappa : \text{some } q \leq p \text{ forces } \check{\alpha} \in \mathscr{X}_\xi\}$. Clearly $p \Vdash \mathscr{X}_\xi \subseteq \check{Y}_\xi$, and so $Y_\xi \notin I$. By the κ-compactness of I, we have $Y = \bigcap_{\xi < \omega_1} Y_\xi \notin I$. Thus $Y \neq \emptyset$, and let α be some element of Y. For each $\xi < \omega_1$, let $q_\xi \leq p$ be such that $q_\xi \Vdash \check{\alpha} \in \mathscr{X}_\xi$. Since P satisfies the countable chain condition, there are ξ, η such that q_ξ and q_η are compatible. Let q be stronger than both q_ξ and q_η; then $q \Vdash \check{\alpha} \in \mathscr{X}_\xi \cap \mathscr{X}_\eta$, a contradiction. ∎

Exercise 34.4. Let I be a σ-saturated κ-complete ideal over κ, and let $\{Y_\xi : \xi < \omega_1\}$ be a family of sets of positive measure. Then there is an uncountable $W \subseteq \omega_1$ such that $\bigcap_{\xi \in W} Y_\xi$ is nonempty.

[Assume that $\{Y_\xi : \xi < \omega_1\}$ is a counterexample. For each $\nu < \omega_1$, let $Z_\nu = \bigcap_{\xi \geq \nu} (\kappa - Y_\xi)$. Show that $Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_\nu \subseteq \cdots$ and that $\bigcup_{\nu < \omega_1} Z_\nu = \kappa$. Hence there is $\gamma < \omega_1$ such that if $Z = \bigcup_{\nu < \gamma} Z_\nu$, then $\kappa - Z \in I$. This is a contradiction since $Y_\gamma \subseteq \kappa - Z$.]

*Exercise 34.5.** If I is a σ-saturated κ-complete ideal over κ (in $\mathfrak{M}$) and P is a σ-saturated notion of forcing then in $\mathfrak{M}[G]$, the ideal generated by I is a σ-saturated κ-complete ideal.

[Proceed as in Lemma 34.7 and use Exercise 34.4 to show that J is σ-saturated.]

If we adjoin $\lambda \geq \kappa$ Cohen reals to a model $\mathfrak{M}$ in which κ is measurable, then κ carries a σ-saturated κ-complete ideal but is not real-valued measurable:

*Exercise 34.6.** Show that in $\mathfrak{M}[G]$ there are functions $f_\alpha : \omega \to \omega$, $\alpha < \lambda$, such that whenever $g : \omega \to \omega$, then for at most countably many α's we have $f_\alpha(n) \leq g(n)$ for all n.

[$\mathfrak{M}[G]$ is also obtained by forcing with the product of λ copies of the notion of forcing that adjoins a generic function $f : \omega \to \omega$, thus $\mathfrak{M}[G] = \mathfrak{M}[\langle f_\alpha : \alpha < \lambda \rangle]$. Show that if $g : \omega \to \omega$, then there is a countable $A \subset \lambda$ such that $g \in \mathfrak{M}[\langle f_\alpha : \alpha \in A \rangle]$; if $\beta \notin A$, use the genericity of f_β over $\mathfrak{M}[\langle f_\alpha : \alpha \in A \rangle]$ to show that $f_\beta(n) > g(n)$ for some n.]

Exercise 34.7. In $\mathfrak{M}[G]$, κ is not real-valued measurable.

[Use Exercise 34.6 and the proof of Lemma 27.9.]

Proof of Theorem 83

We shall prove that if $2^{\aleph_0}$ carries a σ-saturated $2^{\aleph_0}$-complete ideal then $2^\lambda = 2^{\aleph_0}$ for all $\lambda < 2^{\aleph_0}$.

Let λ be a regular cardinal; two functions f, g on λ are *almost disjoint* if there is γ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \gamma$.

Lemma 34.8. *Let κ carry a σ-saturated κ-complete ideal, and let $\lambda < \kappa$ be a regular cardinal. If $\mathscr{F}$ is a family of almost disjoint functions $f : \lambda \to \kappa$, then $|\mathscr{F}| \leq \kappa$.*

Proof. If $|\mathscr{F}| > \kappa$, then because every $f\colon \lambda \to \kappa$ is bounded by some $\beta < \kappa$, there exists $\mathscr{G} \subseteq \mathscr{F}$, and some $\beta < \kappa$ such that $|\mathscr{G}| = \kappa$ and every $f \in \mathscr{G}$ is bounded by β (i.e., $f \in {}^\lambda\beta$).

Let $F\colon [\mathscr{G}]^2 \to \beta$ be the following partition: $F(\{f, g\}) =$ some γ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \gamma$. By Lemma 34.4, there exists $\mathscr{H} \subseteq \mathscr{G}$ of size κ such that the image A of $[\mathscr{H}]^2$ under F is at most countable. Let $\alpha > \sup(A)$. Then $f(\alpha) \neq g(\alpha)$ whenever $f, g \in \mathscr{H}$, which is a contradiction since $|\mathscr{H}| = \kappa$ and $f(\alpha) < \beta$ for all $f \in \mathscr{H}$. ■

Now we are ready to prove the theorem. Let $\kappa = 2^{\aleph_0}$. We prove $2^\lambda = \kappa$ by induction on $\lambda < \kappa$. If λ is a singular cardinal and $2^\nu = \kappa$ for all $\nu < \lambda$, then $2^\lambda = \kappa$ by Theorem 18. Thus let $\lambda < \kappa$ be regular and let us assume that $2^\nu = \kappa$ for all $\nu < \lambda$.

For each $X \subseteq \lambda$, let $f_X = \langle X \cap \alpha : \alpha < \lambda \rangle$. If $X \neq Y$, then f_X and f_Y are almost disjoint. For each $\alpha < \lambda$, the set $\{f_X(\alpha) : X \subseteq \lambda\}$ has size κ, and hence $\{f_X : X \subseteq \lambda\}$ yields a family $\mathscr{F}$ of 2^λ almost disjoint functions from λ into κ. By Lemma 34.8 we get $|\mathscr{F}| \leq \kappa$ and therefore $2^\lambda = \kappa$. ■

35. SATURATION OF IDEALS AND GENERIC ULTRAPOWERS

In Section 34 we investigated countably saturated ideals over an uncountable regular cardinal κ. In particular, we have shown that if such an ideal exists, then κ is at least weakly Mahlo, and we have also shown that existence of such ideals is equiconsistent with existence of measurable cardinals. The present section deals with a related concept. We define the notion of a λ-saturated ideal over κ and investigate these ideals in the cases when $\lambda = \kappa$ or $\lambda = \kappa^+$ (or generally when $\lambda \leq 2^\kappa$). Aside from the general question of existence of λ-saturated ideals for various values of λ, we are particularly interested in saturation of the two "natural" ideals over κ: The ideal of small (i.e., of size $< \kappa$) subsets of κ and the ideal of thin subsets of κ. In the first case, the problem is the already familiar problem of the number of almost disjoint subsets; while in the second case, we obtain some results on stationary sets. Many of the results in this section are proved using the method of generic ultrapowers, which generalizes the method of ultrapowers used in Chapter 5.

Let κ be a regular uncountable cardinal. Throughout this section, let I be a κ-complete ideal over κ containing all singletons; thus $X \in I$ whenever $X \subset \kappa$ is such that $|X| < \kappa$. As in Section 34, I shall be using the following terminology: X has *measure zero* if $X \in I$, *measure one* if $\kappa - X \in I$, and *positive measure* if $X \notin I$; the phrase *almost all* α means that the set of all contrary α's has measure 0.

There are two natural examples of such ideals: the ideal of *small* sets $\{X \subset \kappa : |X| < \kappa\}$ (the least possible such ideal) and the ideal of *thin* sets

$\{X \subset \kappa : \kappa - X$ contains a closed unbounded set$\}$, which is (see Lemma 7.7) the least normal κ-complete ideal over κ.

Let us consider the Boolean algebra $B = P(\kappa)/I$. Recall that if λ is a cardinal, then B is called *λ-saturated* if every pairwise disjoint family of elements of B has size less than λ; $\operatorname{sat}(B)$ is the least λ such that B is λ-saturated. Let us say that I is *λ-saturated* if B is λ-saturated and let

$$\operatorname{sat}(I) = \operatorname{sat}(B)$$

In other words, I is λ-saturated just in case there exists no collection W of size λ of subsets of κ such that $X \notin I$ for all $X \in W$ and $X \cap Y \in I$ whenever X and Y are distinct members of W. If $\operatorname{sat}(I)$ is finite, then κ is the union of finitely many atoms of I; if $\operatorname{sat}(I)$ is infinite, then it is uncountable and regular, by Lemma 17.6. If $\lambda \le \kappa$, then I is λ-saturated if and only if there is no *disjoint* collection W of size λ of subsets X of κ such that $X \notin I$ (see Exercise 27.4). Clearly, every I over κ is $(2^\kappa)^+$-saturated. Thus if I is atomless, then $\operatorname{sat}(I)$ is a regular cardinal and

$$\aleph_1 \le \operatorname{sat}(I) \le (2^\kappa)^+$$

Since I is κ-complete, it follows that $B = P(\kappa)/I$ is a κ-complete Boolean algebra (Exercise 17.16).

Exercise 35.1. If I is κ^+-saturated, then $P(\kappa)/I$ is a complete Boolean algebra.

[By Exercise 17.23 it suffices to show that B is κ^+-complete. Show that $\sum W$ exists in B for every incompatible $W \subset B$. Extend W to a partition Z of B; $Z = \{[X_\alpha] : \alpha < \kappa\}$. Let $Y_\alpha = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$, and $Y = \bigcup \{Y_\alpha : [X_\alpha] \in W\}$. Show that $[Y] = \sum W$ in B.]

Exercise 35.2. If the GCH holds and $B = P(\kappa)/I$ is complete, then I is κ^+-saturated.

[If B is not κ^+-saturated, let W be an incompatible subset of B of size κ^+. For each $X \subseteq W$ let $u_X = \sum X$. It follows that $|B| \ge 2^{\kappa^+}$, but clearly $|B| \le 2^\kappa$; a contradiction.]

Exercise 35.3. If I is normal, then $P(\kappa)/I$ is κ^+-complete.

[Let X_α, $\alpha < \kappa$, be disjoint subsets of κ such that $X_\alpha \notin I$ for all α. For each $\alpha < \kappa$ let Y_α be X_α without the least element of X_α; let $Y = \bigcup_{\alpha<\kappa} Y_\alpha$. On the one hand, $[Y] \ge [X_\alpha]$ for all α; on the other hand, if $Z \subseteq Y$ and $Z \notin I$, let f be the function on Z defined such that for all $x \in Y_\alpha$, $f(x) =$ the least element of X_α. Since f is regressive, and I is normal, f is constant on some $S \notin I$, and hence $Z \cap Y_\alpha \notin I$ for some α. Thus $[Y] = \sum_{\alpha<\kappa} [X_\alpha]$.]

We proved in Section 27 that if there exists a κ-saturated ideal I over κ, then κ is weakly inaccessible. Furthermore, the proof of Lemma 27.5 shows that if $\operatorname{sat}(I) = \lambda < \kappa$, then either κ is measurable or κ is not strongly inaccessible because $\kappa \le 2^{<\lambda}$. In the last section we proved that existence of a countably saturated ideal over $2^{\aleph_0}$ is equiconsistent with existence of a measurable cardinal.

Our next observation is that Lemmas 34.1 and 34.2 remain valid if σ-saturated is replaced by κ-saturated; the proof is exactly the same. Thus we have:

Lemma 35.1.

(a) *If there exists a κ-saturated κ-complete ideal over an uncountable cardinal κ, then there exists a normal κ-saturated κ-complete ideal over κ.*

(b) *Let I be a normal κ-saturated κ-complete ideal over κ. If $S \notin I$ and if $f: S \to \kappa$ is regressive on S, then there is $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for almost all $\alpha \in S$.* ■

A consequence of (b) is that κ is a weakly Mahlo cardinal. (I shall return to this later in the present section.)

Exercise 35.4. If I is a κ-complete ideal over κ with the property that every regressive function is bounded almost everywhere (i.e., if $f(\alpha) < \alpha$ for almost all α, then there is $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for almost all α), then I is κ-saturated (and normal).

[Otherwise, let X_α, $\alpha < \kappa$, be a partition of κ into disjoint sets of positive measure. For $\alpha > 0$, let $Y_\alpha = X_\alpha - \{a_\alpha\}$ where $a_\alpha = \min X_\alpha$, and let $Y_0 = X_0 \cup \{a_\alpha : \alpha > 0\}$. The function f that has value a_α on each Y_α is regressive almost everywhere but is not bounded almost everywhere.]

The proof I presented in Section 34 that if $\operatorname{sat}(I) < \kappa$, then κ is measurable in a transitive model does not go through if $\operatorname{sat}(I) = \kappa$. It is nevertheless true that κ is measurable in a transitive model, and later in this section I shall give a proof of a more general theorem.

In Section 34 we gave an example of a generic model in which $\kappa = 2^{\aleph_0}$ carries an ideal I with $\operatorname{sat}(I) < \kappa$. A different forcing construction gives a generic extension (assuming that κ is measurable in the ground model) in which $\kappa = 2^{\aleph_0}$ has an ideal I with $\operatorname{sat}(I) = \kappa$.

If $\operatorname{sat}(I) < \kappa$ and κ is inaccessible, then I cannot be atomless (and κ is measurable). However, it is consistent (relative to ZFC + there is a measurable cardinal) that an inaccessible cardinal κ carries an ideal I with $\operatorname{sat}(I) = \kappa$. There is a restriction on κ though:

Exercise 35.5. If I is an atomless κ-complete κ-saturated ideal over an inaccessible cardinal κ, then κ is not weakly compact.

[Show that κ does not have the tree property. Use I to construct a tree $(T, \supseteq)$ whose elements are sets of positive measure. At successor steps, split each X on the top level into two disjoint sets of positive measure. At limit steps, take all those intersections along branches that have positive measure. Since I is κ-saturated, each level has size $< \kappa$; each level $\alpha < \kappa$ is nonempty because κ is inaccessible and I is κ-complete. Then use $\operatorname{sat}(I) \le \kappa$ to show that T has no branch of length κ.]

(As a consequence, no weakly compact κ carries an ideal I such that $\operatorname{sat}(I) = \kappa$.)

Now, let us consider the question of whether a regular uncountable cardinal κ can carry a κ-complete ideal I such that $\operatorname{sat}(I) = \kappa^+$. I shall prove later in this section that if that is the case, then κ is measurable in a transitive model. Conversely, if κ is measurable, then there exists a generic extension in which κ is inaccessible and carries an ideal I with $\operatorname{sat}(I) = \kappa^+$.

Existence of a κ^+-saturated ideal over κ does not necessarily imply that κ is a limit cardinal. Thus let us consider the problem of existence of an $\aleph_2$-saturated ideal over ω_1. It has been shown that this assumption implies that for every α there exists a transitive model with α measurable cardinals. And conversely, if κ is a huge cardinal (cf. Section 33), then there is a generic extension in which $\kappa = \aleph_1$ and $\aleph_1$ carries an $\aleph_2$-saturated ideal.

In the particular case when I is the ideal of small subsets of κ, then I is not κ^+-saturated (see Lemma 24.7) and so $\kappa^{++} \leq \text{sat}(I) \leq (2^\kappa)^+$. If I is the ideal of thin subsets of κ, then I is not κ-saturated, as we shall prove presently, and so $\kappa^+ \leq \text{sat}(I) \leq (2^\kappa)^+$. It is not known whether the thin ideal can be κ^+-saturated and the assumption that $\text{sat}(I) = \kappa^+$ seems to be quite strong (if not inconsistent).

If $2^\kappa > \kappa^+$ then a new possibility arises: an ideal I over κ such that $\kappa^{++} \leq \text{sat}(I) \leq 2^\kappa$. As we shall see, this assumption does not entail large cardinal properties; we shall show that there is a generic extension of L in which the ideal of countable subsets of ω_1 is $\aleph_3$-saturated (while $2^{\aleph_1}$ is large). However, there is a relation between saturation of ideals and cardinal exponentiation, and I shall start with a theorem that deals with this relation.

Almost Disjoint Families of Subsets of κ

Let I be the ideal of small subsets of κ: $I = \{X \subseteq \kappa : |X| < \kappa\}$. Thus sets of positive measure are just the subsets of κ of size κ, and "almost everywhere" means everywhere except for a set of size $< \kappa$. To say that I is λ-saturated means that there exists no family W of subsets of κ such that $|W| = \lambda$ and that each $X \in W$ has size κ and $|X \cap Y| < \kappa$ whenever X and Y are distinct elements of W. Let us call such a W an *almost disjoint* family of subsets of κ (and let us call X, Y *almost disjoint* if $|X \cap Y| < \kappa$.

The proof of Lemma 24.7 shows that there exists an almost disjoint family of size κ^+, and hence the ideal of small sets is not κ^+-saturated. Also, Exercise 24.6 shows that if $2^{<\kappa} = \kappa$, then there is a family of 2^κ almost disjoint subsets of κ. We shall now state a more general theorem and then show that the ideal of small sets does not necessarily have the property that $\text{sat}(I) = (2^\kappa)^+$. The theorem establishes a relation between saturation of the ideal of small sets (and, in part (b), between saturation of arbitrary ideals) and the continuum function.

Although the following results can be formulated for an arbitrary uncountable regular cardinal κ, we shall consider only the representative case $\kappa = \omega_1$.

Theorem 84. *Let us assume that* $2^{\aleph_0} < 2^{\aleph_1}$ *and also that* $2^{\aleph_0} < \aleph_{\omega_1}$. *Then:*

(a) *there exists an almost disjoint family of size* $2^{\aleph_1}$ *of uncountable subsets of* ω_1;
(b) *if* I *is a* σ*-complete ideal over* ω_1, *then* $\text{sat}(I) \geq 2^{\aleph_1}$.

Part (b) can be also formulated as follows: If I is a σ-complete ideal over ω_1 and if $2^{\aleph_0} < \aleph_{\omega_1}$, then

$$2^{\aleph_1} \leq 2^{\aleph_0} \cdot \text{sat}(I) \tag{35.1}$$

If there exists an $\aleph_2$-saturated ideal over ω_1, then we get

$$\text{(35.2)} \qquad \text{if} \quad 2^{\aleph_0} = \aleph_1, \qquad \text{then} \quad 2^{\aleph_1} = \aleph_2$$

and

$$\text{(35.3)} \qquad \text{if} \quad \aleph_1 < 2^{\aleph_0} < \aleph_{\omega_1}, \qquad \text{then} \quad 2^{\aleph_0} = 2^{\aleph_1}$$

I shall now prove only part (a) of the theorem. Part (b) can be proved in a similar manner, but I will postpone the proof until later in this section and then give a proof using generic ultrapowers.

Lemma 35.2. *Assume* $2^{\aleph_0} < \aleph_{\omega_1}$. *If* κ *is a regular cardinal such that* $2^{\aleph_0} < \kappa \le 2^{\aleph_1}$, *then there exists a family of* κ *almost disjoint functions from* ω_1 *into* ω_1.

Proof. Let $\mathscr{F}$ be a family of almost disjoint functions on ω_1; we call $\mathscr{F}$ a *branching family* if whenever $f, g \in \mathscr{F}$ and α is such that $f(\alpha) = g(\alpha)$, then $f(\xi) = g(\xi)$ for all $\xi \le \alpha$.

For each $X \subseteq \omega_1$, let $f_X = \langle X \cap \alpha : \alpha < \omega_1 \rangle$. The family $\mathscr{F} = \{f_X : X \in P(\omega_1)\}$ is a branching family of functions on ω_1, $|\mathscr{F}| = 2^{\aleph_1}$; and for each $\alpha < \omega_1$, the functions in $\mathscr{F}$ take values in $P(\alpha)$. Thus there exists a branching family of $2^{\aleph_1}$ functions from ω_1 into $2^{\aleph_0}$.

Let κ be a regular cardinal such that $2^{\aleph_0} < \kappa \le 2^{\aleph_1}$. We shall show that for every $\aleph_\gamma$ such that $\aleph_1 < \aleph_\gamma \le 2^{\aleph_0}$, if there is a branching family of κ functions from ω_1 into ω_γ, then there is a branching family of κ functions from ω_1 into some $\omega_\delta < \omega_\gamma$. Then the lemma clearly follows.

First let $\aleph_\gamma = \aleph_{\delta+1}$ where $\aleph_1 \le \aleph_\delta$, and let $\mathscr{F}$ be a branching family of κ functions from ω_1 into $\omega_{\delta+1}$. Each $f \in \mathscr{F}$ is bounded below $\omega_{\delta+1}$ (because $\omega_{\delta+1} > \omega_1$), and because κ is regular and $\kappa > \omega_{\delta+1}$, there exists $\alpha < \omega_{\delta+1}$ such that $\operatorname{ran}(f) \subseteq \alpha$ for κ functions in $\mathscr{F}$. Thus there exists a branching family of κ functions from ω_1 into α; and since $|\alpha| \le \aleph_\delta$, there is also a branching family of κ functions from ω_1 into ω_δ.

If $\aleph_\gamma$ is a limit cardinal, then $\operatorname{cf}(\omega_\gamma) = \omega$ because $\aleph_\gamma < \aleph_{\omega_1}$. Let $\mathscr{F}$ be a branching family of κ functions from ω_1 into ω_γ. For each $f \in \mathscr{F}$ there exists an ordinal $\eta_f < \omega_\gamma$ such that $f(\alpha) < \eta_f$ for uncountably many α's. Since κ is a regular cardinal and $\kappa > \aleph_\gamma$, there exists $\aleph_\delta$ such that $\aleph_1 \le \aleph_\delta < \aleph_\gamma$, and a family $\mathscr{G} \subseteq \mathscr{F}$ of size κ such that for every $f \in \mathscr{G}$, $f(\alpha) < \omega_\delta$ for uncountably many α's.

For each $\alpha < \omega_1$, let $S_\alpha = \{f(\alpha) : f \in \mathscr{G}\}$. Since $\mathscr{G}$ is a branching family and $\mathscr{G} \subseteq \prod_{\alpha<\omega_1} S_\alpha$, it suffices to show that $|S_\alpha| \le \aleph_\delta$ for all $\alpha < \omega_1$. Thus let $\alpha < \omega_1$. We define a function $t\colon S_\alpha \to \omega_1 \times \omega_\delta$ as follows: For each $x \in S_\alpha$, we first pick some $f \in \mathscr{G}$ such that $x = f(\alpha)$. Then there exists some $\xi > \alpha$ such that $f(\xi) < \omega_\delta$, and we let

$$t(x) = (\xi, f(\xi))$$

We shall now complete the proof by showing that the function t is one-to-one, and hence $|S_\alpha| \le \aleph_\delta$. Let $x, y \in S_\alpha$ be such that $t(x) = t(y)$. Let $\xi > \alpha$ and

$f, g \in \mathscr{G}$ be such that $x = f(\alpha)$, $y = g(\alpha)$, and $t(x) = t(y) = (\xi, f(\xi)) = (\xi, g(\xi))$. Since $\mathscr{G}$ is a branching family and $f(\xi) = g(\xi)$, we necessarily have $f(\alpha) = g(\alpha)$, and hence $x = y$. ■

Part (a) of Theorem 84 now follows easily from Lemma 35.2. Every function from ω_1 into ω_1 is an uncountable subset of $\omega_1 \times \omega_1$ and almost disjoint functions are almost disjoint subsets of $\omega_1 \times \omega_1$. Thus for every regular $\kappa \leq 2^{\aleph_1}$, there exists a family of κ almost disjoint uncountable subsets of ω_1. If $2^{\aleph_1}$ is regular, we are done; if κ is singular, then it is easy to construct an almost disjoint family of $2^{\aleph_1}$ subsets of ω_1 by taking first $\mathrm{cf}(2^{\aleph_1})$ almost disjoint sets, etc. (or simply invoke Lemma 17.6 saying that $\mathrm{sat}(I)$ is regular and hence $\mathrm{sat}(I) > 2^{\aleph_1}$).

Part (b) of Theorem 84 can be proved along the same lines. If I is a σ-complete ideal over ω_1, let us call an *I-function* on ω_1 a function whose domain is a subset of ω_1 of positive measure. Two I-functions f and g are *almost disjoint* if the set of all α such that $f(\alpha) = g(\alpha)$ is at most countable.

If we assume that $\lambda = \mathrm{sat}(I) < 2^{\aleph_1}$, then by Lemma 35.2, there is a family of λ^+ almost disjoint functions from ω_1 into ω_1. It is not difficult then to obtain a family of λ almost disjoint I-functions from ω_1 into ω. This is, however, a contradiction:

Exercise 35.6. Let $\lambda \geq \aleph_2$ be a regular cardinal. If there are λ almost disjoint I-functions from ω_1 into ω, then there exists an almost disjoint family of λ subsets of ω_1 of positive measure.

[As in Exercise 24.7.]

The following example shows that the assumption $2^{\aleph_0} < 2^{\aleph_1}$ in Theorem 84 is necessary.

Example I. A model in which the ideal of countable subsets of ω_1 is $\aleph_3$-saturated while $2^{\aleph_1}$ is large.

Let $\mathfrak{M}$ be a transitive model of ZFC + GCH and let $\mathfrak{M}[G]$ be the generic extension of $\mathfrak{M}$ which adjoins to $\mathfrak{M}$ κ Cohen reals (see (19.1)). We shall show that in $\mathfrak{M}[G]$, every almost disjoint family of uncountable subsets of ω_1 has size at most $\aleph_2$. In fact, we shall use only the facts that $\mathfrak{M}$ satisfies the GCH and that the notion of forcing satisfies the countable chain condition.

Let us assume that $\mathscr{F} \in \mathfrak{M}[G]$ is an almost disjoint family of uncountable subsets of ω_1, $\mathscr{F} = \{A_i : i < \omega_3\}$, and let $\mathscr{F}$ be a name for $\mathscr{F}$. We may as well assume that every condition forces that $\mathscr{F}$ be an almost disjoint family, etc.

Let $i \neq j < \omega_3$. For every condition p there is $q \leq p$ and $\gamma < \omega_1$ such that

$$q \Vdash A_i \cap A_j \subseteq \check{\gamma} \tag{35.4}$$

Thus there is a maximal incompatible set W of conditions, and for every $q \in W$ there is $\gamma = \gamma_q$ with the property (35.4). Since W is at most countable, we can take $\gamma = \sup\{\gamma_q : q \in W\}$ and γ is a countable ordinal with the property that (35.4) holds for *every* condition q. Let us denote $\gamma = f(i, j)$.

The function f is a partition of $[\omega_3]^2$ into $\aleph_1$ pieces. By the Erdös–Rado theorem (Exercise 29.1), and by the GCH, we have

$$\aleph_3 \to (\aleph_2)^2_{\aleph_1}$$

and therefore there exists $H \subset \omega_3$ of size $\aleph_2$, and $\gamma < \omega_1$ such that (35.4) holds for every condition q and every pair $i, j \in H$.

Now we argue in $\mathfrak{M}[G]$: For every pair $i, j \in H$ we have $A_i \cap A_j \subseteq \gamma$, and so $\{A_i - \gamma : i \in H\}$ is a *disjoint* collection of $\aleph_2$ nonempty subsets of ω_1; a contradiction. ■

The Ideal of Thin Sets

The ideal of thin subsets of a regular uncountable cardinal κ is another natural example of a κ-complete ideal. Moreover, it is the least normal κ-complete ideal over κ containing all singletons. We shall now investigate saturation of the thin ideal.

We proved in Lemma 7.6 that κ can be partitioned into κ stationary sets. We shall now prove a stronger theorem: Every stationary subset of κ can be partitioned into κ stationary sets. Thus no restriction

$$I \,|\, S = \{X \subseteq S : X \in I\}$$

(where S is stationary) is a κ-saturated ideal.

As a digression, let me point out that one consequence is that not every subset of ω_1 has the property that either itself or its complement has a closed unbounded subset. Let us consider the following problem: Does every subset of ω_2 have the property that either itself or its complement (in ω_2) has a subset X of order-type ω_1 that is ω-closed (i.e., $\lim_{n\to\omega} a_n \in X$ for every sequence $\{a_n\}_{n<\omega} \subset X$)? This is known to be false in L, and it is open whether this can hold in some model of ZFC.

Theorem 85 (Solovay). *Let κ be a regular uncountable cardinal. Then every stationary subset of κ is the disjoint union of κ stationary subsets.*

We shall first prove the following lemma. Recall that if α is a limit ordinal of uncountable cofinality, then the notion of closed unbounded and stationary subsets of α is defined in the natural way, and the collection of closed unbounded subsets of α generates a filter over α (and the filter is cf α-complete).

Lemma 35.3. *Let S be a stationary subset of κ, and let*

$$T = \{\alpha \in S : \textit{either } \operatorname{cf}\alpha = \omega, \textit{ or } \operatorname{cf}\alpha > \omega \textit{ and } S \cap \alpha \textit{ is not stationary in } \alpha\}$$

Then T is a stationary set.

Recall the Mahlo operation defined in (28.23):

$$\mathrm{M}(X) = \{\alpha < \kappa : X \cap \alpha \text{ is a stationary subset of } \alpha\}$$

Thus Lemma 35.3 states that for every stationary $S \subseteq \kappa$, the set $S - \mathrm{M}(S)$ is stationary.

Proof. We prove that T intersects every closed unbounded subset of κ. Let C be closed unbounded. The set C' of all limit points of C is also closed unbounded, and hence $S \cap C' \neq \emptyset$. Let α be the least element of $S \cap C'$. If $\mathrm{cf}\,\alpha = \omega$, then $\alpha \in T$; and since $\alpha \in C' \subseteq C$, we are done. If $\mathrm{cf}\,\alpha > \omega$, then $C' \cap \alpha$ is a closed unbounded subset of α and is disjoint from $S \cap \alpha$, and so $S \cap \alpha$ is not a stationary subset of α. Hence $\alpha \in T$, and again $C \cap T$ is nonempty. ■

Proof of Theorem 85. I shall try to follow the proof of Lemma 7.6 as much as possible. Let A be a stationary subset of κ. We let W be the set of all $\alpha \in A$ such that either $\mathrm{cf}\,\alpha = \omega$, or $\mathrm{cf}\,\alpha > \omega$ and $A \cap \alpha$ is not stationary. Thus there exists for each $\alpha \in W$ a continuous increasing sequence $\langle a_\xi^\alpha : \xi < \mathrm{cf}\,\alpha \rangle$ such that $a_\xi^\alpha \notin W$, for all α and ξ, and $\alpha = \lim_{\xi \to \mathrm{cf}\,\alpha} a_\xi^\alpha$.

First we show that there is ξ such that for all $\eta < \kappa$, the set

$$\{\alpha \in W : a_\xi^\alpha \geq \eta\} \tag{35.5}$$

is stationary. Otherwise, there is for each ξ some $\eta(\xi)$ and a closed unbounded set C_ξ such that $a_\xi^\alpha < \eta(\xi)$ for all $\alpha \in C_\xi \cap W$ if a_ξ^α is defined. Let C be the diagonal intersection of the C_ξ. Thus if $\alpha \in C \cap W$, then

$$a_\xi^\alpha < \eta(\xi) \qquad \text{for all} \quad \xi < \mathrm{cf}\,\alpha$$

Now let D be the closed unbounded set of all $\gamma \in C$ such that $\eta(\xi) < \gamma$ for all $\xi < \gamma$. Since W is stationary, $W \cap D$ is also stationary; let $\gamma < \alpha$ be two ordinals in $W \cap D$.

Now if $\xi < \gamma$ (and $\xi < \mathrm{cf}\,\alpha$), then $a_\xi^\alpha < \eta(\xi) < \gamma$ and it follows that $a_\gamma^\alpha = \gamma$. This is a contradiction since $\gamma \in W$ and $a_\gamma^\alpha \notin W$.

Once we have found ξ such that (35.5) is stationary for all $\eta < \kappa$, we proceed as in Lemma 7.6. Let f be the function on W defined by

$$f(\alpha) = a_\xi^\alpha$$

The function f is regressive; and so for every $\eta < \kappa$, we find by Fodor's theorem a stationary subset S_η of (35.5) and $\gamma_\eta \geq \eta$ such that $f(\alpha) = \gamma_\eta$ on S_η. If $\gamma_\eta \neq \gamma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$; and since κ is regular, we have $|\{S_\eta : \eta < \kappa\}| = |\{\gamma_\eta : \eta < \kappa\}| = \kappa$. ■

Since the thin ideal is not κ-saturated, the question is whether it can be κ^+-saturated, or in general, 2^κ-saturated. While the first question is still open; the following example shows that we can have $\mathrm{sat}(I) \leq 2^\kappa$ for the thin ideal over κ.

Example II. A model in which the ideal of thin subsets of ω_1 is $\aleph_4$-saturated while $2^{\aleph_1}$ is large.

(A refinement of the following argument shows that the thin ideal in this model is in fact $\aleph_3$-saturated.) As in Example I, let $\mathfrak{M}$ be a transitive model of ZFC + GCH and let $\mathfrak{M}[G]$ be the generic extension that adjoins κ Cohen reals. We shall show that in $\mathfrak{M}[G]$, if $\mathscr{F}$ is a family of stationary subsets of ω_1 such that the intersection of any two distinct elements of $\mathscr{F}$ is thin, then $|\mathscr{F}| \leq \aleph_3$. Again, we shall use only the fact that $\mathfrak{M}$ satisfies the GCH and that the notion of forcing satisfies the c.c.c.

First we need a lemma:

Lemma 35.4. *Let $\mathfrak{M}[G]$ be a generic extension of $\mathfrak{M}$ by a c.c.c. notion of forcing. Then every closed unbounded $C \subseteq \omega_1$ in $\mathfrak{M}[G]$ has a closed unbounded subset D such that $D \in \mathfrak{M}$. Consequently, if $S \in \mathfrak{M}$ is stationary (in $\mathfrak{M}$), then $\mathfrak{M}[G] \vDash S$ is stationary.*

Proof. Let $\mathbf{C}$ be a name for C such that every condition forces that $\mathbf{C}$ is a closed unbounded subset of ω_1. Let

$$D = \{\alpha : \|\check{\alpha} \in \mathbf{C}\| = 1\}$$

Clearly, D is a subset of C and is closed; we have to prove only that D is unbounded.

Let $\alpha_0 < \omega_1$; we wish to find $\alpha > \alpha_0$ such that every condition forces $\check{\alpha} \in \mathbf{C}$. For every p, there is $q \leq p$ and some $\beta > \alpha_0$ such that $q \Vdash \check{\beta} \in \mathbf{C}$. Thus there is a maximal incompatible set W of conditions, and for each $q \in W$ an ordinal $\beta = \beta_q$ such that $q \Vdash \check{\beta} \in \mathbf{C}$. Since W is at most countable, we let $\alpha_1 = \sup\{\beta_q : q \in W\}$; the ordinal α_1 is countable, and we have

$$p \Vdash (\exists \beta \in \mathbf{C})\, \check{\alpha}_0 < \beta \leq \check{\alpha}_1$$

for all conditions p. Similarly, we find $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$ such that for every n and every condition p,

$$p \Vdash (\exists \beta \in \mathbf{C})\, \check{\alpha}_n < \beta \leq \check{\alpha}_{n+1}$$

If we let $\alpha = \lim_n \alpha_n$, it is clear that $\|\check{\alpha} \in \mathbf{C}\| = 1$. ■

Now let us assume that $\mathscr{F} \in \mathfrak{M}[G]$ is a family of stationary subsets of ω_1, $\mathscr{F} = \{A_i : i < \omega_4\}$ such that $A_i \cap A_j$ is thin whenever $i \neq j$, and let $\boldsymbol{\mathscr{F}}$ be a name for $\mathscr{F}$.

Let $i \neq j < \omega_4$. By Lemma 35.4, $A_i \cap A_j$ is included in a thin set that is in $\mathfrak{M}$. Thus for every p there is $q \leq p$ and a thin set $X \in \mathfrak{M}$ such that

$$q \Vdash \mathbf{A}_i \cap \mathbf{A}_j \subseteq \check{X} \tag{35.6}$$

Thus there is a maximal incompatible set W of conditions, and for every $q \in W$ there is $X = X_q$ with the property (35.6). Since $|W| \leq \aleph_0$, we can take

$X = \bigcup \{X_q : q \in W\}$; and X is a thin set such that (35.6) holds for every condition q. Let us denote $X = f(i, j)$.

The function f is a partition of $[\omega_4]^2$ into $2^{\aleph_1}$ pieces. By the Erdös–Rado theorem and by the GCH, we have

$$\aleph_4 \to (\aleph_3)^2_{\aleph_2}$$

and therefore there is $H \subset \omega_4$ of size $\aleph_3$ and a thin set X such that (35.6) holds for every q and every pair $i, j \in H$.

Now arguing in $\mathfrak{M}[G]$, we have $A_i \cap A_j \subseteq X$ for every $i, j \in H$, and so $\{A_i - X : i \in H\}$ is a disjoint collection of $\aleph_3$ stationary subsets of ω_1; a contradiction. ■

Exercise 35.7. Let I be a normal κ-complete ideal over κ. If I is not κ^+-saturated, then there exists an almost disjoint family of κ^+ sets of positive measure.

[Let X_i, $i < \kappa^+$, be sets of positive measure such that $X_i \cap X_j$ has measure zero. For each $i < \kappa^+$, enumerate $\{X_j : j < i\}$ by $\{Z_\alpha : \alpha < \kappa\}$, and let Y_i be the diagonal intersection of $\{X_i - Z_\alpha : \alpha < \kappa\}$. Now Y_i contains almost all elements of X_i, and $Y_i \cap Z_\alpha \subseteq \alpha + 1$ for every $\alpha < \kappa$. Thus any Y_i, Y_j are almost disjoint.]

Exercise 35.8. If the ideal of thin subsets of κ is κ^+-saturated, then every normal κ-complete ideal over κ containing all singletons is κ^+-saturated.

[Use Exercise 35.7 and the fact that every set of positive measure is stationary.]

There are several results describing situations when the thin ideal is not κ^+-saturated. One such result is contained in Theorem 84b; for instance, if $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_2$, then the thin ideal over ω_1 is not $\aleph_2$-saturated. By another result (cf. Exercise 22.11) Jensen's $\diamondsuit$ implies that there are $2^{\aleph_1}$ almost disjoint stationary sets. (And both these results generalize to other regular uncountable cardinals.) The following lemma shows that for some large cardinals κ, the thin ideal over κ is not κ^+-saturated.

Lemma 35.5. *Assume that the thin ideal over κ is κ^+-saturated and let I be a normal κ-complete ideal oveer κ containing all singletons. Then there is a set $A \subseteq \kappa$ of measure one (i.e., $\kappa - A \in I$) such that every stationary subset of A has positive measure.*

[Since every set of positive measure is stationary, this means that $I|A = I_0|A$ where $I|A = \{X \subseteq A : X \in I\}$ and I_0 is the thin ideal.]

Proof. Let $\mathscr{F}$ be a maximal almost disjoint (i.e., $|S \cap T| < \kappa$) family of stationary sets of measure zero; we have $|\mathscr{F}| \leq \kappa$. If $|\mathscr{F}| = \kappa$, $\mathscr{F} = \{S_\alpha : \alpha < \kappa\}$, let A be the diagonal intersection of $\{\kappa - S_\alpha : \alpha < \kappa\}$; if $|\mathscr{F}| < \kappa$, let $A = \bigcap \{\kappa - S : S \in \mathscr{F}\}$. In any case, A has measure one, and $|A \cap S| < \kappa$ for every $S \in \mathscr{F}$ (because $A \cap S_\alpha \subseteq \alpha + 1$).

Now if $S \subseteq A$ is stationary, then S has positive measure, for otherwise $\mathscr{F}$ could not be maximal. ■

Corollary 1. *If κ is measurable, then the thin ideal over κ is not κ^+-saturated.*

Proof. Let D be a normal measure on κ and let I be the dual ideal. Every $A \in D$ can be divided into two disjoint stationary sets. ■

More generally, we have:

Corollary 2. *If κ carries a κ-saturated κ-complete ideal, then the thin ideal over κ is not κ^+-saturated.*

Proof. Apply Lemma 35.4 to a normal κ-saturated κ-complete ideal I over κ, and use Theorem 85 to get a contradiction. ■

The lemma can be further used to show, e.g., that if κ is weakly compact, then the thin ideal is not κ^+-saturated.

Generic Ultrapowers

We shall now introduce a powerful method for dealing with ideals over regular uncountable cardinals. The method is very similar to the method of ultrapowers that we used in Section 28.

Let κ be a regular uncountable cardinal and let I be an ideal over κ. Let us view the universe as a ground model, let us denote this ground model $\mathfrak{M}$, and let us consider the generic extension of $\mathfrak{M}$ given by the completion of the Boolean algebra $P(\kappa)/I$. In other words, consider the notion of forcing $(P, \subseteq)$, where P is the collection of all subsets of κ of positive measure:

(35.7) $X \in P$ iff $X \subseteq \kappa$ and $X \notin I$;
X is stronger than Y iff $X \subseteq Y$.

Let G be a generic filter on P.

Lemma 35.6.

(i) *G is an $\mathfrak{M}$-ultrafilter over κ extending the filter dual to I.*
(ii) *If I is κ-complete (in $\mathfrak{M}$), then G is $\mathfrak{M}$-κ-complete.*
(iii) *If I is normal, then G is $\mathfrak{M}$-normal.*

Proof. (i) If $X \subseteq \kappa$ has measure one, then $\{Y \in P : Y \subseteq X\}$ is dense in P and hence $X \in G$. That G is an $\mathfrak{M}$-ultrafilter is obvious.

(ii) If $\{X_\alpha : \alpha < \gamma\}$, $\gamma < \kappa$, is (in $\mathfrak{M}$) a partition of κ, then by the κ-completeness of I, the set $\{Y \in P : Y \subseteq \text{some } X_\alpha\}$ is dense in P and hence $X_\alpha \in G$ for some α.

(iii) If $X \in G$ and if $f \in \mathfrak{M}$ is a regressive function on X, then $\{Y \subseteq X : f \text{ is constant on } Y\}$ is dense below X, and hence f is constant on some $Y \in G$. ■

From now on assume that I is a κ-complete ideal over κ containing all singletons. Then G is a nonprincipal $\mathfrak{M}$-κ-complete $\mathfrak{M}$-ultrafilter over κ. Note that if I is atomless, then $G \notin \mathfrak{M}$ (but otherwise G may be in $\mathfrak{M}$; e.g., if I is prime, then G is the dual of I).

Let us consider (in $\mathfrak{M}[G]$) the ultrapower $Ult_G(\mathfrak{M})$; let us call this ultrapower a *generic ultrapower*. The generic ultrapower is a model of ZFC, but is not necessarily well-founded. We have of course the fundamental theorem, in this form

$$(35.8)\ Ult_G(\mathfrak{M}) \models \varphi([f_1], \ldots, [f_n]) \quad \text{iff} \quad \{\alpha : \mathfrak{M} \models \varphi(f_1(\alpha), \ldots, f_n(\alpha)\} \in G$$

whenever $f_1, \ldots, f_n \in \mathfrak{M}$ are functions defined on a set $X \in G$. In particular, we have an elementary embedding, the *natural embedding* $j_G\colon \mathfrak{M} \to Ult_G(\mathfrak{M})$, defined by

$$j_G(x) = [c_x]$$

where c_x is the constant function on κ with value x, and $[c_x]$ is its equivalence class in the ultrapower.

Let us denote the generic ultrapower by $\mathfrak{N}$, and $j_G = j$. The ordinal numbers of the model $\mathfrak{N}$ form a linearly ordered class, not necessarily well-ordered, but we shall show that (because I is κ-complete), $Ord^{\mathfrak{N}}$ has an initial segment of order-type κ. If $x \in Ord^{\mathfrak{N}}$, let us call the *order-type* of x the order-type of the set $\{y \in Ord^{\mathfrak{N}} : y <^{\mathfrak{N}} x\}$. If the order-type of x is an ordinal number, let us take the liberty of identifying x with this ordinal.

Lemma 35.7.

(i) *For every $\gamma < \kappa, j(\gamma) = \gamma$; hence $Ord^{\mathfrak{N}}$ has an initial segment of order-type κ.*
(ii) $j(\kappa) \neq \kappa$.
(iii) *If I is normal, then there exists $x \in Ord^{\mathfrak{N}}$ such that $x = \kappa$; in fact $[d] = \kappa$, where d is the diagonal function $d(\alpha) = \alpha$.*

Proof. Routine. G is $\mathfrak{M}$-κ-complete, nonprincipal, and in case (iii) $\mathfrak{M}$-normal. ∎

Let me mention again the fact that was already mentioned and that is fairly easy to verify: If P is the notion of forcing defined above (35.7), then r.o. $P =$ r.o.$(P(\kappa)/I)$; the mapping $X \mapsto [X]$ gives the natural correspondence. This means that sat$(P) =$ sat(I), the fact that we are going to use in the applications.

An important question is: When is the generic ultrapower well-founded? We shall now formulate a necessary and sufficient condition on I for this to be the case.

Let S be a set of positive measure. An *I-partition* of S is a maximal family W of subsets of S of positive measure such that $X \cap Y \in I$ for any distinct $X, Y \in W$. An I-partition W_1 of S is a *refinement* of an I-partition W_2 of S, $W_1 \leq W_2$, if every $X \in W_1$ is a subset of some $Y \in W_2$. A *functional* on S is a collection F of functions such that $W_F = \{\mathrm{dom}(f) : f \in F\}$ is an I-partition of S and $\mathrm{dom}(f) \neq \mathrm{dom}(g)$ whenever $f \neq g \in F$.

We define $F < G$ for two functionals on S to mean that:

(i) each $f \in F \cup G$ is a function into the ordinals;
(ii) $W_F \leq W_G$; and
(iii) if $f \in F$ and $g \in G$ are such that $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$, then $f(\alpha) < g(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$.

The reason we define functionals is that they represent functions in the Boolean-valued model $\mathfrak{M}^B$ (and so are canonical representatives for elements of $Ult_G(\mathfrak{M})$): Let $f \in \mathfrak{M}^B$ be such that

$$S \Vdash f \text{ is a function with } \mathrm{dom}(f) \in G \text{ and } f \in \mathfrak{M} \tag{35.9}$$

Then there is clearly an I-partition W of S, and for each $X \in W$ a function f_X on X such that for all $X \in W$, $X \Vdash f \,|\, \check{X} = \check{f}_X$. Thus the functional $\{f_X : X \in W\}$ "represents" the Boolean-valued f on S.

Conversely, if F is functional on S, then there is $f \in \mathfrak{M}^B$ such that (35.9) holds; and for each $f \in F$, if $X = \mathrm{dom}(f)$, then $X \Vdash f \,|\, \check{X} = \check{f}$.

Note also that if $F < G$ are functionals on S and f, g are the corresponding Boolean-valued elements, then

(35.10)

$$S \Vdash f, g \in \mathfrak{M} \text{ and } \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \text{ and } f(\alpha) < g(\alpha) \text{ for all } \alpha \in \mathrm{dom}(f)$$

Conversely, if f and g satisfy (35.10), then there are functionals F and G that represent f and g, and $F < G$.

Let I be a κ-complete ideal over κ containing all singletons. The ideal I is *precipitous* if whenever S is a set of positive measure and $\{W_n : n < \omega\}$ are I-partitions of S such that

$$W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots$$

then there exists a sequence of sets

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$$

such that $X_n \in W_n$ for each n, and $\bigcap_{n=0}^{\infty} X_n$ is nonempty.

Lemma 35.8. *The following are equivalent:*

(i) *I is precipitous.*
(ii) *For no set S of positive measure is there a sequence of functionals on S such that $F_0 > F_1 > \cdots > F > \cdots$.*
(iii) *If G is a generic filter on P (cf. 35.7), then the generic ultrapower $Ult_G(\mathfrak{M})$ is well-founded.*

Proof. In view of the preceding discussion on functionals, (ii) is equivalent to (iii): If $F_0 > F_1 > \cdots$ are functionals on S, and $f_0, f_1, \ldots,$ the corresponding elements of $\mathfrak{M}^B$, then S forces that $[f_0], [f_1], \ldots$ is a descending sequence of ordinals in the generic ultrapower. Conversely, if S forces that $Ult_G(\mathfrak{M})$ has a descending sequence of ordinals, we construct $F_0, F_1, \ldots$ on S such that $F_0 > F_1 > \cdots$.

The implication (i) → (ii) is easy. If $F_0 > F_1 > \cdots$ are functionals on S, then the partitions $W_{F_0}, W_{F_1}, \ldots$ constitute a counterexample to precipitousness: If $X_0 \supseteq X_1 \supseteq \cdots$ are elements of $W_0, W_1, \ldots$, let $f_0 \in F_0$ be the function with domain X_0, $f_1 \in F_1$ with domain X_1, etc; now if $\bigcap_{n=0}^{\infty} X_n$ were nonempty, we would get $f_0(\alpha) > f_1(\alpha) > \cdots$ for $\alpha \in \bigcap_{n=0}^{\infty} X_n$.

To show that (ii) → (i), let us assume that I is not precipitous and let $W_0 \geq W_1 \geq \cdots$ be partitions of some $S \notin I$ that attest to the fact that I is not precipitous. We shall construct functionals on S such that $F_0 > F_1 > \cdots$.

Without loss of generality, let us assume that if $X \in W_{n+1}$, $Y \in W_n$, and $X \subseteq Y$, then $X \neq Y$. Let $T = \bigcup_{n=0}^{\infty} W_n$; note that the partially ordered set $(T, \subset)$ is an upside-down tree (of height ω).

For each $z \in S$, let us consider the set $T_z = \{X \in T : z \in X\}$. Since every descending sequence $X_0 \supset X_1 \supset \cdots$ in T has empty intersection, it follows that for every z, T_z has no infinite descending sequence $X_0 \supset X_1 \supset \cdots$; hence the relation $\subset$ on T_z is well-founded. Thus there is, for each z, an ordinal function ρ_z on T (the rank function, cf. Section 9) such that $\rho_z(X) < \rho_z(Y)$ when $X \subset Y$. It is clear that if $X \in W_{n+1}$, $Y \in W_n$, and $z \in X \subset Y$, then $\rho_z(X) < \rho_z(Y)$.

Thus we define, for each $X \in T$, a function f_X on X as follows:

$$f_X(z) = \rho_z(X) \qquad (\text{all } z \in X)$$

Now it is clear that if we let $F_n = \{f_X : X \in W_n\}$ for each n, then $F_0, F_1, \ldots$ are functionals on S and $F_0 > F_1 > \cdots$. ■

The next lemma shows that the ideal of small sets is not precipitous. Then we show that a κ^+-saturated ideal over κ is precipitous. Later (in Theorem 86a)) we show that if there exists a precipitous ideal, then there is a transitive model with a measurable cardinal.

Lemma 35.9. *Let κ be a regular uncountable cardinal. The ideal $I = \{X \subseteq \kappa : |X| < \kappa\}$ is not precipitous.*

Proof. Let $I = \{X \subseteq \kappa : |X| < \kappa\}$. A set $X \subseteq \kappa$ has positive measure just in case $|X| = \kappa$. For each such X, let f_X be the unique order-preserving function from X onto κ.

For each set X of positive measure there exists a set $Y \subset X$ of positive measure such that $f_Y(\alpha) < f_X(\alpha)$ for all $\alpha \in Y$; namely if we let $Y = \{\alpha \in X : f_X(\alpha)$ is a successor ordinal$\}$, then $f_X(\alpha) = f_Y(\alpha) + 1$ for all $\alpha \in Y$. Thus for each $X \notin I$ there is an I-partition W_X of X such that for all $Y \in W_X$, $f_Y(\alpha) < f_X(\alpha)$ on Y.

Now we construct I-partitions $W_0 \geq W_1 \geq \cdots$ as follows: We let $W_0 = \{\kappa\}$, and for each n, we let $W_{n+1} = \bigcup \{W_X : X \in W_n\}$. For each n, we let F_n be the functional $F_n = \{f_X : X \in W_n\}$. It is clear that $F_0 > F_1 > \cdots > F_n > \cdots$, and therefore I is not precipitous. ■

Lemma 35.10. *Let κ be a regular uncountable cardinal. Every κ^+-saturated κ-complete ideal over κ is precipitous.*

Proof. Let I be a κ^+-saturated κ-complete ideal over κ. Let S be a set of positive measure and let $W_0 \geq W_1 \geq \cdots$ be I-partitions of S. We shall find $X_0 \supseteq X_1 \supseteq \cdots$ in $W_0, W_1, \ldots$ such that $\bigcap_{n=0}^{\infty} X_n$ is nonempty.

We shall first modify each W_n to obtain a new I-partition W'_n that is almost like W_n but is disjoint. We proceed by induction on n. Since $|W_0| \leq \kappa$, let $W_0 = \{X_\alpha : \alpha < \theta\}$ where $\theta \leq \kappa$, and for each $\alpha < \theta$, let $X'_\alpha = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$; then we let $W'_0 = \{X' : X \in W\}$. Since I is κ-complete, we have $X - X' \in I$ for all $X \in W_0$ and thus W'_0 is an I-partition of S; moreover, W'_0 is disjoint, and is a partition of $S_0 = \bigcup W'_0$ and $S - S_0 \in I$. Having constructed W'_n, we enumerate $W_{n+1} = \{X_\alpha : \alpha < \theta\}$ where $\theta \leq \kappa$, and for each $\alpha < \theta$, let $X'_\alpha = (X_\alpha - \bigcup_{\beta<\alpha} X_\beta) \cap Z$ where Z is the unique $Z \in W'_n$ that is almost all of the unique $Y \in W_n$ such that $X_\alpha \subseteq Y$. We let $W'_{n+1} = \{X' : X \in W_{n+1}\}$; W'_n is a partition of $S_{n+1} = \bigcup W'_{n+1}$, $S - S_{n+1} \in I$, and $X - X' \in I$ for all $X \in W_{n+1}$.

Since each S_n is almost all of S, the set $\bigcap_{n=0}^{\infty} S_n$ is nonempty; let z be an element of this intersection. For each n there is a unique $Y_n \in W'_n$ such that $z \in Y_n$; let X_n be the unique $X_n \in W_n$ such that $Y_n \subseteq X_n$. It is clear that $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$, and $\bigcap_{n=0}^{\infty} X \neq \emptyset$. ■

In the following exercises, "minimal unbounded function" and "least unbounded function" have the meaning defined in the proof of Lemma 34.1.

Exercise 35.9. If I is a precipitous ideal over κ, then there exists a minimal unbounded function.

[There is a set X of positive measure and a function f on X such that X forces that f represents κ in the generic ultrapower.]

Exercise 35.10. If there exists a κ^+-saturated ideal over κ, then there exists a normal κ^+-saturated ideal over κ.

[Let g be a minimal unbounded function. Show that $g_*(I)$ is κ^+-saturated.]

If I is κ^+-saturated, then the Boolean-valued names for functions on κ in $\mathfrak{M}$ can be represented not by functionals but by ordinary functions: Let F be a functional (on κ). Let $W = \{\mathrm{dom}(f) : f \in F\}$; since I is κ^+-saturated, W can be replaced by a *disjoint* W' such that for each $X \in W$ there $X' \in W'$ such that $X' \subseteq X$ and $X - X' \in I$. If we replace each $f \in F$ by its restriction to the corresponding $X' \in W'$, we get a functional F' whose elements have disjoint domains. Then $f = \bigcup F'$ is a function, and if $f \in \mathfrak{M}^B$ is the name corresponding to F, then $\|f = \check{f}\| = 1$.

Exercise 35.11. If I is a κ^+-saturated ideal over κ, then there exists a least unbounded function.

[Let $f \in \mathfrak{M}^B$ be such that $\|f$ represents $\check{\kappa}$ in the generic ultrapower$\| = 1$, and let $f\colon \kappa \to \kappa$ be such that $\|f = \check{f}\| = 1$. Show that f is the least unbounded function.]

Applications of Generic Ultrapowers

1. Proof of Theorem 23

Let me consider this typical special case: Let κ be a singular cardinal of cofinality $\aleph_1$, and assume that $2^\lambda = \lambda^+$ for all $\lambda < \kappa$. We shall show that $2^\kappa = \kappa^+$, using a generic ultrapower.

Let I be the ideal of thin sets over ω_1, let P be the corresponding notion of forcing (i.e., forcing conditions are stationary sets) and let G be a generic filter on P. Note that since $|P| = 2^{\aleph_1} < \kappa$ (in $\mathfrak{M}$), all cardinals $\geq \kappa$ remain cardinals in $\mathfrak{M}[G]$.

Let us work in $\mathfrak{M}[G]$. G is an $\mathfrak{M}$-normal $\mathfrak{M}$-countably-complete $\mathfrak{M}$-ultrafilter over $\omega_1^{\mathfrak{M}}$. Let $\mathfrak{N} = Ult_G(\mathfrak{M})$ be the generic ultrapower and let $j\colon \mathfrak{M} \to \mathfrak{N}$ be the corresponding elementary embedding. $\mathfrak{N}$ is not necessarily well-founded.

Let $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ be (in $\mathfrak{M}$) an increasing continuous sequence of cardinals converging to κ. Let e be the cardinal number in $\mathfrak{N}$ represented by the function $e(\alpha) = \kappa_\alpha$. Let e^+ denote the successor cardinal of e in $\mathfrak{N}$.

For each $X \subseteq \kappa$ in $\mathfrak{M}$ let f_X be the function on $\omega_1^{\mathfrak{M}}$ defined by $f_X(\alpha) = X \cap \kappa_\alpha$. Clearly, each f_X represents in $\mathfrak{N}$ a subset of e. Moreover, if $X \neq Y$, then f_X and f_Y are almost disjoint and hence represent distinct subsets of e. It follows that $|P^{\mathfrak{M}}(\kappa)| \leq |P^{\mathfrak{N}}(e)|$, where $P^{\mathfrak{N}}(e)$ denotes the collection of all subsets of e in $\mathfrak{N}$.

Now $\mathfrak{N} \vDash 2^e = e^+$ (because $\mathfrak{M} \vDash 2^{\kappa_\alpha} = \kappa_\alpha^+$ for all α), which means that in the model $\mathfrak{N}$ there is a one-to-one correspondence between the power set of e and e^+. It follows that there is a one-to-one correspondence between $P^{\mathfrak{N}}(e)$ and the set $\mathrm{ext}(e^+) = \{x \in Ord^{\mathfrak{N}} : x <^{\mathfrak{N}} e^+\}$. Thus we have so far $|P^{\mathfrak{M}}(\kappa)| \leq |\mathrm{ext}(e^+)|$.

Next we observe that $e = \sup\{j(\kappa_\gamma) : \gamma < \omega_1^{\mathfrak{M}}\}$. This is because if f represents an ordinal less than e, then there is a set of limit ordinals $X \in G$ such that $f(\alpha) < \kappa_\alpha$ for all $\alpha \in X$; thus $f(\alpha) < \kappa_{\gamma(\alpha)}$ for some $\gamma(\alpha) < \alpha$, and by normality of G, there is γ such that $[f] <^{\mathfrak{N}} \kappa_\gamma$. Now for each $\gamma < \omega_1^{\mathfrak{M}}$, $|\mathrm{ext}(j(\kappa_\gamma))| \leq |(\kappa_\gamma^{\aleph_1})^{\mathfrak{M}}| < \kappa$, and therefore $|\mathrm{ext}(e)| \leq \kappa$.

If $x <^{\mathfrak{N}} e^+$, then there is in $\mathfrak{N}$ a one-to-one mapping of x into e, and therefore, $|\mathrm{ext}(x)| \leq |\mathrm{ext}(e)| \leq \kappa$. Thus $\mathrm{ext}(e^+)$ is a linearly ordered set whose each initial segment has size at most κ. Therefore $|\mathrm{ext}(e^+)| \leq \kappa^+$, and we have

$$|P^{\mathfrak{M}}(\kappa)| \leq \kappa^+$$

We have argued so far in $\mathfrak{M}[G]$; in other words, we have proved that $|P^{\mathfrak{M}}(\kappa)|^{\mathfrak{M}[G]} \leq (\kappa^+)^{\mathfrak{M}[G]}$. But since all cardinals $\geq \kappa$ in $\mathfrak{M}$ remain cardinals in $\mathfrak{M}[G]$, it is necessary that $|P^{\mathfrak{M}}(\kappa)|^{\mathfrak{M}} \leq (\kappa^+)^{\mathfrak{M}}$; in other words we have proved that $2^\kappa = \kappa^+$ (in $\mathfrak{M}$). ■

2. *Proof of Theorem 84b*

Let $2^{\aleph_0} = \aleph_\gamma < \aleph_{\omega_1}$ and let I be a σ-complete λ-saturated ideal over ω_1. We shall show that $2^{\aleph_1} \leq \aleph_\gamma \cdot \lambda$.

Let P be the notion of forcing corresponding to I, and let G be generic on P. Since $\operatorname{sat}(P) = \operatorname{sat}(I) \leq \lambda$, all cardinals $\geq \lambda$ in $\mathfrak{M}$ are cardinals in $\mathfrak{M}[G]$.

Let us work in $\mathfrak{M}[G]$, and let $\mathfrak{N} = Ult_G(\mathfrak{M})$ and $j = j_G\colon \mathfrak{M} \to \mathfrak{N}$ be as before. For each $X \subseteq \omega_1$ in $\mathfrak{M}$ let f_X be the function on $\omega_1^{\mathfrak{M}}$ defined by $f_X(\alpha) = X \cap \alpha$. Each f_X represents in $\mathfrak{N}$ a subset of the countable ordinal d represented by the function $d(\alpha) = \alpha$; moreover, if $X = Y$, then f_X and f_Y are almost disjoint and hence $[f_X] \neq [f_Y]$. It follows that $|P^{\mathfrak{M}}(\omega_1^{\mathfrak{M}})| \leq |P^{\mathfrak{N}}(d)|$. Let e be the cardinal number in $\mathfrak{N}$ such that $\mathfrak{N} \vDash 2^{\aleph_0} = e$ (we recall that $\omega^{\mathfrak{N}} = \omega$). Since $\mathfrak{N} \vDash |P(d)| = e$, we have $|P^{\mathfrak{N}}(d)| = |\operatorname{ext}(e)|$ and so

$$|P^{\mathfrak{M}}(\omega_1^{\mathfrak{M}})| \leq |\operatorname{ext}(e)| \tag{35.11}$$

Next we shall compute the size of $\operatorname{ext}(e)$. Since $\mathfrak{M} \vDash 2^{\aleph_0} = \aleph_\gamma$ and $j\colon \mathfrak{M} \to \mathfrak{N}$ is elementary, we have $e = j(\omega_\gamma)$. We shall now prove by induction on $\gamma < \omega_1^{\mathfrak{M}}$ that

$$|\operatorname{ext}(j(\omega_\gamma^{\mathfrak{M}}))| \leq \lambda \cdot |\omega_\gamma^{\mathfrak{M}}| \tag{35.12}$$

Let me denote $j(\omega_\gamma^{\mathfrak{M}}) = e_\gamma$ for all $\gamma < \omega_1^{\mathfrak{M}}$. By Lemma 35.7, the ordinals of $\mathfrak{N}$ have an initial segment of order-type $\omega_1^{\mathfrak{M}}$; thus the *infinite cardinals* of $\mathfrak{N}$ also have an initial segment of order-type $\omega_1^{\mathfrak{M}}$, namely $\{e_\gamma : \gamma < \omega_1^{\mathfrak{M}}\}$.

If $\gamma = 0$, then $e_\gamma = \omega$ and (35.12) is true. If γ is a limit ordinal, then $e_\gamma = \sup\{e_\delta : \delta < \gamma\}$ and (35.12) is again true provided it is true for all $\delta < \gamma$. If $\gamma = 1$, then $\operatorname{ext}(e_1)$ is a linearly ordered set whose each initial segment is countable, and hence $|\operatorname{ext}(e_1)| \leq \aleph_1$. Since λ is a cardinal (now we are in $\mathfrak{M}[G]$), we have $\aleph_1 \leq \lambda$, and (35.12) holds.

Let us assume that (35.12) holds for γ and let us show that it also holds for $\gamma + 1$. Every function $f\colon \omega_1 \to \omega_{\gamma+1}$ in $\mathfrak{M}$ is bounded by some constant function, and therefore $j(\omega_{\gamma+1}) = \sup\{j(\xi) : \xi < \omega_{\gamma+1}\}$. Hence the linearly ordered set $\operatorname{ext}(e_{\gamma+1})$ has a cofinal set of order type $\omega_{\gamma+1}^{\mathfrak{M}}$ and each its initial segment has size $\leq \lambda \cdot |\omega_\gamma^{\mathfrak{M}}|$ (because if $\xi < \omega_{\gamma+1}$, then $|\operatorname{ext}(j(\xi))| \leq |\operatorname{ext} e_\gamma| \leq \lambda \cdot |\omega_\gamma^{\mathfrak{M}}|$). It follows that $|\operatorname{ext}(e_{\gamma+1})| \leq \lambda \cdot |\omega_{\gamma+1}^{\mathfrak{M}}|$.

Now we put (35.11) and (35.12) together and get

$$|P^{\mathfrak{M}}(\omega_1^{\mathfrak{M}})| \leq \lambda \cdot \aleph_\gamma^{\mathfrak{M}}$$

This we proved in $\mathfrak{M}[G]$; but since all cardinals $\geq \lambda$ in $\mathfrak{M}$ remain cardinals in $\mathfrak{M}[G]$, the same must be true in $\mathfrak{M}$. Hence (in $\mathfrak{M}$)

$$2^{\aleph_1} \leq \lambda \cdot \aleph_\gamma. \quad \blacksquare$$

3. *The GCH at a Cardinal κ that Carries a κ^+-saturated Ideal*

Let κ be a regular uncountable cardinal and let I be a κ^+-saturated ideal over κ. We shall prove

$$\text{if} \quad 2^\lambda = \lambda^+ \text{ for all } \lambda < \kappa, \qquad \text{then} \quad 2^\kappa = \kappa^+ \tag{35.13}$$

(We proved this for a measurable κ in Lemma 28.13.) Let P be the notion of forcing corresponding to I, let G be generic on P, and let $\mathfrak{N} = Ult_G(\mathfrak{M})$. Since I is κ^+-saturated, $\mathfrak{N}$ is well-founded and hence we identify it with a transitive model $\mathfrak{N} \subseteq \mathfrak{M}[G]$. Let j: $\mathfrak{M} \to \mathfrak{N}$ be the natural embedding. We have $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and $j(\kappa) > \kappa$. If $X \subseteq \kappa$ and $X \in \mathfrak{M}$, then $X \in \mathfrak{N}$ because $X = j(X) \cap \kappa$. Thus $P^{\mathfrak{M}}(\kappa) \subseteq P^{\mathfrak{N}}(\kappa)$.

We assume that $\mathfrak{M} \models (2^\lambda = \lambda^+$ for all $\lambda < \kappa)$ and hence $\mathfrak{N} \models (2^\lambda = \lambda^+$ for all $\lambda < j(\kappa))$, and in particular, $\mathfrak{N} \models |P(\kappa)| = \kappa^+$, where α^+ denotes the least cardinal greater than α. Now $(\kappa^+)^{\mathfrak{N}} \le (\kappa^+)^{\mathfrak{M}[G]}$; and because sat$(P) = \kappa^+$ (in $\mathfrak{M}$), $(\kappa^+)^{\mathfrak{M}}$ is a cardinal in $\mathfrak{M}[G]$ and we have also $(\kappa^+)^{\mathfrak{M}[G]} = (\kappa^+)^{\mathfrak{M}}$. Thus we have, in $\mathfrak{M}[G]$,

$$|P^{\mathfrak{M}}(\kappa)| \le (\kappa^+)^{\mathfrak{M}}$$

and since all cardinals above κ^+ in $\mathfrak{M}$ are preserved, the last formula is also true in $\mathfrak{M}$, and we have $2^\kappa = \kappa^+$. ■

4. *Some Consequences of an $\aleph_2$-saturated Ideal Over ω_1*

Let I be an $\aleph_2$-saturated ideal over ω_1. Then

(35.14) (a) If $2^{\aleph_0} = \aleph_1$, then $2^{\aleph_1} = \aleph_2$.
(b) If $\aleph_1 < 2^{\aleph_0} < \aleph_{\omega_1}$, then $2^{\aleph_1} = 2^{\aleph_0}$,
(c) If $2^{\aleph_0} = \aleph_{\omega_1}$, then $2^{\aleph_1} \le \aleph_{\omega_2}$.
(d) If $\aleph_{\omega_1}$ is a strong limit, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$.

Let $\Phi(\alpha)$ denote the αth member of the class $\{\kappa : \aleph_\kappa = \kappa\}$.

(e) If $\Phi(\omega_1)$ is a strong limit, then $2^{\Phi(\omega_1)} < \Phi(\omega_2)$.

(a) and (b) are consequences of Theorem 84 (cf. (35.2) and (35.3)) (and (a) is also a consequence of (35.13)). (d) gives a better bound on $2^{\aleph_{\omega_1}}$ than that in Theorem 24, which gives $2^{\aleph_{\omega_1}} < \aleph_\gamma$ where $\gamma = (2^{\aleph_1})^+$. (e) gives a bound for $2^{\Phi(\omega_1)}$, which is an open question in ZFC.

As before, let G be a generic ultrafilter over ω_1, let $\mathfrak{N} = Ult_G(\mathfrak{M})$ and let j: $\mathfrak{M} \to \mathfrak{N}$. $\mathfrak{N}$ is a transitive model, $\mathfrak{N} \subseteq \mathfrak{M}[G]$.

Let us denote $\kappa = \omega_1^{\mathfrak{M}}$. We have $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and $j(\kappa) > \kappa$. Thus κ is a countable ordinal in $\mathfrak{N}$. Moreover, every f: $\kappa \to Ord$ in $\mathfrak{M}$ belongs to $\mathfrak{N}$ (as in Lemma 28.9a), and so every $\gamma < \omega_2^{\mathfrak{M}}$ is countable in $\mathfrak{N}$. Since sat$(I) = \aleph_2$, $\omega_2^{\mathfrak{M}}$ is a cardinal in $\mathfrak{M}[G]$, hence in $\mathfrak{N}$, and so

$$j(\omega_1^{\mathfrak{M}}) = \omega_2^{\mathfrak{M}}$$

We shall now prove (c), (d), and (e). To prove (c), let us assume that $\mathfrak{M} \models 2^{\aleph_0} = \aleph_{\omega_1}$. Since $\mathfrak{N} \models |\omega_1^{\mathfrak{M}}| = \aleph_0$, and j is elementary, we have $\mathfrak{N} \models |P^{\mathfrak{M}}(\kappa)| = 2^{\aleph_0} = \aleph_{j\kappa}$. Now $\aleph_{j\kappa} \le \aleph_{j\kappa}^{\mathfrak{M}[G]}$; and since $j(\kappa) = \omega_2^{\mathfrak{M}}$, $\aleph_{j\kappa}^{\mathfrak{M}[G]}$ is the $\omega_2^{\mathfrak{M}}$th cardinal in $\mathfrak{M}[G]$. However, all cardinals $\ge \aleph_2^{\mathfrak{M}}$ are preserved and hence $\aleph_{j\kappa}^{\mathfrak{M}[G]} = \aleph_{\omega_2}^{\mathfrak{M}}$. Thus

$$|P^{\mathfrak{M}}(\omega_1^{\mathfrak{M}})| \le \aleph_{\omega_2}^{\mathfrak{M}}$$

holds in $\mathfrak{N}$, therefore in $\mathfrak{M}[G]$; and because cardinals above $\aleph_2$ are preserved, this also holds in $\mathfrak{M}$.

To prove (d) or (e), note that if $\aleph_{\omega_1}$ (or $\Phi(\omega_1)$) is a strong limit, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1}^{\aleph_1}$ $(2^{\Phi(\omega_1)} = (\Phi(\omega_1))^{\aleph_1})$. Let λ denote $\aleph_{\omega_1}$ in (d) and $\Phi(\omega_1)$ in (e). It is easy to see that $j(\lambda) > \lambda$. Now $\mathfrak{N} \vDash \forall\alpha < j(\lambda)\, |^{\kappa}\alpha| < j(\lambda)$, and because $(^{\kappa}\lambda)^{\mathfrak{M}} \subseteq (^{\kappa}\lambda)^{\mathfrak{N}}$, we have $\mathfrak{N} \vDash |(^{\kappa}\lambda)^{\mathfrak{M}}| < j(\lambda)$.

In case (d) we have $j(\lambda) \le \aleph_{\omega_2}^{\mathfrak{M}}$ as in (c); and in case (e) we obtain similarly $j(\lambda) \le \Phi(j(\kappa)) \le \Phi(\omega_2)$. The rest of the proof of either (d) or (e) is as before. ∎

Exercise 35.12. If I is a precipitous ideal over ω_1 and if $\Phi(\omega_1)$ is a strong limit, then $2^{\Phi(\omega_1)} < \Phi((2^{\aleph_1})^+)$.

[Proceed as in (e) above and show that $j(\Phi(\omega_1)) \le \Phi(j(\omega_1))$ and $j(\omega_1) < (2^{\aleph_1})^+$.]

As I mentioned, the bound on $2^{\aleph_{\omega_1}}$ in (d) is better than that in Theorem 24. The same bound $\aleph_{\omega_2}$ can be obtained directly from Lemma 8.7 if there is no function $\varphi\colon \omega_1 \to \omega_1$ such that $\|\varphi\| = \omega_2$ (see (8.7) for the definition of $\|\varphi\|$).

For every $\gamma < \omega_2$ there is $\varphi_\gamma\colon \omega_1 \to \omega_1$ such that $\|\varphi_\gamma\| = \gamma$. In fact, let us define functions φ_γ, $\gamma < \omega_2$, as follows: $\varphi_0(\alpha) = 0$, $\varphi_{\gamma+1}(\alpha) = \varphi_\gamma(\alpha) + 1$; if $\gamma = \lim_n \gamma_n$, let $\varphi_\gamma(\alpha) = \sup_n \{\varphi_{\gamma_n}(\alpha)\}$; if $\gamma = \lim_{\xi\to\omega_1} \gamma_\xi$, let $\varphi_\gamma(\alpha) = \sup\{\varphi_{\gamma_\xi}(\alpha) : \xi < \alpha\}$. Then $\varphi_\gamma < \varphi_\delta$ on a closed unbounded set whenever $\gamma < \delta$, and $\|\varphi_\gamma\|_S = \gamma$ for all $\gamma < \omega_2$ and all stationary S. It follows that if $\|\varphi\| = \omega_2$, then for every $\gamma < \omega_2$, $\varphi_\gamma(\alpha) < \varphi(\alpha)$ on a closed unbounded set.

Exercise 35.13. If there is $\varphi\colon \omega_1 \to \omega_1$ such that $\|\varphi\| = \omega_2$, then there is a family of $\aleph_2$ functions from ω_1 into ω such that any two differ on a closed unbounded set.

[Consider $\{f_\gamma : \gamma < \omega_2\} \subseteq \prod_{\alpha<\omega_1}(\varphi(\alpha) + 1)$, where $f_\gamma(\alpha) = \min\{\varphi_\gamma(\alpha), \varphi(\alpha)\}$.]

Let me mention that Chang's conjecture implies that there is no family of $\aleph_2$ functions from ω_1 into ω such that any two differ on a closed unbounded set; the proof is analogous to the one in Exercise 29.14. Thus Chang's conjecture implies that if $\aleph_{\omega_1}$ is a strong limit then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$. (This is because by Lemma 8.7, $2^{\aleph_{\omega_1}} < \aleph_\theta$ where θ is the norm of the constant function c_{ω_1} on ω_1.)

5. *κ-Saturated Ideals Over κ*

Let κ be a regular uncountable cardinal and assume that κ carries a κ-saturated ideal. We shall prove:

(35.15) (a) κ is weakly Mahlo;
(b) $\{\alpha < \kappa : \alpha \text{ is weakly Mahlo}\}$ is stationary;
(c) if $X \subseteq \kappa$ has measure one then $X \cap \mathrm{M}(X)$ has measure one, where

$$\mathrm{M}(X) = \{\alpha < \kappa : \operatorname{cf}\alpha < \omega \text{ and } X \cap \alpha \text{ is stationary in } \alpha\}$$

If there exists a κ-saturated ideal over κ, then κ is weakly inaccessible by Lemma 27.8, and there exists a normal κ-saturated ideal over κ (by Lemma 35.1). Let I be a normal κ-saturated ideal over κ. We first prove:

Lemma 35.11. *If $S \subseteq \kappa$ is stationary, then for almost all $\alpha < \kappa$, $S \cap \alpha$ is stationary in α.*

Proof. If not, then there is a set X of positive measure such that $S \cap \alpha$ is not stationary in α (or $\operatorname{cf} \alpha = \omega$) for all $\alpha \in X$. Let G be a generic ultrafilter over κ (corresponding to I) such that $X \in G$. Let $\mathfrak{N} = Ult_G(\mathfrak{M})$. $\mathfrak{N}$ is a transitive model. Since I is normal, κ is represented in $\mathfrak{N}$ by the function $d(\alpha) = \alpha$. Since $S = j(S) \cap \kappa$, we have $\mathfrak{N} \vDash S$ is not stationary.

However, the notion of forcing is κ-saturated and hence κ is a regular cardinal in $\mathfrak{M}[G]$, and by Lemma 35.4 (with ω_1 replaced by κ) $\mathfrak{M}[G] \vDash S$ is stationary. Now $\mathfrak{N} \subseteq \mathfrak{M}[G]$ and so $\mathfrak{N} \vDash S$ is stationary. A contradiction. ■

Now we prove (35.15). Since I is normal, every set of positive measure is stationary. Thus (c) follows since if X has measure one then $\mathrm{M}(X)$ has measure one by Lemma 35.11, and so does $X \cap \mathrm{M}(X)$.

To prove (a), it suffices to show that almost all $\alpha < \kappa$ are regular cardinals. Otherwise, let X be a set of positive measure such that all $\alpha \in X$ are singular. Let $G \ni X$ be generic and let $\mathfrak{N} = Ult_G(\mathfrak{M})$. Then $\mathfrak{N} \vDash \kappa$ is singular, contrary to the fact that κ is regular in $\mathfrak{M}[G]$ and $\mathfrak{N} \subseteq \mathfrak{M}[G]$.

Now (b) follows by an application of (c): Let $X = \{\alpha < \kappa : \alpha \text{ is regular}\}$, then $X \cap \mathrm{M}(X) = \{\alpha < \kappa : \alpha \text{ is weakly Mahlo}\}$. ■

6. *Proof of Theorem 85*

Let κ be a regular uncountable cardinal and let S be a stationary subset of κ. We shall give a quick proof that S is the disjoint union of κ stationary subsets.

Otherwise, the ideal $I = \{X \subseteq \kappa : X \cap S \text{ is thin}\}$ is a normal κ-saturated ideal. By Lemma 35.11, $S - \mathrm{M}(S)$ has measure zero and hence is thin, which contradicts Lemma 35.3. ■

7. *Changing Cofinalities of Cardinals*

Let κ and λ be regular cardinals, $\lambda < \kappa$. Let us assume that the GCH holds and that the thin ideal over κ is κ^+-saturated. Then there exists a generic extension in which all cardinals less than κ remain cardinals, and $\operatorname{cf} \kappa = \lambda$.

Compare this with Corollary 3 of Jensen's covering theorem (Theorem 74). For instance, if $\kappa = \aleph_3$ and $\lambda = \aleph_1$, then $\mathfrak{M}[G] \vDash (|\kappa| = \aleph_2 \text{ and } \operatorname{cf} \kappa = \omega_1)$; if κ is inaccessible, then it becomes a singular cardinal of cofinality λ.

Let I be the ideal of thin subsets of κ, and let $(P, \subseteq)$ be the corresponding notion of forcing. Since I is κ^+-saturated, the algebra $B = P(\kappa)/I$ is complete, and $B = \text{r.o.}\, P$. Every $b \in B$ is represented by a stationary set: $b = [S]$; moreover, if $\{b_\xi : \xi < \gamma\}$ is a partition of B, then we can find *disjoint* S_ξ such that $b_\xi = [S_\xi]$ for all $\xi < \gamma$. Let

$$E = \{\alpha < \kappa : \operatorname{cf} \alpha = \lambda\}$$

We shall show that in the generic extension, all cardinals less than κ are preserved, and that

$$E \Vdash \operatorname{cf} \check{\kappa} = \check{\lambda} \tag{35.16}$$

To show that the cardinals below κ are preserved, it suffices to prove that if ν is a cardinal and $\nu^+ < \kappa$, then $P^{\mathfrak{M}[G]}(\nu) = P^{\mathfrak{M}}(\nu)$. For this, it is enough to show that B is (ν, ν)-distributive.

Let $W_\eta = \{b^\eta_\xi : \xi < \nu\}$, $\eta < \nu$, be ν partitions of B into ν pieces. Each b^η_ξ can be represented by a stationary set S^η_ξ such that each $Z_\eta = \{S^\eta_\xi : \xi < \nu\}$, $\eta < \nu$, is a partition of κ. For each f: $\nu \to \nu$, consider $S_f = \bigcap_\eta S^\eta_{f(\eta)}$; since the number of f's is $2^\nu = \nu^+ < \kappa$, we conclude by κ-completeness of I that those S_f that are stationary constitute a partition refining all the W_η. Thus B is (ν, ν)-distributive.

Now let me show that E forces that there is an increasing function on λ converging to κ. For each $\alpha \in E$, let f_α be an increasing function on λ with limit α. Let us define a B-valued name f for a function from λ to κ as follows:

$$\|f(\check{\xi}) = \check{\eta}\| = [\{\alpha \in E : f_\alpha(\xi) = \eta\}] \qquad (\xi < \lambda, \quad \eta < \kappa)$$

It is clear that f is, with B-value 1, a function with domain $\subseteq \lambda$ and range $\subseteq \kappa$. It is easy to prove that f is increasing. Let me show that $E \Vdash \operatorname{dom} f = \check{\lambda}$ ($E \Vdash$ (ran f is unbounded in $\check{\kappa}$) is verified in a similar way). Let $\xi < \lambda$ and let $S \subseteq E$ be stationary; we want to find a stationary $T \subseteq S$ and $\eta < \kappa$ such that $T \Vdash f(\check{\xi}) = \check{\eta}$. For $\alpha \in S$, let $g(\alpha) = f_\alpha(\xi)$. Since g is regressive, there exists $T \subseteq S$ stationary such that g is constant on T. Obviously, this T will do.

It remains to show that cf κ is exactly λ (in a generic extension $\mathfrak{M}[G]$ such that $E \in G$). We have in fact shown that cf κ = cf λ in $\mathfrak{M}[G]$; thus if cf $\kappa < \lambda$, then there is $\nu < \lambda$, a cardinal in $\mathfrak{M}$, such that cf $\lambda = \nu$ in $\mathfrak{M}[G]$. But this is impossible since an argument analogous to the one above shows that B is (ν, λ)-distributive. ■

Precipitous Ideals

I shall conclude this section with the following theorem, showing that existence of precipitous ideals, even over a successor cardinal, is equiconsistent with existence of measurable cardinals. It has also been shown that it is consistent, relative to existence of measurable cardinals, that the ideal of thin subsets of ω_1 is precipitous.

Theorem 86.

(a) *If κ is a regular uncountable cardinal that carries a precipitous ideal, then κ is measurable in some transitive model of* ZFC.

(b) *If κ is a measurable cardinal, then there exists a generic extension in which $\kappa = \aleph_1$, and κ carries a precipitous ideal.*

The proof of (a) uses the technique of iterated ultrapowers from Section 31 (see also the proof of Theorem 80, in particular (33.8)–(33.12)).

Let κ be a regular uncountable cardinal, and let I be a precipitous ideal over κ. Let K be the class of all strong limit cardinals $\nu > 2^\kappa$ such that cf $\nu > \kappa$. Let $\gamma_0 < \gamma_1 < \cdots < \gamma_n < \cdots$ ($n < \omega$), be elements of K such that $|\gamma_n \cap K| = \gamma_n$, let $A = \{\gamma_n : n = 0, 1, \ldots\}$ and let $\lambda = \sup(A)$.

Lemma 35.12. *There exists an $L[A]$-ultrafilter W over κ such that W is nonprincipal, $L[A]$-κ-complete, $L[A]$-normal, iterable, and every iterated ultrapower $Ult_W^{(\alpha)}(L[A])$ is well-founded.*

Proof. Since I is precipitous, the generic ultrapower is well-founded, and so the diagonal function $d(\alpha) = \alpha$ represents some ordinal number in $Ult_G(\mathfrak{M})$. Thus there is a set S of positive measure, and an ordinal γ such that

$$S \Vdash \check{d} \text{ represents } \check{\gamma} \text{ in } Ult_G(\mathfrak{M}) \tag{35.17}$$

We shall first show that for every $X \in L[A]$, $X \subseteq \kappa$, either $S \cap X$ or $S - X$ has measure 0, and so

$$U = \{X \in P(\kappa) \cap L[A] : X \cap S \text{ has positive measure}\} \tag{35.18}$$

is an $L[A]$-ultrafilter.

Let $H = \mathscr{H}^{L[A]}(\kappa \cup K \cup \{A\})$ be the class of all sets definable in $L[A]$ from elements of $\kappa \cup K \cup \{A\}$ (this is expressible in ZF similarly to the way in which ordinal definability is; or we can use $L_\theta[A]$ for some large θ as in Lemma 31.13). Since $|K \cap \gamma_n| = \gamma_n$ for each n, it follows that if π is the transitive collapse of H, then $\pi(A) = A$ and $\pi[H] = L[A]$. Now if $X \subseteq \kappa$ and $X \in L[A]$, then because π is the identity on κ, we have $X = \pi(X)$ and therefore $X \in H$. Thus for every $X \in P(\kappa) \cap L[A]$ there is a formula φ and a finite set $E \subset \kappa \cup K$ such that

$$X = \{\xi < \kappa : L[A] \vDash \varphi(\xi, E, A)\} \tag{35.19}$$

We shall now show that (35.18) defines an $L[A]$-ultrafilter. Recall that for any generic ultrafilter G over κ, j_G is the identity on κ, and moreover, $j_G(\nu) = \nu$ for all $\nu \in K$ (this follows from the definition of K).

If $X \in L[A]$ and $X \cap S$ has positive measure, then because $X \Vdash \check{X} \in G$ and because (35.17) holds, we have

$$X \cap S \Vdash \check{\gamma} \in j_G(\check{X})$$

Now using (35.19), and the fact that $j_G(A) = A$ and $j_G(E) = E$, we have

$$X \cap S \Vdash (L[\check{A}] \vDash \varphi(\check{\gamma}, \check{E}, \check{A})) \tag{35.20}$$

But the formula forced by $X \cap S$ in (35.20) is about $\mathfrak{M}$, and thus true. Hence

$$\|\check{\gamma} \in j_G(\check{X})\| = 1$$

and by (35.17),

$$S \Vdash \check{X} \in G$$

This, however, means that $S - X$ has measure 0.

Since I is κ-complete, it is clear that U is $L[A]$-κ-complete, and moreover the intersection of any countable family of elements of U is nonempty. It is less clear that U is iterable: Let $\langle X_\alpha : \alpha < \kappa\rangle \in L[A]$; it suffices to show that

$S \Vdash (\{\alpha : X_\alpha \in U\} \in L[A])$. If G is generic such that $S \in G$, then $\{\alpha < \kappa : X_\alpha \in U\} = \{\alpha < \kappa : \gamma \in j_G(X_\alpha)\}$, but this is in $L[A]$ because $j_G(\langle X_\alpha : \alpha < \kappa\rangle) \in L[j_G(A)]$ and $j_G(A) = A$.

By Exercise 31.9, $Ult_U(L[A])$ is well-founded; let $f: \kappa \to \kappa$ be the function that represents κ in $Ult_U(L[A])$. Let $W = f_*(U)$.

It is easy to verify that W is an $L[A]$-normal, $L[A]$-κ-complete, iterable $L[A]$-ultrafilter over κ, and that the intersection of any countable family of elements of W is nonempty. By Exercise 31.9, every iterated ultrapower $Ult_W^{(\alpha)}(L[A])$ is well-founded. ■

Proof of Theorem 86a. Let $A = \{\gamma_n : n = 0, 1, \ldots\}$ be as above, let $\lambda = \sup(A)$, and let W be an $L[A]$-ultrafilter as in Lemma 35.12. Let us define in $L[A]$

$$F = \{X \subseteq \lambda : \exists n_0\ \forall n \geq n_0\ \gamma_n \in X\} \tag{35.21}$$

(compare with (33.9)). I claim that $D = F \cap L[F]$ is a normal measure on λ in $L[D]$.

For each α, let $i_{0\alpha}$ be the elementary embedding $i_{0\alpha}$: $L[A] \to Ult_W^{(\alpha)}(L[A])$. It follows from the definition of the class K that:

(i) if $\alpha < \gamma_n$, then $i_{0\alpha}(\gamma_n) = \gamma_n$;
(ii) $i_{0\gamma_n}(\kappa) = \gamma_n$;
(iii) if $\alpha < \lambda$, then $i_{0\alpha}(\lambda) = \lambda$.

Hence for all $\alpha < \lambda$, $i_{0\alpha}(L[A]) = L[A]$, $i_{0\alpha}(F) = F$, and $i_{0\alpha}(D) = D$.

We shall now prove that D is an ultrafilter in $L[D]$. Otherwise, let $X \subseteq \lambda$ be the least X (in the canonical well-ordering of $L[D]$) such that $X \notin D$ and $\lambda - X \notin D$. Since $i_{0\alpha}(D) = D$ for all $\alpha < \lambda$, we have $i_{0\alpha}(X) = X$ for all $\alpha < \lambda$; in particular, $i_{0\gamma_n}(X) = X$ for all n. Now for any n, if $\gamma_n \in X$, then $i_{0\gamma_n}(\kappa) \in i_{0\gamma_n}(X)$ and hence $\kappa \in X$, and vice versa. Hence either all γ_n are in X or none, and so either $X \in F$ or $\lambda - X \in F$, a contradiction.

The proof that D is κ-complete (in $L[D]$) and normal is similar and is left to the reader.

Thus we have proved that the model $L[A]$ satisfies the following formula $\varphi(\lambda)$:

$$\varphi(\lambda)\text{: } \exists D \text{ such that } L[D] \vDash D \text{ is a normal measure on } \lambda \tag{35.22}$$

The proof of Theorem 86a will be complete if we find a transitive model M and an elementary embedding i: $M \to L[A]$ such that $i(\kappa) = \lambda$. Then $M \vDash \varphi(\kappa)$ and hence κ is measurable in some transitive model.

Let me recall that for each α, $i_{0\alpha}$ is the elementary embedding $i_{0\alpha}$: $L[A] \to Ult_W^{(\alpha)}(L[A])$. As we have seen, if $\alpha < \lambda$, then $i_{0\alpha}(\lambda) = \lambda$ and $i_{0\alpha}(L[A]) = L[A]$. Let C be a proper class of ordinals greater than λ such that $i_{0\lambda}(\nu) = \nu$ for all $\nu \in C$.

Let $H = \mathscr{H}^{L[A]}(\kappa \cup \{\lambda\} \cup C)$ be the class of all sets definable in $L[A]$ from elements of $\kappa \cup \{\lambda\} \cup C$. (As before, the problem of expressibility of H in ZF can be overcome by replacing $L[A]$ by a suitable large segment $L_\theta[A]$.) H is an elementary submodel of $L[A]$.

If $\alpha < \lambda$, then $i_{0\alpha}(\nu) = \nu$ for all $\nu \in \kappa \cup \{\lambda\} \cup C$; it follows that $i_{0\alpha}(x) = x$ for all $x \in H$. Observing that for every ν such that $\kappa \leq \nu < \lambda$ there exists $\alpha < \lambda$ such that $i_{0\alpha}(\nu) > \nu$, we conclude that H contains no ordinal ν such that $\kappa \leq \nu < \lambda$. Hence if π is the transitive collapse of H, and $M = \pi[H]$, then $\pi(\lambda) = \kappa$; thus $i = \pi^{-1}$ is an elementary embedding of some transitive model M into $L[A]$, and $i(\kappa) = \lambda$. ■

The proof of Theorem 86b uses the notion of forcing which collapses all $\alpha < \kappa$ onto ω and makes $\kappa = \aleph_1$ (the Lévy collapsing algebra of Section 20). The proof that if κ is measurable in $\mathfrak{M}$, then in $\mathfrak{M}[G]$ it carries a precipitous ideal uses some methods that we shall be dealing with in the next section. For this reason, and because this section is already quite long, I postpone the proof of Theorem 86b until Section 37.

36. MEASURABLE CARDINALS AND THE GENERALIZED CONTINUUM HYPOTHESIS

The model $L[U]$ that we discussed in Section 31 establishes consistency of the generalized continuum hypothesis with the existence of measurable cardinals. The obvious question is whether we can use the method of forcing to obtain models in which measurable cardinals exist and the continuum function $2^{\aleph_\alpha}$ behaves in some prescribed manner.

We are already aware of one limitation on the behavior of the continuum function if κ is a measurable cardinal: The value of 2^κ depends on the values 2^α for $\alpha < \kappa$; for instance, if $2^\alpha = \alpha^+$ for all $\alpha < \kappa$, then $2^\kappa = \kappa^+$. We shall show below (Theorem 87) that this is not the only restriction on 2^κ. It turns out that in order that $2^\kappa > \kappa^+$, the universe must have very strong large cardinal properties; the theorem shows that if $2^\kappa > \kappa^+$, then for every θ there is a model with θ measurable cardinals.

As one may guess, the behavior of the continuum function above κ (on regular cardinals) is not influenced by measurability of κ; we shall also show that for any $\lambda < \kappa$, the values of 2^α can be arbitrarily prescribed for all regular $\alpha < \lambda$. Then we introduce a method of forcing that will enable us (under a strong large cardinal assumption) to construct models in which the generalized continuum hypothesis fails at a measurable cardinal.

The two main theorems of this section are the following:

Theorem 87 (Kunen). *If there exists a measurable cardinal κ such that $2^\kappa > \kappa^+$, then for every ordinal number θ there exists a transitive model M of* ZFC *with at least θ measurable cardinals.*

Theorem 88 (Silver). *If there exists a supercompact cardinal κ, then there is a generic extension in which κ is a measurable cardinal and $2^\kappa > \kappa^+$.*

The proof of Theorem 87 is very much like the proof of Theorem 80, and I shall closely follow that proof. In the proof of Theorem 80, the key step was provided by Lemma 33.2. The following lemma plays a similar role in the present proof:

Lemma 36.1. *Let κ be a measurable cardinal and let us assume that $2^\kappa > \kappa^+$. Let U be a κ-complete nonprincipal ultrafilter over κ. If X is a set of ordinals of size $< \kappa$, and if δ is an ordinal, $\delta < \kappa^{++}$, then there is a transitive model N of* ZFC *such that $X \in N$, that $W = U \cap N \in N$ and if we denote j_W the natural embedding j_W: $N \to Ult_W(N)$, then*

$$\delta < j_W(\kappa) < \kappa^{++} \tag{36.1}$$

Proof. Let me first state explicitly what is implicit in the statement of the lemma: If $W \in N$, then in N, W is a κ-complete nonprincipal ultrafilter, and one can construct, within N, the ultrapower $Ult_W(N)$.

Let X be a set of ordinals, $|X| < \kappa$, and let $\delta < \kappa^{++}$. Let j_U: $V \to Ult_U(V)$. Since $j_U(\kappa) > 2^\kappa$ and $2^\kappa \geq \kappa^{++}$, there are functions f_ξ: $\kappa \to \kappa$, $\xi \leq \delta$, such that $f_\xi < f_\eta$ almost everywhere (mod U) whenever $\xi < \eta$. Let $Y \subset \kappa^+$ be some set of ordinals such that $f_\xi \in L[Y]$ for all $\xi \leq \delta$. Let $N = L[X, Y, U]$.

Clearly, $W = U \cap N \in N$, and $X \in N$ and $Y \in N$. Thus $f_\xi \in N$ for all $\xi \leq \delta$, and if $\xi < \eta$, then $f_\xi < f_\eta$ almost everywhere (mod W). Thus $j_W(\kappa) > \delta$.

The standard proof shows that $N \models 2^\kappa = \kappa^+$ (see, e.g., Lemma 31.1 and Exercise 15.3). Since $N \models j_W(\kappa) < (2^\kappa)^+$, we have $j_W(\kappa) < \kappa^{++}$. ∎

Proof of Theorem 87 Let me follow closely the proof of Theorem 80. Again, I shall prove only the case $\theta = 2$. Let K be the class of ordinals defined in (33.8), let K_α be as in (33.17) and let $A = \{\gamma_n\}_{n=0}^\infty$ and $A' = \{\gamma'_n\}_{n=0}^\infty$ be the sets defined in (33.18). Let $\lambda = \lim_{n\to\infty} \gamma_n$ and $\lambda' = \lim_{n\to\infty} \gamma'_n$, and let $F = F(A)$, $F' = F(A')$ be the filters over λ and λ' defined in (33.9). We claim that $L[F, F']$ has two measurable cardinals, λ and λ', and that $F \cap L[F, F']$ and $F' \cap L[F, F']$ are normal measures.

As in the proof of Theorem 80, one verifies that $F \cap L[F, F']$ is a normal measure on λ in $L[F, F']$, and additional work is required to show that $F' \cap L[F, F']$ is a normal measure on λ'. For each n, let $M_n \prec L[A, A']$ be the Skolem hull of K_n in $L[A, A']$; cf. (33.19); let π_n be the transitive collapse of M_n and let j_n: $L[\pi_n(A), \pi_n(A')] \to L[A, A']$ be the elementary embedding $j_n = \pi_n^{-1}$.

The next step in the proof is to prove that

$$\pi_n(\gamma_n) < \kappa^{++} \qquad \text{for all} \quad n < \omega \tag{36.2}$$

As in the proof of Lemma 33.4, this is done by induction on n. For $n = 0$, we have $\pi_0(\gamma_0) \leq \kappa$ exactly as in the proof of Lemma 33.4. Thus let us assume that $\pi_n(\gamma_n) < \kappa^{++}$ and let us show that $\pi_{n+1}(\gamma_{n+1}) < \kappa^{++}$. By Lemma 36.1 there exists a transitive model N of ZFC such that $\pi_n(A) \in N$ and $A' \in N$, and $W \in N$ such that if j_W: $N \to Ult_W(N)$, then $\pi_n(\gamma_n) < j_W(\kappa) < \kappa^{++}$. It suffices to show that $\pi_n(\alpha) < j_W(\kappa)$ for all $\alpha < \gamma_{n+1}$ in M_{n+1}, since then $\pi_{n+1}(\gamma_{n+1}) = \sup\{\pi_{n+1}(\alpha): \alpha < \gamma_{n+1} \text{ and } \alpha \in M_{n+1}\} \leq j_W(\kappa) < \kappa^{++}$.

Let $\alpha < \gamma_{n+1}$ be in M_{n+1}. To show that $\pi_n(\alpha) < j_W(\kappa)$, we proceed exactly as in the proof of Lemma 33.4, except that we argue inside the model $Ult_W(N)$ (instead of $Ult_U(V)$ as there); here we use the fact that $Ult_W(N)$ contains both $\pi_n(A)$ and A'.

We complete the proof of the theorem (for $\theta = 2$) as in the proof of Theorem 80. We consider the model $M_\omega \prec L[A, A']$, the Skolem hull of K_ω; $M_\omega = L[B, A']$, where $B = \pi_\omega(A)$, $A' = \pi_\omega(A')$, and

$$j_\omega: L[B, A'] \to L[A, A']$$

where $j_\omega = \pi_\omega^{-1}$ and π_ω is the transitive collapse of M_ω.

We use the fact that $\pi_\omega(\lambda) < \kappa^{++}$ (by 36.2)). Let N be a transitive model of ZFC such that $B \in N$ and $A' \in N$, and let $W \in N$ be such that $\pi_\omega(\lambda) < j_W(\kappa)$, where $j_W: N \to Ult_W(N)$.

In $Ult_W(N)$, B is a subset of $j_W(\kappa)$ and A' is a subset of the class K. Thus we apply (33.16) and get

$$Ult_W(N) \vDash (L[F(B), F(A')] \vDash F(A') \cap L[F(B), F(A')] \text{ is a normal measure})$$

and upon application of j_ω to $L[F(B), F(A')]$, we obtain, as in the proof of Theorem 80,

$$L[F, F'] \vDash F' \cap L[F, F'] \text{ is a normal measure on } \lambda'$$

This completes the proof of Theorem 87 for $\theta = 2$. For the general case, see the concluding remarks in the proof of Theorem 80. ■

Preservation of Measurable Cardinals in Generic Extensions

Our goal now is to construct a model of ZFC in which there is a measurable cardinal κ with the property $2^\kappa > \kappa^+$. By Theorem 87, we have to start with a model of something stronger than just a measurable cardinal, and we also have to make sure that $2^\alpha > \alpha^+$ holds for many cardinals α below κ in the resulting model. We shall proceed gradually, introducing more and more sophisticated methods of getting generic extensions in which a measurable cardinal remains measurable.

As a starting point, we prove:

Lemma 36.2. *Let κ be a measurable cardinal in the ground model $\mathfrak{M}$. Let $(P, <)$ be a notion of forcing such that $|P| < \kappa$. Then κ is measurable in the generic extension.*

Proof. I shall give a proof using elementary embeddings since similar arguments will be used in subsequent constructions; for a direct proof, see Exercise 36.1. Let $B = \text{r.o. } P$; since $|B| < \kappa$, we may as well assume that $B \in V_\kappa$. We can also assume that P is a dense subset of B. Let G be an $\mathfrak{M}$-generic ultrafilter on B; let us work in $\mathfrak{M}[G]$.

Since κ is measurable in $\mathfrak{M}$, there is an elementary embedding $j\colon \mathfrak{M} \to \mathfrak{N}$ (where $\mathfrak{N}$ is the ultrapower of $\mathfrak{M}$ by some U over κ) such that κ is the least ordinal moved. We shall extend j to an elementary embedding (denoted also j) of $\mathfrak{M}[G]$ into $\mathfrak{N}[G]$, thus showing that κ is measurable in $\mathfrak{M}[G]$.

Since $B \in V_\kappa$, we have $j(B) = B$, and B is a complete Boolean algebra in $\mathfrak{N}$. Since $\mathfrak{N} \subseteq \mathfrak{M}$ and G is generic over $\mathfrak{M}$, G is also generic over $\mathfrak{N}$. Let $i_G^{\mathfrak{M}}$ denote the G-interpretation of $\mathfrak{M}^B$, $i_G^{\mathfrak{M}}\colon \mathfrak{M}^B \to \mathfrak{M}[G]$, and let $i_G^{\mathfrak{N}}$ denote the G-interpretation of $\mathfrak{N}^B$, $i_G^{\mathfrak{N}}\colon \mathfrak{N}^B \to \mathfrak{N}[G]$. We define $j(x)$ for $x \in \mathfrak{M}[G]$ as follows: Let $\mathbf{x} \in \mathfrak{M}^B$ be a name for x, $x = i_G^{\mathfrak{M}}(\mathbf{x})$. Let

$$j(x) = i_G^{\mathfrak{N}}(j(\mathbf{x})) \tag{36.3}$$

Since $\mathbf{x} \in \mathfrak{M}^B$, we have $j(\mathbf{x}) \in \mathfrak{N}^B$ and so $i_G^{\mathfrak{N}}(j(\mathbf{x})) \in \mathfrak{N}[G]$. However, we have to show that the definition (36.3) does not depend on which name for x we choose.

Let $\mathbf{y}$ be another B-valued name and let $p \in G$ be such that

$$p \Vdash \mathbf{x} = \mathbf{y} \tag{36.4}$$

That is, (36.4) holds in $\mathfrak{M}$; we apply j to (36.4), and have (in $\mathfrak{N}$)

$$j(p) \Vdash j(\mathbf{x}) = j(\mathbf{y})$$

But $j(p) = p \in G$ and therefore

$$i_G^{\mathfrak{N}}(j\mathbf{x}) = i_G^{\mathfrak{N}}(j\mathbf{y}).$$

Finally we show that $j\colon \mathfrak{M}[G] \to \mathfrak{N}[G]$ is elementary. Let φ be a formula such that

$$\mathfrak{M}[G] \vDash \varphi(x, \ldots)$$

Let $\mathbf{x}, \ldots$ be such that $i_G(\mathbf{x}) = x, \ldots$. There is some $p \in G$ such that

$$p \Vdash \varphi(\mathbf{x}, \ldots) \tag{36.5}$$

Applying j to (36.5), we get (in $\mathfrak{N}$)

$$p \Vdash \varphi(j\mathbf{x}, \ldots)$$

(because $j(p) = p$). Hence

$$\mathfrak{N}[G] \vDash \varphi(j(x), \ldots)$$

and since φ was arbitrary, j is elementary. ■

Exercise 36.1. Let $\mathfrak{M}$, κ, P be as in Lemma 36.2; let U be (in $\mathfrak{M}$) a κ-complete ultrafilter over κ. Then in $\mathfrak{M}[G]$, the filter

$$W = \{X \subseteq \kappa : X \supseteq Y \text{ for some } Y \in U\}$$

is a κ-complete ultrafilter over κ.

[For instance, to show that W is an ultrafilter, consider $X \in \mathfrak{M}^B$ such that $\|X \subseteq \check{\kappa}\| \in G$. The function $\alpha \mapsto \|\check{\alpha} \in X\|$ is a partition of κ into $|B| < \kappa$ pieces, and by κ-completeness of U there is $Y \in U$ such that $\|\check{\alpha} \in X\|$ is the same for all $\alpha \in Y$. Now either $X \in W$ or $\kappa - X \in W$ according to whether this B-value is in G or not.]

It is an immediate consequence of Lemma 36.2 that if κ is a measurable cardinal, $\lambda < \kappa$, and F is a function on regular cardinals below λ such that (i) $F(\alpha) \leq F(\beta)$ if $\alpha \leq \beta$, (ii) $\operatorname{cf}(F(\alpha)) > \alpha$, and (iii) $F(\alpha) < \kappa$ for all α in its domain, then there is a model in which κ is measurable and $2^\alpha = F(\alpha)$ for all $\alpha \in \operatorname{dom}(F)$.

One can also prescribe the values of the continuum function above a measurable cardinal (that is, for regular $\alpha < \kappa$); this follows from this easy observation:

Exercise 36.2. If κ is a measurable cardinal and P is a κ-closed notion of forcing (or just κ-distributive), then κ is measurable in the generic extension.

Exercise 36.3. Let κ be a measurable cardinal. There is a generic extension in which κ is measurable and such that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\aleph_\alpha \geq \kappa$.

[Use Exercise 20.8. The notion of forcing is κ-closed.]

In the following lemma we introduce a method that preserves measurability of κ while adding subsets to an unbounded set of cardinals below κ. It is vital however that the set $A \subseteq \kappa$ has a normal measure 0.

Lemma 36.3. *Let $\mathfrak{M}$ be a transitive model of* ZFC + GCH *and let κ be a measurable cardinal in $\mathfrak{M}$. Let (in $\mathfrak{M}$) D be a normal measure on κ and let A be a set of regular cardinals below κ such that $A \notin D$; let F be a function on A such that $F(\alpha) < \kappa$ for all $\alpha \in A$, and:*

(i) $\operatorname{cf} F(\alpha) > \alpha$;
(ii) $F(\alpha_1) \leq F(\alpha_2)$ *whenever* $\alpha_1 \leq \alpha_2$.

Then there is a generic extension $\mathfrak{M}[G]$ of $\mathfrak{M}$ with the same cardinals, such that κ is measurable in $\mathfrak{M}[G]$, and for every $\alpha \in A$,

$$\mathfrak{M}[G] \vDash 2^\alpha = F(\alpha) \tag{36.6}$$

Proof. If we did not care about measurability of κ in $\mathfrak{M}[G]$, we would of course use the Easton product as in the proof of Theorem 46 (see (20.11)). The actual construction is a modification of that method.

Let $j\colon \mathfrak{M} \to \mathfrak{N}$ be the elementary embedding given by the ultrapower $Ult_D(\mathfrak{M})$. As we assume that $A \notin D$, we have $\kappa \notin j(A)$.

Let $(P, <)$ be the Easton product of P_α, $\alpha \in A$, where each P_α is the notion of forcing that adjoins $F(\alpha)$ subsets of α; cf. (19.10), (20.10), and (20.11). Thus conditions are 0, 1-functions whose domain consists of triples (α, ξ, η) where $\alpha \in A$, $\xi < \alpha$, and $\eta < F(\alpha)$, and such that for every regular cardinal γ,

$$|\{(\alpha, \xi, \eta) \in \operatorname{dom}(p) : \alpha \leq \gamma\}| < \gamma$$

In particular, $|p| < \kappa$ for all $p \in P$, hence $P \subset V_\kappa$ and so $j(p) = p$ for each $p \in P$.

We shall however use not P but $j(P)$ as our notion of forcing. Thus $j(P)$ is, in $\mathfrak{N}$, the Easton product of P_α for $\alpha \in j(A)$. Note that $P \subset j(P)$ and that $j(P)$ is isomorphic to $P \times Q$ where (in $\mathfrak{N}$) $P = (jP)^{<\kappa}$ and $Q = (jP)^{\geq \kappa}$.

Let G be an $\mathfrak{M}$-generic filter on $j(P)$. We claim that $\mathfrak{M}[G]$ has the same cardinals and cofinalities as $\mathfrak{M}$ and satisfies (36.6) and that κ is a measurable cardinal in $\mathfrak{M}[G]$. Let $G_1 = G \cap P$; since $j(P)$ is isomorphic to $P \times Q$, there is an $\mathfrak{M}[G_1]$-generic filter G_2 on Q such that $\mathfrak{M}[G] = \mathfrak{M}[G_1 \times G_2]$.

As we have noted before, $\kappa \notin j(A)$, and so $Q = (jP)^{\geq \kappa}$ is in fact $= (jP)^{>\kappa}$, and hence is, in $\mathfrak{N}$, κ-closed. But since ${}^\kappa\mathfrak{N} \subseteq \mathfrak{N}$, Q is κ-closed in $\mathfrak{M}$. Moreover, we have $|P| = \kappa$ and $|jP| = |j\kappa| = \kappa^+$. Thus for each regular $\lambda \leq \kappa$, we can break $j(P)$ into a product of two notions of forcing, one that satisfies the λ^+-chain condition and one that is λ-closed, and hence all cardinals $\leq \kappa^+$ are preserved. Since $|jP| = \kappa^+$, all cardinals $> \kappa^+$ are also preserved.

We prove (36.6) similarly to Theorem 46: For $\alpha \in A$, we regard $j(P)$ as a product $(jP)^{>\alpha} \times (jP)^{\leq \alpha}$; and since $(jP)^{\leq \alpha} = P^{\leq \alpha}$ and $(jP)^{>\alpha}$ is α-closed, we conclude that $(2^\alpha)^{\mathfrak{M}[G]} = F(\alpha)$.

Thus it remains to show that κ is a measurable cardinal in $\mathfrak{M}[G]$. We shall first extend $j\colon \mathfrak{M} \to \mathfrak{N}$ to an elementary embedding

$$j\colon \mathfrak{M}[G_1] \to \mathfrak{N}[G] \tag{36.7}$$

Let $B = \text{r.o. } P$ and let $j(B) = (\text{r.o.}(jP))^{\mathfrak{N}}$; let $i^{\mathfrak{M}}_{G_1}$ denote the G_1-interpretation of $\mathfrak{M}^B$ and let $i^{\mathfrak{N}}_G$ denote the G-interpretation of $\mathfrak{N}^{j(B)}$. We define $j(x)$ for $x \in \mathfrak{M}[G_1]$ as follows: Let $\mathbf{x} \in \mathfrak{M}^B$ be such that $x = i^{\mathfrak{M}}_{G_1}(\mathbf{x})$. Let

$$j(x) = i^{\mathfrak{N}}_G(j(\mathbf{x})) \tag{36.8}$$

As in the proof of Lemma 36.7, we show that (36.8) does not depend on the choice of $\mathbf{x}$. Since $j(p) = p$ for all $p \in P$ and because $G_1 \subset G$, it follows that if some $p \in G_1$ forces $\mathbf{x} = \mathbf{y}$, then (in $\mathfrak{N}$) $p \Vdash j\mathbf{x} = j\mathbf{y}$ and therefore $i^{\mathfrak{N}}_G(\mathbf{x}) = i^{\mathfrak{N}}_G(\mathbf{y})$. The same reasoning shows that $j\colon \mathfrak{M}[G_1] \to \mathfrak{N}[G]$ is elementary.

Using (36.7), we define an $\mathfrak{M}[G_1]$-ultrafilter U over κ as follows:

$$X \in U \quad \text{iff} \quad \kappa \in j(X)$$

for all $X \subseteq \kappa$ in $\mathfrak{M}[G_1]$. A standard argument shows that U is $\mathfrak{M}[G_1]$-κ-complete; and since j extends the original $j = j_D$, U is nonprincipal.

Now we use again the fact that $j(P)$ is isomorphic to $P \times Q$, where $|P| = \kappa$ and Q is κ-closed. Thus every subset of κ is in $\mathfrak{M}[G_1]$ and therefore U is in $\mathfrak{M}[G]$ a κ-complete nonprincipal ultrafilter over κ. ■

The method of Lemma 36.3 can be used to prove consistency of the GCH with a measurable cardinal by forcing.

*Exercise 36.4.** Let κ be a measurable cardinal. There is a generic extension in which κ is measurable and which satisfies the GCH.

[First get the GCH above κ by Exercise 36.3. Then use $j(P)$, where P is the Easton product from Exercise 20.8 making the GCH true below κ.]

Inverse Limits of Sequences of Complete Boolean Algebras

In Section 23 we defined direct limits of sequences of complete Boolean algebras. We shall now define a notion that subsumes the notion of a direct limit and will enable us to perform a new kind of iteration of forcing.

Let α be a limit ordinal number and let for each $i < \alpha$, B_i be a complete Boolean algebra such that if $i < j$, then B_i is a complete Boolean subalgebra of B_j. We defined the *direct limit*

$$\lim_{i \to \alpha} \operatorname{dir} B_i$$

as the completion of the Boolean algebra $\bigcup_{i<\alpha} B_i$. More generally, let B_i, $i < \alpha$, be complete Boolean algebras and let $\{e_{ij} : i \le j < \alpha\}$ be a commutative system of complete embeddings, e_{ij}: $B_i \to B_j$. Then

$$\lim_{i \to \alpha} \operatorname{dir} \{B_i, e_{ij}\}_{i,\, j<\alpha}$$

is the unique complete Boolean algebra such that there exist complete embeddings $e_{i\alpha}$: $B_i \to B$ commuting with the e_{ij} and that $\bigcup_{i<\alpha} e_{i\alpha}[B_i]$ is dense in B.

Let me recall the definition (23.21) of the *canonical projection*: If B is a complete Boolean subalgebra of D, then for $d \in D$,

$$h(d) = \prod\{b \in B : d \le b\}$$

Let us generalize this to the case when e: $B \to D$ is a fixed complete embedding of B into D; then we let

$$h(d) = \prod\{b \in B : d \le e(b)\} \tag{36.9}$$

Now let B_i, $i < \alpha$, be complete Boolean algebras such that B_i is a complete subalgebra of B_j whenever $i \le j < \alpha$. Let h_{ij} be, for any $i \le j$, the canonical projection of B_j onto B_i. Let $\mathscr{F}$ be a collection of functions $\langle b_i : i < \alpha\rangle \in \prod_{i<\alpha} (B_i - \{0\})$ such that

(36.10) (i) $b_i = h_{ij}(b_j) \qquad (i \le j)$;

(ii) $\mathscr{F}$ contains all eventually constant functions c_b for all nonzero $b \in \bigcup_{i<\alpha} B_i$, where

$$c_b = \langle h_{0i}(b), h_{1i}(b), \ldots, b, b, b, \ldots\rangle \qquad (b \in B_i) \tag{36.11}$$

We partially order $\mathscr{F}$ componentwise, i.e., $\langle b_i : i < \alpha\rangle \le \langle c_i : i < \alpha\rangle$ if $b_i \le c_i$ for all i. Let $B = \text{r.o.}(\mathscr{F},)$, we call B an *inverse limit* of B_i, $i < \alpha$,

$$B = \lim_{i \to \alpha} \operatorname{inv}^{(\mathscr{F})} B_i \tag{36.12}$$

[It should be clear how to define

$$\lim_{i\to\alpha} \operatorname{inv}^{(\mathscr{F})}\{B_i, e_{ij}\}_{i,\, j<\alpha}$$

in the general case when the embeddings e_{ij} are not just identity functions.]

For each $i < \alpha$, there is a complete embedding $e_{i\alpha}$ of B_i into the inverse limit (36.12), namely

$$e_{i\alpha}(b) = c_b \tag{36.13}$$

It is easy to verify that (36.13) is indeed a complete embedding of B_i in B. Moreover, if $i \leq j$, then $e_{i\alpha}$ and $e_{j\alpha}$ agree on B_i (in the general case, $e_{i\alpha}$ commutes with the e_{ij}'s). Thus we can identify each $b \in C = \bigcup_{i<\alpha} B_i$ with $c_b \in B$; then each B_i is a complete Boolean subalgebra of B (and so C is a subalgebra of B, but the completion of C may not be embeddable in B).

If $\mathscr{F}$ is the collection of all eventually constant functions, then (36.12) is the direct limit of B_i, $i < \alpha$; otherwise, various other choices of $\mathscr{F}$ may give various inverse limits of the sequence B_i, $i < \alpha$.

Let us again assume that B_i, $i < \alpha$, are complete Boolean algebras and that B_i is a complete subalgebra of B_j whenever $i \leq j < \alpha$; and let for each $i < \alpha$, P_i be a dense subset of B_i such that for any $i \leq j$,

(36.14) (i) $P_i \subseteq P_j$;
(ii) if $p \in P_j$, then $h_{ij}(p) \in P_i$.

Let P be the set of all functions $\langle p_i : i < \alpha \rangle$ such that

$$p_i = h_{ij}(p_j) \tag{36.15}$$

for all $i \leq j$. We partially order P componentwise, and call it the *inverse limit* of P_i, $i < \alpha$:

$$P = \lim_{i\to\alpha} \operatorname{inv} P_i$$

(and r.o. P is an inverse limit of B_i, $i < \alpha$).

Iterated Forcing

Iteration of generic extensions was introduced in Section 73: We defined two-step iterations and introduced direct limits of sequences of complete Boolean algebras. We shall now generalize the technique by considering other than direct limits at limit steps of the iteration.

Recall the two-step iteration. (To simplify matters, let us consider only notions of forcing that are separative and have a greatest element 1.) Let P be a notion of forcing and let $B = \text{r.o. } P$. In $\mathfrak{M}^B$, let $\dot{Q}$ be a notion of forcing, and let $\hat{Q}$ be a set of representatives of the class $\{x \in \mathfrak{M}^B : \|x \in \dot{Q}\| = 1\}$ in the equivalence relation $\|x = y\| = 1$. Let us denote

$$P * \dot{Q} \tag{36.16}$$

the set of all pairs $(p, q) \in P \times \hat{Q}$, partially ordered by

$$(36.17) \qquad (p_1, q_1) \leq (p_2, q_2) \quad \text{iff} \quad p_1 \leq p_2 \text{ and } p_1 \Vdash q_1 \leq q_2$$

(In $P * Q$, we identify any two pairs (p, q_1), (p, q_2) such that $p \Vdash q_1 = q_2$.) As we have shown in Section 23, the algebra B is embedded in the algebra $D = \text{r.o.}(P * Q)$ by the embedding $e\colon B \to D$ defined in (23.7). It is obvious that for each $p \in P$,

$$(36.18) \qquad e(p) = (p, 1)$$

(where 1 denotes the greatest element of Q). Similarly, if h denotes the canonical projection of D onto B, then

$$(36.19) \qquad h(p, q) = p$$

We shall now define the general notion of iterated forcing. Let us assume that in each Boolean-valued universe $\mathfrak{M}^B$ we are given a fixed B-valued notion of forcing $Q \in \mathfrak{M}^B$. By *iterated forcing*, we understand a transfinite sequence of forcing notions P_α, together with a commutative system of complete embeddings $e_{\alpha\beta}\colon B_\alpha \to B_\beta$ where $B_\alpha = \text{r.o.}(P_\alpha)$ for each α such that:

(36.20) (i) $P_{\alpha+1} = P_\alpha * Q_\alpha$, where Q_α is the given notion of forcing in $\mathfrak{M}^{B_\alpha}$, and $e_{\alpha,\alpha+1}$ is the embedding satisfying (36.18);

(ii) if α is a limit ordinal, then P_α is either the direct limit $\lim \text{dir}_{i \to \alpha} \{P_i; e_{ij}\}$ or the inverse limit $\lim \text{inv}_{i \to \alpha} \{P_i; e_{ij}\}$ (and $e_{i\alpha}$ are the corresponding embeddings).

There is a more intuitive representation of iterated forcing. The verification of the following facts may not be immediate but is routine:

Elements of each P_α can be identified with functions $p = \langle p_i : i < \alpha \rangle$ such that:

(36.21) (i) for every $\gamma < \alpha$, the restriction $p|\gamma = \langle p_i : i < \gamma \rangle$ belongs to P_γ;

(ii) for every $\gamma < \alpha$, if Q_γ denotes the given B_γ-valued notion of forcing, then $\|p_\gamma \in Q_\gamma\|_{B_\gamma} = 1$;

(iii) the partial ordering $\leq_\alpha$ of P_α is defined as follows: $p \leq_\alpha q$ iff for every $\gamma < \alpha$,

$$p|\gamma \leq_\gamma q|\gamma \quad \text{and} \quad p|\gamma \Vdash_\gamma p_\gamma \leq_{Q_\gamma} q_\gamma$$

where $\Vdash_\gamma$ denotes the forcing relation associated with $(P_\gamma, \leq_\gamma)$.

If α is a successor ordinal, then P_α consists of all functions that satisfy (36.21).

If α is a limit ordinal and P_α is the inverse limit, then P_α consists of all functions that satisfy (36.21).

If α is a limit ordinal and P_α is the direct limit, then P_α consists of all functions $p = \langle p_i : i < \alpha \rangle$ that satisfy (36.21) and such that there is some $\gamma < \alpha$ such that $p_i = 1$ for all $i \geq \gamma$.

The embeddings $e_{\alpha\beta}\colon B_\alpha \to B_\beta$ satisfy

$$e_{\alpha\beta}(p) = p^\frown 1^\frown 1^\frown 1^\frown \cdots \qquad (p \in P_\alpha) \tag{36.22}$$

and similarly, the canonical projections $h_{\alpha\beta}$ satisfy

$$h_{\alpha\beta}(p) = p \,|\, \alpha \qquad (p \in P_\beta) \tag{36.23}$$

[Instead of the verification of (36.21)–(36.23), the reader can take them for the definition of iterated forcing, and verify that $\{e_{\alpha\beta} : \alpha \le \beta\}$ is a commutative system of complete embeddings, that each $h_{\alpha\beta}$ is the canonical projection, that each $P_{\alpha+1}$ is isomorphic to $P_\alpha * \boldsymbol{Q}_\alpha$, and that for a limit α, P_α is isomorphic either to $\lim \mathrm{dir}_{i\to\alpha} \{P_i;\, e_{ij}\}$ or $\lim \mathrm{inv}_{i\to\alpha} \{P_i,\, e_{ij}\}$.]

We proved in Section 23 that the two-step iteration preserves both the chain condition and the closedness of forcing (cf. Lemma 23.6 and Exercise 23.5). Also, if the only limits are direct limits, then iteration of forcing preserves chain conditions (cf. Lemma 23.7; see also Exercise 23.8). We shall now generalize these results.

A notion of forcing $(P, <)$ is *strongly λ-closed* if whenever $D \subset P$ (D for *directed*) is such that $|D| \le \lambda$ and for any $d_1, d_2 \in D$ there is $e \in D$ such that $e \le d_1$ and $e \le d_2$, then there is $p \in P$ such that $p \le d$ for all $d \in D$.

Lemma 36.4.

(a) *If P is strongly λ-closed, and if $\|\boldsymbol{Q}$ is strongly $\check{\lambda}$-closed$\|_{\mathrm{r.o.}P}$, then $P * \boldsymbol{Q}$ is strongly λ-closed.*

(b) *If $P = \lim \mathrm{dir}_{i\to\alpha} P_i$, if each P_i is strongly λ-closed, and if* $\operatorname{cf} \alpha > \lambda$, *then P is strongly λ-closed.*

(c) *Let us consider iterated forcing; for each α, let P_α, B_α, $\boldsymbol{Q}_\alpha$ be as above. Let λ be a cardinal and assume that for each α, $\boldsymbol{Q}_\alpha$ is a strongly $\check{\lambda}$-closed notion of forcing in $\mathfrak{M}^{B_\alpha}$. Let us further assume that if α is a limit ordinal and* $\operatorname{cf}\alpha \le \lambda$, *then P_α is the inverse limit of P_β, $\beta < \alpha$. Then each P_α is strongly λ-closed.*

Proof. (a) Let $D = \{(p_\alpha \boldsymbol{q}_\alpha) : \alpha < \lambda\}$ be a directed subset of $P * \boldsymbol{Q}$. Clearly, $D_1 = \{p_\alpha : \alpha < \lambda\}$ is a directed subset of P and hence there is $p \in P$ stronger than all p_α, $\alpha < \lambda$. Since for any $\alpha, \beta < \lambda$ there is $\gamma < \lambda$ such that $(p_\gamma \boldsymbol{q}_\gamma) \le (p_\alpha \boldsymbol{q}_\alpha)$ and $(p_\gamma \boldsymbol{q}_\gamma) \le (p_\beta \boldsymbol{q}_\beta)$, it is clear that $p \Vdash (\boldsymbol{q}_\gamma \le \boldsymbol{q}_\alpha$ and $\boldsymbol{q}_\gamma \le \boldsymbol{q}_\beta)$ and thus p forces that $\{\boldsymbol{q}_\alpha : \alpha < \lambda\}$ is a directed subset of $\boldsymbol{Q}$. Hence

$$p \Vdash \exists q \in \boldsymbol{Q} \text{ stronger than all the } \boldsymbol{q}_\alpha$$

and therefore there is $\boldsymbol{q}$ such that $\|\boldsymbol{q} \in \boldsymbol{Q}\| = 1$ and

$$p \Vdash \boldsymbol{q} \le \boldsymbol{q}_\alpha \text{ for all } \alpha < \check{\lambda}$$

It follows that $(p\boldsymbol{q}) \le (p_\alpha \boldsymbol{q}_\alpha)$ for all $\alpha < \lambda$.

(b) Let D be a directed subset of P, $|D| \le \lambda$. For each $d \in D$ there is $\gamma_d < \alpha$ such that if $d = \langle p_i : i < \alpha\rangle$, then $p_i = 1$ for all $i \ge \gamma_d$. Since $\lambda < \operatorname{cf}\alpha$, there is $\gamma < \alpha$ such that each $d \in D$ is as follows:

$$d = (d\,|\,\gamma)^\frown 1^\frown 1^\frown 1^\frown \cdots$$

Now $D_\gamma = \{(d \mid \gamma) : d \in D\}$ is a directed subset of P_γ; and since P_γ is strongly λ-closed, D_γ has a lower bound $p \in P_\gamma$. Then

$$p^\frown 1^\frown 1^\frown 1^\frown \cdots$$

is a lower bound for D in P.

(c) By induction on α. It follows from (a) and (b) that the assertion is true if α is a successor or if P_α is the direct limit. Thus assume that $P_\alpha = \lim \text{inv}_{i \to \alpha} P_i$.

Let $D = \langle p^\nu : \nu < \lambda\}$ be a directed subset of P_α; for each ν let $p^\nu = \langle p_i^\nu : i < \alpha\rangle$. We shall construct, by induction on $i < \alpha$, a function $p = \langle p_i : i < \alpha\rangle \in P_\alpha$ stronger than all p^ν, $\nu < \lambda$.

We construct p such that for each $i < \alpha$, $\langle p_j : j < i\rangle$ is in P_i and is (in P_i) stronger than all $p^\nu \mid i$, $\nu < \lambda$. Having constructed $\langle p_j : j < i\rangle$, we let p_i be such that

$$\langle p_j : j < i\rangle \Vdash p_i \le p_i^\nu \text{ for all } \nu < \lambda$$

Moreover, if $p_i^\nu = 1$ for all $\nu < \lambda$, we let $p_i = 1$ too.

If $\gamma \le \alpha$ is a limit ordinal, we have to show that $\langle p_i : i < \gamma\rangle \in P_\gamma$. If P_γ is the inverse limit, then there is nothing to prove, so let us assume that P_γ is the direct limit. By the assumption, we have $\operatorname{cf} \gamma > \lambda$ and therefore there is $\delta < \gamma$ such that for all $\nu < \lambda$, $p_i^\nu = 1$ for all i such that $\delta \le i < \gamma$. Hence we have $p_i = 1$ for all i, such that $\delta \le i < \gamma$, and so $\langle p_i : i < \lambda\rangle \in P_\gamma$. Thus we have $p = \langle p_i : i < \alpha\rangle \in P_\alpha$, and it is clear from the construction that $p \le p^\nu$ for all $\nu < \lambda$. ∎

Iterated forcing does not generally preserve chain conditions, but there is one important instance when it does (see Exercise 23.8):

Lemma 36.5. *Let κ be a regular uncountable cardinal and let P be the direct limit of P_α, $\alpha < \kappa$. Assume that there is a stationary set $S \subseteq \kappa$ such that for all $\alpha \in S$, P_α is the direct limit of P_i, $i < \alpha$. If each P_α satisfies the κ-chain condition, then P also satisfies the κ-chain condition.*

Proof. The same proof as in Lemma 23.7. ∎

The most important property of iterated forcing is the following factor lemma. Although what the lemma states is very simple, it is somewhat complicated to formulate.

Let us consider an iteration of forcing. For each α, let P_α be the αth iterate, let $B_\alpha = \text{r.o.}\ P_\alpha$, and let $Q_\alpha \in \mathfrak{M}^{B_\alpha}$ be the B_α-valued notion of forcing that gives the $(\alpha + 1)$th step of the iteration.

Let us fix an ordinal number α, and let us do the following in the Boolean-valued model $\mathfrak{M}^{B_\alpha}$:

Let us consider the following iteration of forcing:

(36.24) (i) $P_0^{(\alpha)} = \{1\}$;

(ii) $P_{\beta+1}^{(\alpha)} = P_\beta^{(\alpha)} * Q_{\alpha+\beta}$;

(iii) if β is a limit ordinal, then $P_\beta^{(\alpha)}$ is either the direct limit or the inverse limit of $P_i^{(\alpha)}$, $i < \beta$, according to whether in $\mathfrak{M}$, $P_{\alpha+\beta}$ is the direct or inverse limit.

Clause (ii) needs an explanation. $\boldsymbol{Q}_{\alpha+\beta}$ is an element of the Boolean-valued model $\mathfrak{M}^{B_{\alpha+\beta}}$. Since B_α is embedded in $B_{\alpha+\beta}$, the model $\mathfrak{M}^{B_{\alpha+\beta}}$ can be considered a Boolean-valued extension of $\mathfrak{M}^{B_\alpha}$ (by the algebra $\boldsymbol{B}_\beta^{(\alpha)} = B_{\alpha+\beta}\colon B_\alpha$) and so $\boldsymbol{Q}_{\alpha+\beta}$ can be considered a $\boldsymbol{B}_\beta^{(\alpha)}$-valued object in $\mathfrak{M}^{B_\alpha}$. Since the factor lemma shows (by induction) that $P_{\alpha+\beta}$ is isomorphic to $P_\alpha * \boldsymbol{P}_\beta^{(\alpha)}$, it follows that in $\mathfrak{M}^{B_\alpha}$, $\boldsymbol{B}_\beta^{(\alpha)} = \text{r.o.}(\boldsymbol{P}_\beta^{(\alpha)})$, and hence the definition (ii) is legitimate.

Lemma 36.6. (Factor Lemma). *Let us consider iterated forcing and let the notation be as above. Let α be an ordinal and let $\boldsymbol{P}_\beta^{(\alpha)}$ be as in* (36.24). *Let us assume that for any limit ordinal β, if* cf $\beta \le |P_\alpha|$, *then $P_{\alpha+\beta}$ is the inverse limit. Then for each β,*

$$P_{\alpha+\beta} \qquad \textit{is isomorphic to} \qquad P_\alpha * \boldsymbol{P}_\beta^{(\alpha)}$$

Proof. By induction on β. Let β be an ordinal number; we shall construct an isomorphism π between $P_\alpha * \boldsymbol{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$.

If $\beta = 0$, then $P_\alpha * \boldsymbol{P}_\alpha^{(\alpha)} = \{(p, 1) : p \in P_\alpha\}$ and we let $\pi(p, 1) = p$. Thus let $\beta > 0$. A typical element of $P_\alpha * \boldsymbol{P}_\beta^{(\alpha)}$ is a pair $(p, \boldsymbol{q})$ where $p \in P_\alpha$ and $\boldsymbol{q}$ is an element of $\mathfrak{M}^{B_\alpha}$ such that in $\mathfrak{M}^{B_\alpha}$, $\boldsymbol{q}$ is a $\check\beta$-sequence and satisfies the conditions on iterated forcing; in particular, for each $i < \check\beta$, $\boldsymbol{q}|i$ is in $\boldsymbol{P}_i^{(\alpha)}$ and the ith term of $\boldsymbol{q}$ is in $\boldsymbol{Q}_{\alpha+i}$.

We shall define a β-sequence $\langle p_{\alpha+i} : i < \beta\rangle$ and let $\pi(p, \boldsymbol{q}) = p \frown \langle p_\alpha, p_{\alpha+1}, \dots, p_{\alpha+i}, \dots\rangle$. This mapping π will be an isomorphism between $P_\alpha * \boldsymbol{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$. For $i < \beta$, let $\boldsymbol{q}_i \in \mathfrak{M}^{B_\alpha}$ be such that with B_α-value 1, $\boldsymbol{q}_i$ is the ith term of $\boldsymbol{q}$. Hence (recall the remark following the definition (36.24) of $\boldsymbol{P}_\beta^{(\alpha)}$)

$$\mathfrak{M}^{B_\alpha} \models (\boldsymbol{q}_i \text{ is } \boldsymbol{B}_i^{(\alpha)}\text{-valued and } \|\boldsymbol{q}_i \in \boldsymbol{Q}_{\alpha+i}\|_{\boldsymbol{B}_i^{(\alpha)}} = 1)$$

By the induction hypothesis, $B_{\alpha+i}$ is isomorphic to $B_\alpha * \boldsymbol{B}_i^{(\alpha)}$, and so $\mathfrak{M}^{B_{\alpha+i}}$ is isomorphic to $(\mathfrak{M}^{B_\alpha})^{\boldsymbol{B}_i^{(\alpha)}}$. Let $p_{\alpha+i} \in \mathfrak{M}^{B_{\alpha+i}}$ be the element corresponding to $\boldsymbol{q}_i$ under the isomorphism between $(\mathfrak{M}^{B_\alpha})^{\boldsymbol{B}_i^{(\alpha)}}$ and $\mathfrak{M}^{B_{\alpha+i}}$. Clearly, $\|p_{\alpha+i} \in \boldsymbol{Q}_{\alpha+i}\|_{B_{\alpha+i}} = 1$.

Let $\pi(p, \boldsymbol{q}) = p \frown \langle p_\alpha, p_{\alpha+1}, \dots, p_{\alpha+i}, \dots\rangle$. All we have to do now is to show that π is an isomorphism between $P_\alpha * \boldsymbol{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$. We shall show that for each $(p, \boldsymbol{q}) \in P_\alpha * \boldsymbol{P}_\beta^{(\alpha)}$, $\pi(p, \boldsymbol{q})$ is in $P_{\alpha+\beta}$ and leave the rest to the reader, namely to show that

$$(p\boldsymbol{q}) \le (p'\boldsymbol{q}') \qquad \text{iff} \qquad \pi(p\boldsymbol{q}) \le \pi(p'\boldsymbol{q}')$$

We want to show that for each $\gamma \le \beta$, $p \frown \langle p_{\alpha+i} : i < \gamma\rangle$ is an element of $P_{\alpha+\gamma}$. Since $\|p_{\alpha+i} \in \boldsymbol{Q}_{\alpha+i}\|_{B_{\alpha+i}} = 1$ for each $i < \beta$, we need only show that if γ is a limit ordinal and $P_{\alpha+\gamma}$ is the direct limit, then there exists $i_0 < \gamma$ such that $p_{\alpha+i} = 1$ for all i, $i_0 \le i < \gamma$.

Thus let $\gamma \le \beta$ be a limit ordinal such that $P_{\alpha+\gamma}$ is the direct limit of $P_{\alpha+i}$, $i < \gamma$. Hence in $\mathfrak{M}^{B_\alpha}$, $\boldsymbol{P}_\gamma^{(\alpha)}$ is the direct limit, and therefore

$$(36.25) \qquad \|(\exists i_0 < \check\gamma)(\forall i \ge i_0) \text{ the } i\text{th term of } \boldsymbol{q} \text{ is } 1\|_{B_\alpha} = 1$$

Now we have made an assumption that if $P_{\alpha+\gamma}$ is the direct limit, then $\mathrm{cf}\,\gamma > |P_\alpha|$. It is easy to see that because $|P_\alpha| < \mathrm{cf}\,\gamma$, (36.25) implies that there exists $i_0 < \gamma$ such that for all $i \geq i_0$

$$\|q_i = 1\|_{B_\alpha} = 1$$

Thus for all $i \geq i_0$,

$$\|p_{\alpha+i} = 1\|_{B_{\alpha+i}} = 1$$

and hence $p_{\alpha+i} = 1$ for all $i \geq i_0$. ■

Proof of Theorem 88 Let $\mathfrak{M}$ be a transitive model of ZFC and assume that κ is a supercompact cardinal in $\mathfrak{M}$ and that $2^\kappa = \kappa^+$ (otherwise there is nothing to prove). We shall construct a generic extension $\mathfrak{M}[G]$ of $\mathfrak{M}$ in which κ is measurable and $2^\kappa = \kappa^{++}$.

We use iterated forcing, and successively adjoin to each inaccessible cardinal $\alpha \leq \kappa$, α^{++} subsets of α. At limit steps of the iteration we use direct limits when the ordinal is a regular cardinal and inverse limits otherwise.

Let us define, by induction on α, the αth iterate P_α (and the corresponding forcing relation $\Vdash_\alpha$ and the algebra $B_\alpha = \text{r.o.}(P_\alpha)$) and the B_α-valued notion of forcing Q_α:

(36.26) (a) If α is an inaccessible cardinal, let Q_α be the notion of forcing in $\mathfrak{M}^{B_\alpha}$ that adjoins α^{++} subsets of α; that is, we let Q_α be, in $\mathfrak{M}^{B_\alpha}$, the set of all functions p whose domain is a subset of size $<\alpha$ of $\alpha \times \alpha^{++}$ (and Q_α is ordered by $\supseteq$). Further, we let $\hat{Q}_\alpha$ be a set of representatives for

$$\{x \in \mathfrak{M}^{B_\alpha} : \|x \in Q_\alpha\| = 1\}$$

in the equivalence relation $\|x = y\| = 1$.

If α is not an inaccessible cardinal, let $\hat{Q}_\alpha = \{1\}$ (as usual, 1 denotes the greatest element of each notion of forcing).

(b) P_α is the set of all α-sequences $\langle p_i : i < \alpha\rangle$ satisfying the following:

(1) for every $\gamma < \alpha$, $\langle p_i : i < \gamma\rangle \in P_\gamma$ and $p_\gamma \in \hat{Q}_\gamma$;
(2) if α is a regular cardinal, then $\exists i_0 \forall i \geq i_0\, p_i = 1$.

(c) If $p, q \in P_\alpha$, then $p \leq_\alpha q$ iff

$$(\forall\gamma < \alpha)\,[p|\gamma \leq_\gamma q|\gamma \text{ and } p|\gamma \Vdash_\gamma p_\gamma \text{ is stronger than } q_\gamma].$$

Finally, let $P = P_{\kappa+1}$, and let $B = \text{r.o. } P$.

Let G be a generic filter on P and let $\mathfrak{M}[G]$ be the generic extension of $\mathfrak{M}$ by G. We shall prove that κ is a measurable cardinal in $\mathfrak{M}[G]$ and that $\mathfrak{M}[G] \vDash 2^\kappa = \kappa^{++}$. To begin, P is isomorphic to the two-step iteration $P_\kappa * Q_\kappa$. Thus $\mathfrak{M}[G] = \mathfrak{M}[G_\kappa][H_\kappa]$, where G_κ is $\mathfrak{M}$-generic on P_κ and H_κ is $\mathfrak{M}[G_\kappa]$-generic on $Q_\kappa = i_{G_\kappa}(Q_\kappa)$. Now P_κ is the direct limit of P_α, $\alpha < \kappa$; and since κ is a Mahlo cardinal, there is a stationary set of $\alpha < \kappa$ such that P_α is also a direct limit. Since clearly $|P_\alpha| < \kappa$ for all $\alpha < \kappa$, it follows by Lemma 36.5 that P_κ satisfies the κ-chain condition and hence κ is a regular cardinal in $\mathfrak{M}[G_\kappa]$. Also, $|P_\kappa| = \kappa$, and hence $\mathfrak{M}[G_\kappa]$ satisfies $(\forall\alpha < \kappa)2^\alpha \leq \kappa$. In $\mathfrak{M}[G_\kappa]$, Q_κ is the standard notion of forcing that adjoins κ^{++} subsets of κ and preserves all cardinals. Thus $\mathfrak{M}[G] \vDash (\kappa$ is a regular cardinal and $2^\kappa = \kappa^{++})$.

It remains to prove that κ is a measurable cardinal in $\mathfrak{M}[G]$. The basic idea is as in Lemma 36.2 or Lemma 36.3: to find an elementary embedding of $\mathfrak{M}[G]$; the actual construction is somewhat more complicated.

Let $\lambda = \kappa^{++}$. Since κ is supercompact in $\mathfrak{M}$, there is an elementary embedding $j\colon \mathfrak{M} \to \mathfrak{N}$ such that κ is the least ordinal moved, $j(\kappa) > \lambda$ and

(36.27) $$^{\lambda}\mathfrak{N} \cap \mathfrak{M} \subseteq \mathfrak{N}$$

It is easy to verify that $|P| = \lambda$, that (by (36.27)) $P \in \mathfrak{N}$, and moreover, P is defined in $\mathfrak{N}$ by the same definition (36.26) as in $\mathfrak{M}$. Also, $(\text{r.o. } P)^{\mathfrak{N}} = B$.

Since $P \in \mathfrak{N}$ and $\mathfrak{N} \subseteq \mathfrak{M}$, G is also an $\mathfrak{N}$-generic filter on P, and we can consider the model $\mathfrak{N}[G]$. Let me prove now the following formula which will be used toward the end of the proof:

(36.28) $$^{\lambda}\mathfrak{N}[G] \cap \mathfrak{M}[G] \subseteq \mathfrak{N}[G]$$

It suffices to show that if $f \in \mathfrak{M}[G]$ is a function from λ into ordinals, then $f \in \mathfrak{N}[G]$. Let $\dot{f}$ be a name for f and let $p_0 \in G$ be a condition that forces that $\dot{f}$ is a function from $\check{\lambda}$ into the ordinals. For each $\alpha < \lambda$, let

$$A_\alpha = \{p \leq p_0 : \exists\beta\ p \Vdash \dot{f}(\check{\alpha}) = \check{\beta}\}$$

Each A_α is dense below p_0 (and hence $A_\alpha \cap G \neq \emptyset$). For each $\alpha < \lambda$ and each $p \in A_\alpha$, let us denote by $g(\alpha, p)$ the unique β such that $p \Vdash \dot{f}(\check{\alpha}) = \check{\beta}$. Since $|P| = \lambda$, we have $|g| = \lambda$ and hence $g \in \mathfrak{N}$. Now it is easy to see that $f \in \mathfrak{N}[G]$ because it is defined in $\mathfrak{N}[G]$ as follows: $f(\alpha) =$ the unique β such that for some $p \in G$, $g(\alpha, p) = \beta$. Q.E.D. (36.28).

Let us now consider $j(P)$. In $\mathfrak{N}$, jP is a notion of forcing obtained by iteration (i.e., via (36.26)) up to $j(\kappa) + 1$. I claim that in $\mathfrak{N}$ we can apply the factor lemma to jP at $\alpha = \kappa + 1$. Indeed, the first nontrivial step above α in the iteration occurs at the least inaccessible cardinal (in $\mathfrak{N}$) above κ, thus the first nontrivial direct limit is taken far above λ and then only at regular cardinals. Since $|P_{\kappa+1}| = \lambda$, the assumption of Lemma 36.6 is satisfied.

Hence jP is isomorphic to a two-step iteration (in $\mathfrak{N}$)

(36.29) $$(jP)_{\kappa+1} * (jP)^{(\kappa+1)}_{j\kappa+1}$$

Now the first factor of (36.29) is equal to $P_{\kappa+1} = P$. Let us denote Q the second factor of (36.29). By the factor lemma, Q is, in $\mathfrak{N}^B$, a notion of forcing obtained by iteration (36.26) from $\kappa + 1$ to $j\kappa + 1$. At each $i > \kappa$, the iteration uses a notion of forcing in $\mathfrak{N}^{B_i}$, that is either trivial or adjoins i^{++} subsets of i (if i is inaccessible in $\mathfrak{N}$); in either case, the notion of forcing is strongly λ-closed in $\mathfrak{N}^{B_i}$. By Lemma 36.4c, $\mathbf{Q}$ is strongly $\check{\lambda}$-closed in $\mathfrak{N}^B$. Thus we can write

$$jP = P * \mathbf{Q} \tag{36.30}$$

where $\mathbf{Q} \in \mathfrak{N}^B$ is a strongly $\check{\lambda}$-closed notion of forcing.

Let $i^{\mathfrak{N}}_G$ be the G-interpretation of $\mathfrak{N}^B$, and let $Q = i^{\mathfrak{N}}_G(\mathbf{Q})$. Thus Q is a strongly λ-closed notion of forcing in $\mathfrak{N}[G]$.

Let $p \in P$. Then by (36.30), jp is (represented by) a pair $(s, \mathbf{q})$ where $s \in P$ and $\mathbf{q} \in \mathfrak{N}^B$ is such that $\|\mathbf{q} \in \mathbf{Q}\| = 1$. By the definition of P, $p = \langle p_i : i < \kappa + 1\rangle$ and there is $i_0 < \kappa$ such that $p_i = 1$ for all i, $i_0 \leq i < \kappa$. Thus $j(p) = \langle p'_i : i < j(\kappa) + 1\rangle$ and $p'_i = 1$ for all i, $i_0 \leq i < j(\kappa)$. In particular, $p'_\kappa = 1$; and since $p'_i = p_i$ for all $i < \kappa$, and $s = j(p)|(\kappa + 1)$, we have $s = (p|\kappa)^\frown 1$. This implies that if $p \in G$ and $jp = (s, \mathbf{q})$, then $s \in G$.

Now let

$$D = \{q \in Q : \text{for some } p \in G,\ q = i^{\mathfrak{N}}_G(\mathbf{q}) \text{ where } j(p) = (s, \mathbf{q})\} \tag{36.31}$$

Since $j|P \in \mathfrak{M}$ and has size λ, we have $j|P \in \mathfrak{N}$ and therefore $D \in \mathfrak{N}[G]$. It is easy to see that D is directed, i.e., if $q_1, q_2 \in D$, then there is $q \in D$ such that $q \leq q_1$ and $q \leq q_2$ (this is because $G \subseteq P$ is directed). We have (in $\mathfrak{N}[G]$), $|D| \leq |G| \leq |P| = \lambda$; and because Q is strongly λ-closed, there exists $a \in Q$ (a *master condition*) such that $a \leq q$ for all $q \in D$.

We shall now consider (as an auxiliary model), a generic extension of $\mathfrak{M}[G]$. Let H be an $\mathfrak{M}[G]$-generic filter on Q such that H contains the master condition a. Since H is also $\mathfrak{N}[G]$-generic, and $jP = P * \mathbf{Q}$, there is an $\mathfrak{N}$-generic filter K on jP such that $\mathfrak{N}[K] = \mathfrak{N}[G][H]$; in fact

$$K = \{(s, \mathbf{q}) : s \in G \text{ and } i^{\mathfrak{N}}_G(\mathbf{q}) \in H\} \tag{36.32}$$

Everything is ready now to extend the elementary embedding j: $\mathfrak{M} \to \mathfrak{N}$ to an embedding of $\mathfrak{M}[G]$ in $\mathfrak{N}[K]$. We work in $\mathfrak{M}[G][H]$ and define, for $x \in \mathfrak{M}[G]$,

$$j(x) = i^{\mathfrak{N}}_K(j(\mathbf{x})) \tag{36.33}$$

where $\mathbf{x} \in \mathfrak{M}^B$ is some name for x, and $i^{\mathfrak{N}}_K$ is the K-interpretation of $\mathfrak{N}^{jB}$.

We have to show that the definition (36.33) does not depend on the choice of the name $\mathbf{x}$; the verification of elementarity of j is then straightforward. Here we use the master condition a. If $p \in G$, then $jp = (s, \mathbf{q})$ where $s \in P$ and $\|\mathbf{q} \in \mathbf{Q}\| = 1$. We have shown that $s \in G$, and if $q = i^{\mathfrak{N}}_G(\mathbf{q})$, then, because $p \in G$, we have $q \geq a$ and therefore $q \in H$. Thus $(s, \mathbf{q}) \in K$, and we have

$$\text{if } \ p \in G, \quad \text{then } \ j(p) \in K \tag{36.34}$$

Now if $p \in G$ forces $\mathbf{x} = \mathbf{y}$, then $jp \in K$ forces $j\mathbf{x} = j\mathbf{y}$ and hence $i^{\mathfrak{N}}_K(j\mathbf{x}) = i^{\mathfrak{N}}_K(j\mathbf{y})$.

Thus we have (in $\mathfrak{M}[G][H]$) an elementary embedding

$$j\colon \mathfrak{M}[G] \to \mathfrak{N}[K]$$

and we can define, in the usual way, an $\mathfrak{M}[G]$-ultrafilter over κ

$$U = \{X \subseteq \kappa : \kappa \in j(X)\}$$

U is nonprincipal and $\mathfrak{M}[G]$-κ-complete. It suffices to show that $U \in \mathfrak{M}[G]$; for then $\mathfrak{M}[G]$ satisfies that κ is a measurable cardinal.

By (36.28), Q is λ-closed not only in $\mathfrak{N}[G]$, but also in $\mathfrak{M}[G]$. Thus the generic extension $\mathfrak{M}[G][H]$ of $\mathfrak{M}[G]$ does not add any new λ-sequences in $\mathfrak{M}[G]$, and because $|U| = \lambda$ we have $U \in \mathfrak{M}[G]$. ∎

37. SOME APPLICATIONS OF FORCING IN THE THEORY OF LARGE CARDINALS

In the preceding sections, in particular in Sections 34 and 36 we used the method of forcing to obtain consistency results concerning large cardinals. I shall now present several other applications of forcing and discuss various consistency results involving large cardinals.

Large Cardinals in Generic Extensions

As we have already seen, various ingenious methods of forcing have been devised to ensure that a given large cardinal remains a large cardinal in the generic extension. There are in fact several quite general results saying that if the notion of forcing used has such and such properties and if κ is a large cardinal of such and such kind, then in the generic extension κ is still a large cardinal of that kind. Conversely, there are results that say that certain notions of forcing cannot make a given cardinal large.

The two lemmas that are presented below are typical examples of such results. We proved (in Lemma 36.2) that if κ is measurable and $|P| < \kappa$, then κ remains measurable in $\mathfrak{M}[G]$. It turns out that practically every large cardinal is preserved under "mild" extensions, i.e., such that $|P| < \kappa$, and conversely κ does not become large in a mild extension if it is not large already. The other lemma is included to show the relation between saturation of Boolean algebras and saturation of ideals in the corresponding Boolean-valued models.

Lemma 37.1. *Let κ be an infinite cardinal in $\mathfrak{M}$, and let $(P, <)$ be a notion of forcing such that $|P| < \kappa$. Let G be an $\mathfrak{M}$-generic filter on $\mathfrak{M}$. Then κ is inaccessible (Mahlo, weakly compact, Ramsey, measurable, compact) in $\mathfrak{M}$ if and only if it is inaccessible (Mahlo, weakly compact, Ramsey, measurable, compact) in $\mathfrak{M}[G]$.*

Proof. If κ is inaccessible in $\mathfrak{M}$, then firstly κ is regular in $\mathfrak{M}[G]$ because all cardinals and cofinalities above $|P|$ are preserved. Secondly, if $\alpha < \kappa$, then $(2^\alpha)^{\mathfrak{M}[G]} \leq |\text{r.o. } P|^\alpha < \kappa$ and hence κ is inaccessible in $\mathfrak{M}[G]$. Conversely, if κ is inaccessible in $\mathfrak{M}[G]$, then κ is inaccessible in $\mathfrak{M}$ because $\mathfrak{M} \subseteq \mathfrak{M}[G]$.

If κ is Mahlo in $\mathfrak{M}$, then firstly κ is inaccessible in $\mathfrak{M}[G]$. Secondly, every $\alpha > |P|$ is a regular cardinal in $\mathfrak{M}[G]$ if and only if it is a regular cardinal in $\mathfrak{M}$ and so the set $S = \{\alpha < \kappa : |P| < \alpha$ and α is regular in $\mathfrak{M}[G]\}$ is stationary in $\mathfrak{M}$. By Lemma 35.4, S is also stationary in $\mathfrak{M}[G]$ and hence κ is Mahlo in $\mathfrak{M}[G]$. Conversely, if κ is Mahlo in $\mathfrak{M}[G]$, then κ is Mahlo in $\mathfrak{M}$ because $\mathfrak{M} \subseteq \mathfrak{M}[G]$.

If κ is weakly compact in $\mathfrak{M}$, let $F: [\kappa]^2 \to \{0, 1\}$ be a partition of $[\kappa]^2$ in $\mathfrak{M}[G]$. Let $\mathbf{F} \in \mathfrak{M}^B$ be its name, such that $\|\mathbf{F} : [\check{\kappa}]^2 \to \{0, 1\}\| = 1$. For each $\alpha \neq \beta \in \kappa$, let $G(\alpha, \beta) = \|\mathbf{F}(\check{\alpha}, \check{\beta}) = \check{0}\|$; G is (in $\mathfrak{M}$) a partition of $[\kappa]^2$ into $|B|$ pieces (where $B =$ r.o. P). Since $|B| < \kappa$, there is $H \subseteq \kappa$ of size κ homogeneous for G, and it is easy to see that H is homogeneous for F.

Conversely, if κ is weakly compact in $\mathfrak{M}[G]$, let $F: [\kappa]^2 \to \{0, 1\}$ be a partition of $[\kappa]^2$ in $\mathfrak{M}$. There is, in $\mathfrak{M}[G]$, a set $K \subseteq \kappa$ of size κ, homogeneous for F. An easy forcing argument shows that K has a subset $H \in \mathfrak{M}$ of size κ (note that $K = \bigcup\{K_p : p \in G\}$ where $K_p = \{\alpha : p \Vdash \check{\alpha} \in \mathbf{K}\}$, and that some K_p must be unbounded in κ); hence F has in $\mathfrak{M}$ a homogeneous set of size κ.

The argument for Ramsey cardinals is exactly the same as for weakly compact cardinals.

If κ is measurable in $\mathfrak{M}$, then it is measurable in $\mathfrak{M}[G]$ by Lemma 36.2. Conversely, if κ is measurable in $\mathfrak{M}[G]$, let $U \in \mathfrak{M}[G]$ be a κ-complete nonprincipal ultrafilter over κ, let J be the dual prime ideal, and let $\mathbf{J} \in \mathfrak{M}^B$ be its name. (Without loss of generality we assume that $\|\mathbf{J}$ is a $\check{\kappa}$-complete nonprincipal prime ideal$\| = 1$.) Let $I = \{X \subseteq \kappa : \|\check{X} \in \mathbf{J}\| = 1\}$. It is easy to verify that I is a κ-complete ideal containing all singletons. We claim that I is $|P|^+$-saturated: If $p \Vdash \check{X} \notin \mathbf{J}$ and $p \Vdash \check{Y} \notin \mathbf{J}$, then $p \Vdash \check{X} \cap \check{Y} \notin \mathbf{J}$ (because $\mathbf{J}$ is prime). Hence if X and Y are such that $X \notin I$, $Y \notin I$, and $X \cap Y \in I$, then $\|\check{X} \notin \mathbf{J}\| \cdot \|\check{Y} \notin \mathbf{J}\| = 0$, and it follows that I is $|P|^+$-saturated. However, since I is ν-saturated for some $\nu < \kappa$ and κ is inaccessible, we conclude by (a generalization of) Lemma 27.5—see also Section 34—that κ is measurable.

If κ is compact, let $\lambda \geq \kappa$ and let us show that in $\mathfrak{M}[G]$, there is a fine measure on $P_\kappa(\lambda)$. Let U be a fine measure on $P_\kappa(\lambda)$ in $\mathfrak{M}$, and let $j = j_U$ be the natural elementary embedding j_U: $\mathfrak{M} \to Ult_U(\mathfrak{M})$. The standard argument shows that $X \in U$ if and only if $H \in j(X)$, where H is the set in $Ult_U(\mathfrak{M})$ represented by the function $d(Z) = Z$ on $P_\kappa(\alpha)$; also, $H \supseteq \{j(\alpha) : \alpha < \lambda\}$ (and is equal to it just in case U is normal). Similarly, as in the proof of Lemma 36.2, we extend j to $\mathfrak{M}[G]$ as follows:

$$j(x) = i_G^{Ult}(j\mathbf{x})$$

where $\mathbf{x}$ is a name for x; the definition is legitimate because we assume, without loss of generality, that $P \in V_\kappa$ and hence $j(p) = p$ for all $p \in P$, and $j(P) = P$. Now we define, in $\mathfrak{M}[G]$, an ultrafilter W over $P_\kappa(\lambda)$ as follows:

$$X \in W \quad \text{iff} \quad H \in j(X)$$

It is routine to verify that W is a fine measure on $P_\kappa(\lambda)$; for instance, if $Z_0 \in P_\kappa(\lambda)$, then $\{Z \in P_\kappa(\lambda) : Z \supseteq Z_0\} \in W$ because $j(Z_0) = \{j(\alpha) : \alpha \in Z_0\} \subseteq H$.

Conversely, if κ is compact in $\mathfrak{M}[G]$, let S be a set in $\mathfrak{M}$ and let F be a κ-complete filter over S (in $\mathfrak{M}$); let us show that there is a κ-complete ultrafilter in $\mathfrak{M}$ extending F. Every set $X \subset F$ of size $< \kappa$ in $\mathfrak{M}[G]$ is included in some $Y \subset F$ of size $< \kappa$ such that $Y \in \mathfrak{M}$ (this is because $|P| < \kappa$). Hence F generates a κ-complete filter in $\mathfrak{M}[G]$ and that in turn is included in a κ-complete ultrafilter on U. Let J be the dual prime ideal. As in the proof for measurable cardinals above, the ideal $I = \{X \subseteq S : \|\check{X} \in \boldsymbol{J}\| = 1\}$ is κ-complete and $|P|^+$-saturated, and clearly $X \in F$ implies $S - X \in I$. Since κ is inaccessible and I is ν-saturated for some $\nu < \kappa$, I has an atom A. If $X \in F$, then $X \cap A \notin I$ and so $\{X \subseteq S : X \cap A \notin I\}$ is a κ-complete ultrafilter extending F. ■

There are of course numerous examples when κ ceases to be large when we use a notion of forcing of size $\geq \kappa$ (a good example is the Lévy collapse). The example given below is quite interesting since obviously inaccessibility of κ is preserved by any notion of forcing that is α-distributive for all $\alpha < \kappa$.

Note that if κ is a Mahlo cardinal, then κ has no unbounded subset A such that $|A \cap \gamma| < \gamma$ for all regular $\gamma < \kappa$ (since the set of all regular $\gamma < \kappa$ is stationary, an application of Fodor's theorem would give $|A| < \kappa$).

Exercise 37.1. Let κ be an inaccessible cardinal. There is a notion of forcing $(P, <)$ such that $|P| = \kappa$ and P is α-distributive for all $\alpha < \kappa$, and such that κ is not a Mahlo cardinal in the generic extension.

[Forcing conditions are sets $p \subseteq \kappa$ such that $|p \cap \gamma| < \gamma$ for every regular $\gamma \leq \kappa$; $p \leq q$ iff p is an end-extension of q, i.e., if $q = p \cap \alpha$ for some α. To show that for any $\alpha < \kappa$, P does not add any new α-sequence, observe that $\forall p\ \exists q \leq p$ such that $P_q = \{r \in P : r \leq q\}$ is α-closed.]

The following lemma shows that there is a relation between saturation of ideals in generic extensions and saturation of the notion of forcing. Recall that in Exercise 34.5 we proved that if κ carries a σ-saturated κ-complete ideal, then so it does in the generic extension if the notion of forcing is σ-saturated. The lemma below is a generalization.

The proof of (a) uses generic ultrapowers (see Section 35). One particular property needed is the following:

Exercise 37.2. Let I be a κ^+-saturated κ-complete ideal over κ. Let G be a corresponding generic ultrafilter and let $\mathfrak{N} = Ult_G(\mathfrak{M})$ be the generic ultrapower. Then every $s\colon \kappa \to \mathfrak{M}$ in $\mathfrak{M}[G]$ is in $\mathfrak{N}$.

[Let $\boldsymbol{s}$ be a name for s; for each $\alpha < \kappa$, let $\boldsymbol{s}_\alpha$ be a name such that $\|\boldsymbol{s}(\check{\alpha}) = \boldsymbol{s}_\alpha\| = 1$. By the remark preceding Exercise 35.11, each $\boldsymbol{s}_\alpha$ is represented by a function $f_\alpha \in \mathfrak{M}$ on κ. Let $h\colon \kappa \to \kappa$ be the least unbounded function (that is, h represents κ in $\mathfrak{N}$). Let f be the function on κ defined by $f(\alpha) = \langle f_\beta(\alpha) : \beta < h(\alpha)\rangle$. Show that f represents $\boldsymbol{s}$ in the generic ultrapower.]

Lemma 37.2. *Let B be a complete Boolean algebra in $\mathfrak{M}$, let G be an $\mathfrak{M}$-generic ultrafilter and let κ be an uncountable regular cardinal in $\mathfrak{M}$.*

(a) *Let $\lambda \le \kappa^+$ and assume that* sat $B \le \lambda$ *and* sat $B < \kappa$. *If in $\mathfrak{M}$, κ carries a λ-saturated κ-complete ideal, then the same is true in $\mathfrak{M}[G]$.*

(b) *If* sat $B \le \lambda$ *and if in $\mathfrak{M}[G]$, κ carries a λ-saturated κ-complete ideal, then the same is true in $\mathfrak{M}$.*

Proof. We shall prove (b) first since the proof is somewhat easier. Let $\boldsymbol{J} \in \mathfrak{M}^B$ be such that $\|\boldsymbol{J}$ is a $\check\kappa$-complete $\check\lambda$-saturated ideal over $\check\kappa\| = 1$. We let $I = \{X \subseteq \kappa : \|\check X \in \boldsymbol{J}\| = 1\}$. It is easy to see that I is a κ-complete ideal containing all singletons. To show that I is λ-saturated, let us consider the Boolean algebra $D = P(\kappa)/I$ and let us show that D is λ-saturated.

In $\mathfrak{M}^B$, let $\boldsymbol{C}$ be the Boolean algebra $P(\check\kappa)/\boldsymbol{J}$. Since B is λ-saturated and $\mathfrak{M}^B \models \boldsymbol{C}$ is $\check\lambda$-saturated, the algebra $B * \boldsymbol{C}$ is λ-saturated (see Lemma 23.6). Thus it suffices to find a homomorphism h: $D \to B * \boldsymbol{C}$ such that $h(d) \ne 0$ for every $d \ne 0$.

If $d \in D$, then d is the equivalence class (mod I) $d = [X]$ for some $X \subseteq \kappa$. Let $h(d) = \boldsymbol{c}$ where $\boldsymbol{c}$ is in $\mathfrak{M}^B$ the equivalence class of $\check X$ mod $\boldsymbol{J}$. It is routine to verify that the definition of h does not depend on the choice of X, that h is a homomorphism from D to $B * \boldsymbol{C}$, and that $h(d) \ne 0$ whenever $d \ne 0$.

I shall now sketch the proof of (a). Let I be, in $\mathfrak{M}$, a κ-complete λ-saturated ideal over κ. Let $J \in \mathfrak{M}[G]$ be the ideal generated by I. Since sat $B \le \kappa$, it is routine to verify that J is κ-complete. Let $\boldsymbol{J} \in \mathfrak{M}^B$ be the canonical name for J, and let $\boldsymbol{C} \in \mathfrak{M}^B$ be the Boolean algebra $\boldsymbol{C} = P(\check\kappa)/\boldsymbol{J}$.

We want to show that $\mathfrak{M}^B \models \boldsymbol{C}$ is $\check\lambda$-saturated; by Lemma 23.6 it suffices to show that $B * \boldsymbol{C}$ is λ-saturated because B is λ-saturated. Let $D = P(\kappa)/I$. We shall find in $\mathfrak{M}^D$ a Boolean algebra $\boldsymbol{E}$ such that $\mathfrak{M}^D \models \boldsymbol{E}$ is $\check\lambda$-saturated, and such that $D * \boldsymbol{E}$ is isomorphic to $B * \boldsymbol{C}$. Since D is λ-saturated, it will follow that $D * \boldsymbol{E}$ is λ-saturated and we shall be done.

In $\mathfrak{M}^D$, consider the generic ultrapower $\mathfrak{N} = Ult_{\boldsymbol{G}}(\mathfrak{M})$, where $\boldsymbol{G}$ is the canonical ultrafilter on $\check D$. Let $\boldsymbol{j}$: $\mathfrak{M} \to \mathfrak{N}$ be the corresponding elementary embedding. Let $\boldsymbol{E} = \boldsymbol{j}(B)$.

Let $\text{sat}(B) = \nu < \kappa$. By elementarity of $\boldsymbol{j}$, we have $\mathfrak{N} \models \text{sat}(\boldsymbol{j}\check B) = \boldsymbol{j}(\check\nu)$; and since $\boldsymbol{j}(\check\nu) = \check\nu$ and by Exercise 37.2 all $\check\nu$-sequences in $\mathfrak{M}^D$ are in $\mathfrak{N}$, we have $\mathfrak{M}^D \models \text{sat}(\boldsymbol{E}) = \check\nu$. Thus $\mathfrak{M}^D \models \boldsymbol{E}$ is $\check\lambda$-saturated.

It remains to show that $B * \boldsymbol{C}$ and $D * \boldsymbol{E}$ are isomorphic. Let $\boldsymbol{c} \in B * \boldsymbol{C}$. Then $\boldsymbol{c} \in \mathfrak{M}^B$ and $\|\boldsymbol{c} \in \boldsymbol{C}\|_B = 1$. Thus there is $\boldsymbol{X} \in \mathfrak{M}^B$ such that $\|\boldsymbol{X} \subseteq \check\kappa\|_B = 1$ and that

$$\|\boldsymbol{c} \text{ is the equivalence class of } \boldsymbol{X} \text{ mod } \boldsymbol{J}\|_B = 1$$

Let f: $\kappa \to B$ be the function $f(\alpha) = \|\check\alpha \in \boldsymbol{X}\|$. Since $f(\alpha) \in B$ for all $\alpha < \kappa$, f represents in $\mathfrak{N} = Ult_{\boldsymbol{G}}(\check B) \subseteq \mathfrak{M}^D$ an element $\boldsymbol{e} \in \boldsymbol{j}(\check B) = \boldsymbol{E}$; and $\boldsymbol{e} \in D * \boldsymbol{E}$. We let $h(\boldsymbol{c}) = \boldsymbol{e}$.

We leave it to the reader to verify that the definition of $h(c)$ does not depend on the choice of X and that $h: \boldsymbol{B} * \boldsymbol{C} \to \boldsymbol{D} * \boldsymbol{E}$ is an isomorphism. ∎

Large Cardinals and the Generalized Continuum Hypothesis

Considering the influence of large cardinals on cardinal arithmetic, it is important to know what behavior of the continuum function $2^{\aleph_\alpha}$ is compatible with various kinds of large cardinals. The most is known for measurable cardinals: First, there is a transitive model of ZFC + "there is a measurable cardinal," namely the model $L[U]$, which satisfies the GCH. Secondly, $2^\kappa > \kappa^+$ for a measurable cardinal implies existence of models with many measurable cardinals, thus showing that consistency of $2^\kappa > \kappa^+$ at a measurable cardinal requires some stronger assumption than just measurability. And thirdly, Theorem 88 provides a model for $2^\kappa > \kappa^+$ at a measurable cardinal.

For some, "small," large cardinals, consistency with the GCH can be established by means of a transitive model. For instance, weakly compact cardinals are consistent with the GCH because every weakly compact cardinal is weakly compact in L. Or, if κ is a Ramsey cardinal, it is known that there exists a transitive model (the *core model* K) that satisfies the GCH and in which κ is Ramsey. On the other hand, no construction of a transitive model is known for compact (or supercompact) cardinals. However, compact and supercompact cardinals are consistent with the GCH, but the consistency proof uses a generic extension by iterated forcing, using the techniques and ideas presented in Section 36.

Iterated forcing as in Section 36 can also be used to establish the consistency of $2^\kappa > \kappa^+$ and "κ is weakly compact (Ramsey, supercompact)" relative to ZFC + "κ is weakly compact (Ramsey, supercompact)."

Changing Cofinality of Cardinals

Let us address ourselves to the following problem: Can one construct a generic extension in which all cardinals are preserved but the cofinality of some cardinals is different from their cofinality in the ground model? Obviously, in order to achieve this we have to change some (weakly) inaccessible cardinal into a singular cardinal. Now Corollary 3 of Jensen's covering theorem tells us that for this, it is necessary to assume at least $0^\#$ in the ground model. Thus we formulate the problem as follows: Let κ be some large cardinal and let $\lambda < \kappa$ be a regular cardinal. Find a generic extension which preserves cardinals and in which $\operatorname{cf} \kappa = \lambda$.

We have seen one method of doing this in Section 35 (application 7 of generic ultrapowers); however, the assumptions we have used there are rather strong and perhaps not as plausible as better known large cardinal assumptions.

Theorem 89 below provides an answer to the problem when κ is measurable and $\lambda = \aleph_0$. A generalization of the construction in Theorem 89 gives the solution for any regular $\lambda < \kappa$ assuming that κ is supercompact.

Theorem 89 (Prikry). *Let κ be a measurable cardinal. There is a generic extension in which* $\operatorname{cf} \kappa = \omega$ *and no cardinals are collapsed. Moreover, every bounded subset of κ in $\mathfrak{M}[G]$ is in the ground model.*

Proof. Let κ be a measurable cardinal and let D be a normal measure on κ. Let $(P, <)$ be the following notion of forcing. A forcing condition is a pair $p = (s, A)$ where $s \in [\kappa]^{<\omega}$, i.e., s is a finite subset of κ, and $A \in D$. A condition (s, A) is stronger than a condition (t, B) if:

(37.1) (a) t is an initial segment of s, i.e., $t = s \cap \alpha$ for some α;
(b) $A \subseteq B$;
(c) $s - t \subseteq B$.

We immediately note that any two conditions (s, A), (s, B) with the same first coordinate are compatible, and hence any incompatible $W \subseteq P$ has size at most κ. We also note that if (s, A) and (t, B) are compatible, then either s is an initial segment of t, or t is an initial segment of s.

Let G be a generic filter on P. We shall show that in $\mathfrak{M}[G]$, κ has cofinality ω, that every bounded subset of κ is in $\mathfrak{M}$, and that all cardinals and cofinalities above κ are preserved.

The last statement is immediate since P satisfies the κ^+-chain condition. It is also easy to show that $\operatorname{cf} \kappa = \omega$ in $\mathfrak{M}[G]$: If (s, A) and (t, B) are in G, then either s is an initial segment of t or vice versa; hence $S = \bigcup \{s : (s, A) \in G \text{ for some } A\}$ is a subset of κ of order type ω. By genericity of G, S is clearly an unbounded subset of κ, and hence $\operatorname{cf} \kappa = \omega$.

It remains to show that if $X \in \mathfrak{M}[G]$ is such that $X \subseteq \lambda$ for some $\lambda < \kappa$, then $X \in \mathfrak{M}$. For this, we need the property of P stated in Lemma 37.3 below (and Lemma 34.4). The proof uses Theorem 70 which states that every partition of $[\kappa]^{<\omega}$ into less than κ pieces has a homogeneous set $H \in D$.

Lemma 37.3. *Let σ be a sentence of the forcing language. There exists $A \in D$ such that the condition $(\emptyset, A)$ decides σ, i.e., either $(\emptyset, A) \Vdash \sigma$, or $(\emptyset, A) \Vdash \neg\sigma$.*

Proof. Let S^+ be the set of all $s \in [\kappa]^{<\omega}$ such that $(s, X) \Vdash \sigma$ for some X and let $S^- = \{s : \exists X(s, X) \Vdash \neg\sigma\}$. Let $T = [\kappa]^{<\omega} - (S^+ \cup S^-)$. By Theorem 70, there is a set $A \in D$ such that for every n, either $[A]^n \subseteq S^+$ or $[A]^n \subseteq S^-$ or $[A]^n \subseteq T$. We shall prove that $(\emptyset, A)$ decides σ.

If not, then there are conditions (s, X) and (t, Y), both stronger than $(\emptyset, A)$ such that one forces σ and the other forces $\neg\sigma$. We may assume that $|s| = |t|$, say $|s| = |t| = n$. Since $(s, X) \leq (\emptyset, A)$, we have $s \in [A]^n$; and similarly, $t \in [A]^n$. But $s \in S^+$ and $t \in S^-$, which is a contradiction since S^+ and S^- are disjoint and therefore cannot both have a nonempty intersection with $[A]^n$. ■

Lemma 37.4. *Let σ be a sentence of the forcing language and let (s_0, A_0) be a condition. Then there exists $A \subseteq A_0$ such that $A \in D$ and such that the condition (s_0, A) decides σ.*

Proof. A slight modification of the preceding proof; we may assume that $\min(A_0) > \max(s_0)$. Let S^+ be the set of all $s \in [A_0]^{<\omega}$ such that $(s_0 \cup s, X) \Vdash \sigma$ for some $X \subseteq A_0$ and let S^- be defined similarly. As before, there exists $A \subseteq A_0$ in D such that for no n, $[A]^n$ intersects both S^+ and S^-. It follows that (s_0, A) decides σ. ■

Now let $\lambda < \kappa$ and let $X \subseteq \lambda$; we will show that $X \in \mathfrak{M}$. Let $\mathbf{X}$ be a name for X, and let $p_0 \in G$ be a condition such that $p \Vdash \mathbf{X} \subseteq \check{\lambda}$. It suffices to show that for each $p \leq p_0$ there is $q \leq p$ and a set $Z \subseteq \lambda$ such that $q \Vdash \mathbf{X} = \check{Z}$.

Let $p \leq p_0$, $p = (s, A)$. For each $\alpha < \lambda$, there is, by Lemma 37.4, a set $A_\alpha \subseteq A$ in D such that (s, A_α) decides the sentence $\check{\alpha} \in \mathbf{X}$. Let $B = \bigcap_{\alpha<\lambda} A_\alpha$; we have $B \in D$ and $q = (s, B)$ decides each $\check{\alpha} \in \mathbf{X}$, $\alpha < \lambda$. Thus if $Z = \{\alpha < \lambda : q \Vdash \check{\alpha} \in \mathbf{X}\}$, we have $q \Vdash \mathbf{X} = \check{Z}$.

This completes the proof of Theorem 89. ■

We proved in Section 29 that every measurable cardinal is a Rowbottom cardinal and that a Rowbottom cardinal is either weakly inaccessible or has cofinality ω. The model $\mathfrak{M}[G]$ of Theorem 89 provides an example of a singular Rowbottom cardinal. The exercise below shows that κ has in $\mathfrak{M}[G]$ the combinatorial property equivalent to being a Rowbottom cardinal.

*Exercise 37.3.** In $\mathfrak{M}[G]$, if $F\colon [\kappa]^{<\omega} \to \lambda$ is a partition into $\lambda < \kappa$ pieces, there exists $H \subseteq \kappa$ of size κ such that F takes at most $\aleph_0$ values on $[H]^{<\omega}$.

[Let $\mathbf{F}$ be a name for F and let (s_0, A_0) be a condition (such that $\max(s_0) < \min(A_0)$). Let g be a partition of $[\kappa]^{<\omega} \times [\kappa]^{<\omega}$ into λ pieces, defined as follows: If $s \in [A_0]^{<\omega}$ and for some $X \subseteq A_0$, $(s_0 \cup s, X) \Vdash \mathbf{F}(\check{t}) = \check{\alpha}$, then let $g(s, t) = \alpha$; otherwise, let $g(s, t) = 0$. Show that there is $A \subseteq A_0$ in D and a countable $S \subseteq \lambda$ such that $g[[A]^{<\omega} \times [A]^{<\omega}] \subseteq S$. Then $(s_0, A) \Vdash \mathbf{F}[[\check{A}]^{<\omega}] \subseteq \check{S}$.]

Singular Cardinals Problems

It follows from Jensen's covering theorem that if the singular cardinals hypothesis fails, then $0^\#$ exists; more recent results then show that, moreover, there is a model with a measurable cardinal. Thus to construct models violating the singular cardinals hypothesis one needs large cardinals in the ground model. And even then we cannot prescribe the behavior of the continuum function at singular cardinals arbitrarily: As we have seen in Section 8, the least singular cardinal at which the SCH fails has cofinality ω, and there are also bounds on 2^κ in some cases.

The last two theorems establish independence of the singular cardinals hypothesis.

Corollary (to Theorems 88 and 89). *It is consistent (relative to existence of a supercompact cardinal) that there is a strong limit singular cardinal κ such that $2^\kappa > \kappa^+$.*

Proof. Let κ be a supercompact cardinal. First we construct a generic extension in which κ is measurable and $2^\kappa > \kappa^+$. Then we extend the model further to make κ a singular cardinal. The new model still satisfies $2^\kappa > \kappa^+$, and κ is a strong limit cardinal. ■

There are several forcing constructions giving answers to various singular cardinals problems. For instance, there is a model in which $2^{\aleph_n} = \aleph_{n+1}$ for all n, and $2^{\aleph_\omega} = \aleph_{\omega+2}$. It is an open problem to construct a model in which $2^{\aleph_\alpha} > \aleph_{\alpha+1}$ for all α.

Exercise 37.4. It is consistent that $2^{\operatorname{cf}\kappa} < \kappa$ and $\kappa^+ < \kappa^{\operatorname{cf}\kappa} < 2^\kappa$.

[Extend the model in the corollary by adjoining a large number of subsets of ω_1.]

More Applications of Forcing

We shall now use the method introduced in Section 36 to show that it is consistent with a measurable cardinal κ to assume that there is more than one normal measure on κ. Then we use the method to prove Theorem 86b, giving a model with a precipitous ideal over ω_1.

Example. Let $\mathfrak{M}$ be a transitive model of ZFC + GCH + there is a measurable cardinal κ. Let $\lambda > \kappa$ be a regular cardinal. There is a generic extension in which $2^\kappa = \kappa^+$, $2^{\kappa^+} = \lambda$, and there are λ normal measures on κ.

We use the method introduced in Lemma 36.3 (which will be followed closely). For each regular cardinal α, let $P(\alpha)$ be the notion of forcing that adjoins a generic subset of α; let P be the Easton product of the $P(\alpha)$'s for all successor cardinals $\alpha < \kappa$. More precisely, P consists of 0, 1-functions p such that $\operatorname{dom}(p) \subseteq \kappa \times \kappa$, and:

(37.2) (i) if $(\alpha, \beta) \in \operatorname{dom}(p)$, then α is a successor cardinal $< \kappa$, and $\beta < \alpha$;
(ii) for every regular γ, $|\{(\alpha, \beta) \in \operatorname{dom} p : \alpha \leq \gamma\}| < \gamma$.

For each ν, let P_ν consist of all $p \in P$ such that $\alpha < \nu$ whenever $(\alpha, \beta) \in \operatorname{dom} p$; similarly P^ν is the set of all p such that $\alpha \geq \nu$ for all $(\alpha, \beta) \in \operatorname{dom} p$. For each ν, P is isomorphic to $P_\nu \times P^\nu$.

Let U be a normal measure on κ, let $\mathfrak{N} = Ult_U(\mathfrak{M})$, and let $j = j_U$ be the corresponding elementary embedding. Let us consider the notion of forcing $j(P)$ in $\mathfrak{N}$. As in Lemma 36.3, we have $(jP)_\kappa = P$; let $Q = (jP)^\kappa$. Clearly, $(\kappa, \beta) \notin \operatorname{dom} p$ for any $\beta < \kappa$ and hence Q is κ-closed.

If $G \times H$ is a generic filter on $P \times Q$, then $\mathfrak{M}[G \times H]$ has a normal measure on κ defined in a similar way as in Lemma 36.3:

$$X \in D \quad \text{iff} \quad \kappa \in i^{\mathfrak{N}}_{G \times H}(j(X)) \tag{37.3}$$

As before, every $X \subseteq \kappa$ in $\mathfrak{M}[G \times H]$ is in $\mathfrak{M}[G]$; X can be any r.o. P-valued name for X, and its choice is irrelevant because $p \in G$ implies $jp \in G \times H$.

Let $\boldsymbol{D}$ be the canonical name for D; thus if $p \in P$, $q \in Q$, and if $X \in \mathfrak{M}^{\text{r.o.}P}$, then

$$p \cup q \Vdash (X \in \boldsymbol{D} \leftrightarrow \check{\kappa} \in j(X)) \tag{37.4}$$

Let $p \in P$ and $q \in Q$; we have $p = j(p)$, and q is represented in the ultrapower $\mathfrak{N}$ by a function $\langle q_\alpha : \alpha < \kappa \rangle$ where $q_\alpha \in P^\alpha$ for each $\alpha < \kappa$. From (37.4) we get

$$p \cup q \Vdash X \in \boldsymbol{D} \quad \text{iff} \quad \text{for almost all } \alpha, \quad p \cup q_\alpha \Vdash \check{\alpha} \in X \tag{37.5}$$

Now we are ready to construct the model. Let R be the κ^+-product of λ copies of Q, i.e., the set of all functions $f \in {}^\lambda Q$ such that $|\{i < \lambda : f(i) \neq \emptyset\}| \leq \kappa$. We consider the notion of forcing $P \times R$.

Let $G \times H$ be an $\mathfrak{M}$-generic filter on $P \times R$. We claim that the model $\mathfrak{M}[G \times H]$ has the required properties.

Since R is κ-closed, all subsets of κ are contributed by G; hence $2^\kappa = \kappa^+$ holds in $\mathfrak{M}[G \times H]$. Standard arguments show that cardinals are preserved and $2^{\kappa^+} \leq \lambda$ in $\mathfrak{M}[G \times H]$.

For each $i < \lambda$, let $\boldsymbol{Q}_i$ denote the ith copy of Q and let $\boldsymbol{D}_i$ be the canonical name for a normal measure using Q_i instead of Q in (37.4). It suffices to show that for any $i \neq k < \lambda$, $\boldsymbol{D}_i$ and $\boldsymbol{D}_k$ denote different measures in $\mathfrak{M}[G \times H]$.

The last assertion follows by a standard argument using genericity, and we leave its proof to the reader: Given $i \neq k$ and a condition (p, r) in $P \times R$, use (37.5) to find a stronger condition (p', r') and some r.o. P-valued name X such that (p', r') forces $X \in \boldsymbol{D}_i$ but $X \notin \boldsymbol{D}_k$. ■

The model $L[U]$ has the property that the only κ-complete ultrafilter over κ that extends the closed unbounded filter is the unique normal measure (see Exercise 31.12). Exercise 37.6 shows that if there are κ distinct normal measures on κ, then there exist nonnormal κ-complete ultrafilters extending the closed unbounded filter.

Exercise 37.5. Let U_α, $\alpha < \kappa$, be κ-complete nonprincipal ultrafilters over κ and let D be a normal measure on κ, $D \neq U_\alpha$ for all α. Then there exists $X \in D$ such that $X \notin U_\alpha$ for all α.

[For each $\alpha < \kappa$ let $X_\alpha \in D$ be such that $X_\alpha \notin U_\alpha$. Let X be the diagonal intersection $\triangle_{\alpha<\kappa} X_\alpha$.]

Exercise 37.6. Assume that there exist κ distinct normal measures D_α, $\alpha < \kappa$, on κ. Then there exists a κ-complete ultrafilter U over κ such that U is not normal and such that $C \in U$ for every closed unbounded set C.

[Let $\{A_\alpha : \alpha < \kappa\}$ be a pairwise disjoint collection of subsets of κ such that $A_\alpha \in D_\alpha$ for all $\alpha < \kappa$; such a collection can be easily constructed using Exercise 37.5. Let W be any κ-complete nonprincipal ultrafilter over κ. We define U as follows: Let $X \in U$ iff $\{\alpha : X \cap A_\alpha \in D_\alpha\} \in W$. U is a κ-complete nonprincipal ultrafilter over κ. If C is closed unbounded, then $C \cap A_\alpha \in D_\alpha$ for all α, and hence $C \in U$. For each α let $a_\alpha = \min A_\alpha$, and let f be a function on κ that has value a_α on each A_α. Then for each $\gamma < \kappa$, $\{\xi : f(\xi) > \gamma\} \in U$ and $\{\xi : f(\xi) < \xi\} \in U$; hence U is not normal.]

Proof of Theorem 86b. We shall now use the method of Lemma 36.3 to show that it is consistent, relative to a measurable cardinal, that $\aleph_1$ carries a precipitous ideal.

Let $\mathfrak{M}$ be a transitive model of ZFC and let κ be a measurable cardinal. We shall show that if $\mathfrak{M}[G]$ is the generic extension of $\mathfrak{M}$ obtained by the Lévy collapse such that κ becomes $\aleph_1$, then $\mathfrak{M}[G]$ has a precipitous ideal over $\aleph_1$.

Let P be the set of all functions p such that $\operatorname{dom}(p)$ is a finite subset of $\kappa \times \omega$ and such that $p(\alpha, n) < \alpha$ for all $(\alpha, n) \in \operatorname{dom}(p)$; p is stronger than q if $p \supseteq q$ (see (20.7)). Let G be an $\mathfrak{M}$-generic filter on P. In $\mathfrak{M}[G]$, κ is $\aleph_1$.

Let D be, in $\mathfrak{M}$, a normal measure on κ, let $\mathfrak{N} = Ult_D(\mathfrak{M})$ and let $j\colon \mathfrak{M} \to \mathfrak{N}$ be the elementary embedding $j = j_D$. In $\mathfrak{M}[G]$, let I be the ideal over κ generated by the dual of D; i.e.,

$$(37.6) \qquad X \in I \quad \text{iff} \quad X \cap Y = \emptyset \quad \text{for some} \quad Y \in D$$

A routine argument (using sat $P = \kappa$) shows that I is in $\mathfrak{M}[G]$ a countably complete ideal containing all singletons. It can be proved that I is precipitous; instead, we shall prove a weaker (but sufficient) property, namely that there exists $S \subseteq \aleph_1$, $S \notin I$, such that $I|S = \{X \subseteq \aleph_1 : X \cap S \in I\}$ is a precipitous ideal.

For that, it suffices to show that there is $S \notin I$ such that (when forcing with sets $X \notin I$) S forces that the generic ultrapower is well-founded. In turn, it suffices to construct an extension of $\mathfrak{M}[G]$ in which there exists an $\mathfrak{M}[G]$-ultrafilter W over κ, generic over $\mathfrak{M}[G]$ (with respect to forcing with sets $X \notin I$) such that the generic ultrapower $Ult_W(\mathfrak{M}[G])$ is well-founded.

Let us go back to $\mathfrak{M}$ and consider again the notion of forcing P. For every ν, let P_ν be the set of all $p \in P$ such that $\alpha < \nu$ whenever $(\alpha, n) \in \operatorname{dom}(p)$, and let $P^\nu = \{p \in P : \alpha \geq \nu \text{ for all } (\alpha, n) \in \operatorname{dom}(p)\}$; P is isomorphic to the product $P_\nu \times P^\nu$.

Let us consider the notion of forcing $j(P)$. Clearly, $(jP)_\kappa = P$, and thus jP is isomorphic to $P \times Q$ where $Q = (jP)^\kappa$. Every $q \in Q$ is represented in the ultrapower $\mathfrak{N}$ by a function $\langle q_\alpha : \alpha < \kappa \rangle$ such that $q_\alpha \in P^\alpha$ for all $\alpha < \kappa$.

Let H be an $\mathfrak{M}[G]$-generic filter on Q; thus $G \times H$ is $\mathfrak{M}$-generic on $P \times Q$. As in Lemma 36.3, we define in $\mathfrak{M}[G \times H]$ an $\mathfrak{M}[G]$-ultrafilter W over κ as follows:

$$(37.7) \qquad X \in W \quad \text{iff} \quad \kappa \in i^{\mathfrak{N}}_{G \times H}(jX)$$

The definition (37.7) does not depend on the choice of the name X because $p \in G$ implies $j(p) \in G \times H$. Let $\boldsymbol{W}$ be the canonical name for W. As in (37.5), we have for any $p \in P$, $q \in Q$,

$$(37.8) \qquad (p, q) \Vdash X \in \boldsymbol{W} \quad \text{iff} \quad \text{for almost all } \alpha, \ \ p \cup q_\alpha \Vdash \check{\alpha} \in X$$

(here X is an r.o. P-valued name and $\langle q_\alpha : \alpha < \kappa \rangle$ represents q in $\mathfrak{N}$; "almost all" refers to the normal measure D).

First we observe that the ultrapower $Ult_W(\mathfrak{M}[G])$ is well-founded. This is because the following commutative diagram holds:

$$\begin{tikzcd}
\mathfrak{M}[G] \arrow[r, "j"] \arrow[d, "j_W"'] & \mathfrak{N}[G \times H] \\
Ult_W(\mathfrak{M}[G]) \arrow[ur, "k"'] &
\end{tikzcd}$$

In the diagram, j is the extension of j: $\mathfrak{M} \to \mathfrak{N}$ defined by

$$j(x) = i^{\mathfrak{N}}_{G \times H}(x)$$

and k is defined as follows: If $f \in \mathfrak{M}[G]$ is a function on κ representing $[f]$ in $Ult_W(\mathfrak{M}[G])$, then

$$k([f]) = (jf)(\kappa)$$

Both j and k are elementary and the diagram commutes.

It remains to show that W is $\mathfrak{M}[G]$-generic with respect to forcing with sets $X \notin I$. It suffices to show that if $\mathscr{X} = \{X_i : i < \theta\}$ is an I-partition of κ, then $X_i \in W$ for some i. Let $\dot{\mathscr{X}} \in \mathfrak{M}^{\text{r.o.}P}$ be a name for $\mathscr{X}$ and let $\dot{X}_i$, $i < \theta$, be names for the X_i. Let us assume that there are conditions $p \in G$ and $q \in H$ such that

$$p \Vdash \dot{\mathscr{X}} \text{ is an } I\text{-partition of } \check{\kappa}$$

and for each $i < \theta$,

$$(p, q) \Vdash \dot{X}_i \notin W$$

We shall derive a contradiction.

Let q be represented in $\mathfrak{N}$ by $\langle q_\alpha : \alpha < \kappa \rangle$. By (37.8) there is for each i a set $A_i \in D$ such that for all $\alpha \in A_i$,

$$p \cup q_\alpha \Vdash \check{\alpha} \notin \dot{X}_i \tag{37.9}$$

Let us define (in $\mathfrak{M}[G]$),

$$T = \{\alpha < \kappa : q_\alpha \in G\} \tag{37.10}$$

We shall prove that $T \notin I$ and that $T \cap X_i \in I$ for all $i < \theta$, thus reaching a contradiction since $\mathscr{X}$ is an I-partition.

For each $i < \theta$, if $\alpha \in T$ and $\alpha \in A_i$, then $p \cup q_\alpha \in G$ and hence, by (37.9), $\alpha \notin X_i$. It follows that $T \cap X_i \cap A_i = \emptyset$, and so by (37.6), $T \cap X_i \in I$.

Let us finally show that $T \notin I$. It suffices to show that $T \cap Z \neq \emptyset$ whenever $Z \in D$. Thus let $Z \in D$, and let us prove that $q_\alpha \in G$ for some $\alpha \in Z$. Let

$$\Delta = \{r \in P : r \leq q_\alpha \text{ for some } \alpha \in Z\}$$

It is easy to see that Δ is dense in P because Z is unbounded and $q_\alpha \in P^\alpha$ for each $\alpha < \kappa$. Thus $\Delta \cap G \neq \emptyset$ and hence $T \cap Z \neq \emptyset$. ■

Large Cardinals and Set Theory Without the Axiom of Choice

Since the methods of the theory of large cardinals use heavily the axiom of choice, most of the problems concerning large cardinals are not very interesting if the axiom of choice is not assumed. An exception is measurability.

In ZF, a measurable cardinal is defined in the usual way: An uncountable aleph κ is measurable if there exists a κ-complete nonprincipal ultrafilter over κ. One of the first properties of measurable cardinals we proved in ZFC in Section 27 is that every measurable cardinal is inaccessible. The same proof in ZF gives only the following: If κ is measurable, then κ is regular, and $2^\lambda \not\geq \kappa$ for any $\lambda < \kappa$.

In particular, it is not excluded that $\aleph_1$ might be measurable. And indeed, we prove in Theorem 90 below that ZF + "$\aleph_1$ is measurable" is consistent. The consistency is relative to ZFC + "there is a measurable cardinal." This is necessary because if κ is measurable (in ZF) and if U is a κ-complete nonprincipal ultrafilter over κ, then $L[U]$ is a model of ZFC + "κ is measurable."

Theorem 90 (Jech). *Let $\mathfrak{M}$ be a transitive model of* ZFC + "*there is a measurable cardinal.*" *There is a symmetric model $\mathfrak{N} \supseteq \mathfrak{M}$ of* ZF *such that $\mathfrak{N} \vDash \aleph_1$ is measurable.*

We shall construct a symmetric extension of $\mathfrak{M}$. Recall the theory of symmetric models from Section 21. We consider a complete Boolean algebra B, a group $\mathscr{G}$ of automorphisms of B, and a normal filter $\mathscr{F}$ on $\mathscr{G}$ (see (21.5)). For every $x \in \mathfrak{M}^B$ we let sym(x) be the symmetry group of x, sym$(x) = \{\pi \in \mathscr{G} : \pi x = x\}$, and call x symmetric if sym$(x) \in \mathscr{F}$. We denote by HS the class of all hereditarily symmetric names (21.14). If G is an $\mathfrak{M}$-generic ultrafilter on B, we let $\mathfrak{N}$ be the G-interpretation of the class HS; $\mathfrak{N}$ is a model of ZF and $\mathfrak{N} \supseteq \mathfrak{M}$.

Let us call a subset $A \subseteq B$ *symmetric* if

$$\{\pi \in \mathscr{G} : \pi a = a \text{ for all } a \in A\} \in \mathscr{F} \tag{37.11}$$

Lemma 37.5. *Let κ be measurable in $\mathfrak{M}$, and let $\mathfrak{N}$ be a symmetric extension of $\mathfrak{M}$ (via B, G, $\mathscr{F}$, $\mathscr{G}$). If every symmetric subset of B has size $< \kappa$, then κ is measurable in $\mathfrak{N}$.*

Proof. Let U be, in $\mathfrak{M}$, a κ-complete nonprincipal ultrafilter over κ. We show that U generates a κ-complete nonprincipal ultrafilter over κ in $\mathfrak{N}$. It suffices to show that if $\gamma < \kappa$ and $\{X_\alpha : \alpha < \gamma\}$ is a partition of κ in $\mathfrak{N}$, then for some $\alpha < \gamma$, X_α includes some $Y \in U$.

I shall give the proof for $\gamma = 2$ since the general case is analogous. Let $X \in \mathfrak{N}$ be a subset of κ, and let $\underset{\sim}{X} \in$ HS be a symmetric name for X. Let $A = \{\|\check{\alpha} \in X\| : \alpha < \kappa\}$. If $\pi \in \mathscr{G}$ is such that $\pi(X) = X$, then (because $\pi\check{\alpha} = \check{\alpha}$ for all α) $\pi a = a$ for all $a \in A$; thus A is a symmetric subset of B.

Hence $|A| < \kappa$. For each $a \in A$, let $Y_a = \{\alpha : \|\check{\alpha} \in X\| = a\}$. Clearly,

$\{Y_a : a \in A\}$ is a partition of κ into less than κ pieces, and hence one $Y = Y_a$ is in U. Now if $a \in G$, then we have $Y \subseteq X$, and if $a \notin G$, then $Y \subseteq \kappa - X$. Hence either X or $\kappa - K$ has a subset that is in U. ■

Proof of Theorem 90. Let κ be a measurable cardinal in $\mathfrak{M}$. Let P be the set of all one-to-one finite sequences $p = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle$ of ordinals less than κ; p is stronger than q if p extends q (this notion of forcing collapses κ and makes it countable). Let $\mathscr{G}$ be the set of all permutations of κ; every $\pi \in \mathscr{G}$ induces an automorphism of $(P, <)$ as follows:

$$\pi(\langle \alpha_0, \ldots, \alpha_{n-1} \rangle) = \langle \pi(\alpha_0), \ldots, \pi(\alpha_{n-1}) \rangle$$

and, in turn, an automorphism of $B = \text{r.o. } P$. Thus we identify $\mathscr{G}$ with the group of automorphisms so induced.

For each $\gamma < \kappa$, let

$$H_\gamma = \{\pi \in \mathscr{G} : \pi\alpha = \alpha \text{ for all } \alpha < \gamma\}$$

and let $\mathscr{F}$ be the normal filter on $\mathscr{G}$ generated by $\{H_\gamma : \gamma < \kappa\}$. Thus $x \in \mathfrak{M}^B$ is symmetric just in case there is $\gamma < \kappa$ such that $\pi x = x$ whenever $\pi\alpha = \alpha$ for all $\alpha < \gamma$.

Let G be an $\mathfrak{M}$-generic ultrafilter on B and let $\mathfrak{N}$ be the symmetric model given by B, G, $\mathscr{F}$, $\mathscr{G}$. We shall show that $\kappa = (\aleph_1)^{\mathfrak{N}}$ and that κ is measurable in $\mathfrak{N}$.

If $\gamma < \kappa$, then γ is countable in $\mathfrak{N}$: Let f be the name such that

$$\|f(\check{n}) = \check{\alpha}\| = \sum\{p \in P : p(n) = \alpha\}$$

for all $n < \omega$ and $\alpha < \gamma$. Clearly, f is symmetric because $\pi(f) = f$ for every $\pi \in H_\gamma$, and hence $f \in \text{HS}$. The interpretation of f is a one-to-one function of ω onto γ.

It remains to show that κ is measurable in $\mathfrak{N}$. By Lemma 37.5 it suffices to show that every symmetric $A \subseteq B$ has size $< \kappa$. Let $A \subseteq B$ be symmetric. There exists $\gamma < \kappa$ such that $\pi a = a$ for all $a \in A$ and all $\pi \in H_\gamma$.

For every $a \in A$, let $S_a = \{p \in P : p \leq a\}$. If $\pi \in H_\gamma$ and $p \in S_a$, then $\pi p \in S_a$ because $\pi p \leq \pi a = a$. Let $T_a = \{p \in S_a : p(n) < \gamma + \omega \text{ for all } n \in \text{dom}(p)\}$. If $\pi \in H_\gamma$ and $p \in T_a$, then $\pi p \in S_a$. Conversely, if $p \in S_a$, there is $\pi \in H_\gamma$ that maps all $\alpha \in \text{ran}(p)$ greater than γ into $\gamma + \omega$; since $\pi \in H_\gamma$ and $p \in S_a$, we have $\pi p \in S_a$ and hence $\pi p \in T_a$. Thus $S_a = \{\pi(p) : p \in T_a \text{ and } \pi \in H_\gamma\}$, and consequently

$$\text{if } a \neq b \in A, \quad \text{then } T_a \neq T_b \tag{37.12}$$

However, each T_a is a set of finite sequences in $\gamma + \omega$; and since κ is inaccessible, (37.12) implies that $|A| < \kappa$. ■

In connection with measurability of $\aleph_1$, we recall the following open question: Is it consistent that the closed unbounded filter over $\aleph_1$ is a countably complete ultrafilter? It has been proved by the method of Theorem 80 that if that is the case, then for every θ there is a transitive model with θ measurable cardinals.

Let me also mention another open problem. In Section 21 (Example IV) we presented a model of ZF in which $\aleph_1$ is a singular cardinal. By Jensen's covering theorem (see Exercise 30.7), if we want both $\aleph_1$ and $\aleph_2$ singular, then we have to use large cardinals. Indeed, a model is known (using $\kappa < \lambda$ such that κ is supercompact and λ is measurable) in which $\operatorname{cf} \omega_2 = \operatorname{cf} \omega_1 = \omega$. The problem of whether $\operatorname{cf} \alpha = \omega$ for all limit α is consistent is still open.

38. MORE ON ULTRAFILTERS

We have used ultrafilters extensively in our study of large cardinals, particularly ultrafilters that are countably complete. In this last section of Part III, we shall investigate ultrafilters that are not countably complete. First we shall consider ultrafilters over ω that have a partition property related to Ramsey's theorem (Lemma 29.1), and then we shall investigate a property of ultrafilters called regularity.

Ramsey Ultrafilters Over ω

In Section 24, we introduced p-points and Ramsey ultrafilters. Recall that an ultrafilter D over ω is a *p-point* if for every partition $\mathscr{A}$ of ω such that $A \notin D$ for all $A \in \mathscr{A}$, there exists $X \in D$ such that $X \cap A$ is finite for each $A \in \mathscr{A}$. D is a *Ramsey ultrafilter* if for every partition $\mathscr{A}$ of ω such that $A \notin D$ for all $A \in \mathscr{A}$ there exists $X \in D$ such that for every $A \in \mathscr{A}$, $X \cap A$ has one element.

If $2^{\aleph_0} = \aleph_1$, then Ramsey ultrafilters exist: Let $\mathscr{A}_\alpha$, $\alpha < \omega_1$, enumerate all partitions of ω and let us construct an ω_1-sequence of infinite subsets of ω as follows: Given X_α, let $X_{\alpha+1} \subseteq X_\alpha$ be such that either $X_{\alpha+1} \subseteq A$ for some $A \in \mathscr{A}_\alpha$, or that $|X_{\alpha+1} \cap A| \leq 1$ for all $A \in \mathscr{A}_\alpha$. If α is a limit, let X_α be such that $X_\alpha \subseteq X_\beta$ mod.f. for all $\beta < \alpha$. Then $D = \{X : X \supseteq X_\alpha \text{ for some } \alpha < \omega_1\}$ is a Ramsey ultrafilter.

We shall show below (Theorem 91) that existence of Ramsey ultrafilters is not provable in ZFC; we shall exhibit a model of ZFC in which there are no Ramsey ultrafilters.

The following lemma explains the name of Ramsey ultrafilters:

Lemma 38.1. *Let D be a nonprincipal ultrafilter over ω. D is Ramsey if and only if for all natural numbers n and k, every partition F: $[\omega]^n \to \{1, \ldots, k\}$ has a homogeneous set $H \in D$.*

Proof. First assume that D has the partition property stated in the lemma. Let $\mathscr{A}$ be a partition of ω such that $A \notin D$ for all $A \in \mathscr{A}$; we shall find $X \in D$ such that $|X \cap A| \leq 1$ for all $A \in \mathscr{A}$. Let F: $[\omega]^2 \to \{0, 1\}$ be as follows: $F(x, y) = 0$ if x and y are in the same $A \in \mathscr{A}$, and $F(x, y) = 1$ if x and y are in different members of $\mathscr{A}$. If $H \in D$ is homogeneous for F, then clearly H has at most one element common with each $A \in \mathscr{A}$.

Now let us assume that D is a Ramsey ultrafilter. We shall first prove that D has the following property:

(38.1) if $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ are sets in D, then there is a sequence $a_0 < a_1 < a_2 < \cdots$ such that $\{a_n\}_{n=0}^{\infty} \in D$ and $a_{n+1} \in X_{a_n}$ for all n.

Thus let $X_0 \supseteq X_1 \supseteq \cdots$ be sets in D. Since D is a p-point, there exists $Y \in D$ such that each $Y - X_n$ is finite. Let us define a sequence $y_0 < y_1 < \cdots$ in Y as follows:

$$y_0 = \text{least } y_0 \in Y \text{ such that } \{y \in Y : y > y_0\} \subseteq X_0$$

$$y_1 = \text{least } y_1 \in Y \text{ such that } y_1 > y_0 \text{ and } \{y \in Y : y > y_1\} \subseteq X_{y_0}$$

$$\vdots$$

$$y_{n+1} = \text{least } y_{n+1} \in Y \text{ such that } y_{n+1} > y_n \text{ and } \{y \in Y : y > y_{n+1}\} \subseteq X_{y_n}$$

For each n, let $A_n = \{y \in Y : y_n < y \le y_{n+1}\}$. Since D is Ramsey, there exists a set $\{z_n\}_{n=0}^{\infty} \in D$ such that $z_n \in A_n$ for all n.

We observe that for each n, $z_{n+2} \in X_{z_n}$: Since $z_{n+2} > y_{n+2}$, we have $z_{n+2} \in X_{y_{n+1}}$, and since $y_{n+1} \ge z_n$, we have $X_{y_{n+1}} \subseteq X_{z_n}$ and hence $z_{n+2} \in X_{z_n}$.

Thus if we let $a_n = z_{2n}$ and $b_n = z_{2n+1}$, for all n, then either $\{a_n\}_{n=0}^{\infty} \in D$ or $\{b_n\}_{n=0}^{\infty} \in D$; and in either case we get a sequence that satisfies (38.1).

Now we use the property (38.1) to prove the partition property; we proceed by induction on n and follow closely the proof of Lemma 29.1. Let F be a function from $[\omega]^{n+1}$ into $\{1, \ldots, k\}$. For each $a \in \omega$, let F_a be the function on $[\omega - \{a\}]^n$ defined by $F_a(x) = F(x \cup \{a\})$. By the induction hypothesis, there exists for each $a \in \omega$ a set $H_a \in D$ such that F_a is constant on $[H_a]^n$. There exists $X \in D$ such that the constant value of F_a is the same for all $a \in X$; say $F_a(x) = r$ for all $a \in X$ and all $x \in [H_a]^n$.

For each n, let $X_n = X \cap H_0 \cap H_1 \cap \cdots \cap H_n$. By (38.1) there exists a sequence $a_0 < a_1 < a_2 < \cdots$ such that $a_{n+1} \in X_{a_n}$ for each n, and that $\{a_n\}_{n=0}^{\infty} \in D$. Let $H = \{a_n\}_{n=0}^{\infty}$. It is clear that for each $i \in \omega$, $a_i \in X$ and $\{a_m : m > i\} \subseteq H_{a_i}$. Hence $F_{a_i}(x) = r$ for all $x \in [\{a_m : m > i\}]^n$, and it follows that F is constant on $[H]^{n+1}$. ■

If D is an ultrafilter over ω, let us call a function f: $\omega \to \omega$ *unbounded* mod D if $\{n : f(n) > k\} \in D$ for every $k \in \omega$; let us call f *one-to-one* mod D if there is $X \in D$ such that f is one-to-one on X. The following exercise is a straightforward application of the definition of Ramsey ultrafilter:

Exercise 38.1. An ultrafilter D over ω is a Ramsey ultrafilter if and only if every function unbounded mod D is one-to-one mod D.

If D and E are ultrafilters over ω, then $D \le E$ means that for some function f: $\omega \to \omega$,

$$D = f_*(E) = \{X : f_{-1}(X) \in E\}$$

*Exercise 38.2.** If $D = f_*(D)$, then $\{n : f(n) = n\} \in D$.

[Let $X = \{n : f(n) < n\}$, $Y = \{n : f(n) > n\}$. For each $n \in X$, let $l(n)$ be the length of the maximal sequence such that $n > f(n) > f(f(n)) > \cdots$. Let $X_0 = \{n \in X : l(n) \text{ is even}\}$ and $X_1 = \{n \in X : l(n) \text{ is odd}\}$. Neither X_1 nor X_2 can be in D since, e.g., $X_0 \cap f_{-1}(X_0) = \emptyset$. The set Y is handled similarly, except that it remains to show that the set Z of all n such that the sequence $n < f(n) < f^2(n) < f^3(n) < \cdots$ is infinite cannot be in D. For $x, y \in Z$ let $x \equiv y$ if $f^k(x) = f^m(x)$ for some k and m. For each $x \in Z$, let a_x be a fixed representative of the class $\{y : y \equiv x\}$; let $l(x)$ be the least $k + m$ such that $f^k(x) = f^m(a_x)$. Let $Z_0 = \{x \in Z : l(x) \text{ is even}\}$ and $Z_1 = \{x \in Z : l(x) \text{ is odd}\}$. Clearly $f_{-1}(Z_1) \cap Z = Z_0$.]

$D \equiv E$ means that there is a one-to-one function of ω onto ω such that $E = f_*(D)$. $\equiv$ is an equivalence relation (and if $D'_1 \equiv D \leq E \equiv E'$, then $D' \leq E'$).

Exercise 38.3. If $D \leq E$ and $E \leq D$, then $D \equiv E$.

[Use Exercise 38.2.]

Thus $\leq$ is a partial ordering of ultrafilters over ω (or rather their equivalence classes mod $\equiv$). A nonprincipal ultrafilter D over ω is *minimal* if there is no nonprincipal E such that $E \leq D$ and $E \not\equiv D$.

Exercise 38.4. An ultrafilter D over ω is minimal if and only if it is Ramsey.

[If D is Ramsey and $E = f_*(D)$ is nonprincipal, then f is unbounded mod D, hence one-to-one mod D and consequently, $E \equiv D$. If D is minimal and f is unbounded mod D, then $D \leq f_*(D)$ and hence $D = g_*(f_*(D))$ for some g. It follows, by Exercise 38.2, that f is one-to-one mod D.]

We shall now outline the proof of the following theorem which shows that it is not provable in ZFC that Ramsey ultrafilters exist.

Theorem 91 (Kunen). *Let $\mathfrak{M}$ be a transitive model of* ZFC *plus* GCH. *There is a generic extension of $\mathfrak{M}$ in which there are no Ramsey ultrafilters.*

We start with the following observation. Let S be an infinite set. A *finitely additive measure* on S is a real-valued function μ on $P(S)$ such that:

(38.2) (i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$;
(ii) if $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$;
(iii) if X and Y are disjoint, then $\mu(X \cup Y) = \mu(X) + \mu(Y)$;

(compare with (27.1)). A finitely additive measure μ on S is *atomless* if for every $X \subseteq S$ of positive measure there is $Y \subseteq X$ such that $0 < \mu(Y) < \mu(X)$.

Lemma 38.2. *If μ is an atomless finitely additive measure on ω, then there is no p-point extending the filter $\{X \subseteq \omega : \mu(X) = 1\}$.*

Proof. Assume that D is a p-point such that $X \in D$ whenever $\mu(X) = 1$. Hence every $X \in D$ has positive measure. We construct an ω_1-sequence $X_\alpha, \alpha < \omega_1$, of sets in D as follows: Given X_α, let $X_{\alpha+1} \subset X_\alpha$ be such that $X_{\alpha+1} \in D$ and $\mu(X_{\alpha+1}) < \mu(X_\alpha)$. If α is a limit, let $X_\alpha \in D$ be such that $X_\alpha \subseteq X_\beta$ mod.f. for all

$\beta < \alpha$ (see Lemma 24.9). Then $\mu(X_0) > \mu(X_1) > \cdots > \mu(X_\alpha) > \cdots$ is a descending ω_1-sequence of real numbers; a contradiction. ■

Now let $\mathfrak{M}$ be the ground model and assume that $\mathfrak{M}$ satisfies the GCH. Let $\kappa = \omega_2$ and let us consider the measure space ${}^{\kappa}\{0, 1\}$ defined in (34.3); let B be the corresponding measure algebra. For every $\alpha < \kappa$, let B_α be the measure algebra of ${}^{\alpha}\{0, 1\}$, and B^α the measure algebra of ${}^{\kappa-\alpha}\{0, 1\}$. We have $B = B_\alpha \oplus B^\alpha$; and if $\operatorname{cf} \alpha = \omega_1$, then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$. Also, $B = \bigcup_{\alpha<\kappa} B_\alpha$.

Let G be an $\mathfrak{M}$-generic ultrafilter on B. We shall show that in $\mathfrak{M}[G]$ there is no Ramsey ultrafilter over ω. Let $D \in \mathfrak{M}[G]$ be an ultrafilter over ω and let $\boldsymbol{D}$ be a name for D. For each $\alpha < \kappa$ we have $|B_\alpha| \le \aleph_1$ and so there are $\aleph_1$ B_α-valued names for subsets of ω. Moreover, if $\operatorname{cf} \alpha = \omega_1$ and X is a B_α-valued name for a subset of ω, then X is B_β-valued for some $\beta < \alpha$.

For each $\alpha < \omega_1$ there is $\gamma < \omega_1$ such that whenever X is a B_α-valued name for a subset of ω then $\|X \in \boldsymbol{D}\| \in B_\gamma$. Consequently, there exists α (of cofinality ω_1) such that for all $X \in \mathfrak{M}^B$, $\|X \subseteq \check{\omega}\| \in B_\alpha$ implies $\|X \in \boldsymbol{D}\| \in B_\alpha$ and hence if we denote $G_\alpha = G \cap B_\alpha$ and $U = D \cap \mathfrak{M}[G_\alpha]$, then $U \in \mathfrak{M}[G_\alpha]$ and U is an ultrafilter.

A somewhat more complicated argument (which we leave out) shows that if D is a Ramsey ultrafilter in $\mathfrak{M}[G]$, then α can be found such that U is a Ramsey ultrafilter in $\mathfrak{M}[G_\alpha]$.

Now $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}[G_\alpha]$, namely $\mathfrak{M}[G] = \mathfrak{M}[G_\alpha][G^\alpha]$ where G^α is $\mathfrak{M}[G_\alpha]$-generic on B^α. We state without proof that the completion of B^α in $\mathfrak{M}[G_\alpha]$ is a measure algebra (in $\mathfrak{M}[G_\alpha]$). Hence if we apply the following lemma in $\mathfrak{M}[G_\alpha]$, we obtain a contradiction, thus showing that $\mathfrak{M}[G]$ has no Ramsey ultrafilters.

Lemma 38.3. *If U is a Ramsey ultrafilter (in $\mathfrak{M}$) and if B is an atomless measure algebra, then in $\mathfrak{M}[G]$ there is no Ramsey ultrafilter extending U.*

Proof. Let m be a measure on B. We shall use U and m to define a finitely additive measure on ω in $\mathfrak{M}[G]$ in very much the same way as we did in Lemma 34.6.

If $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence of real numbers and if U is an ultrafilter over ω, there is a unique real number a such that for every $\varepsilon > 0$, the set $\{n : |a_n - a| < \varepsilon\}$ is in U. We denote this unique a

$$a = \lim_{U} a_n$$

Now let me follow (34.6)–(34.12). If $a \in B$ is nonzero and if $a \Vdash A \subseteq \check{\omega}$, we let

$$f_a(A, n) = \frac{m(a \cdot \|\check{n} \in A\|)}{m(a)} \quad \text{and} \quad \mu_a(A) = \lim_{U} f_a(A, n)$$

As before, we further let

$$\mu_b^*(A) = \inf_{a \le b} \mu_a(A)$$

and finally

$$\mu(A) = \sup_{b \in G} \mu_b^*(A)$$

All we said about μ in the proof of Lemma 34.6 is again true except that μ is finitely additive, not κ-additive. In particular, if $X \in \mathfrak{M}$ is such that $X \in U$, then $\mu(X) = 1$. Also, since B is atomless, it can be shown that μ is atomless.

We want to apply Lemma 38.2 and show that there is no Ramsey ultrafilter extending U. For this reason, we shall show that the filter $\{X : \mu(X) = 1\}$ coincides with the filter generated by U. Thus it suffices to show that if $\mu(X) = 0$, then there is $Y \in U$ such that $X \cap Y = \varnothing$.

Let us assume, on the contrary, that there is $\boldsymbol{X} \in \mathfrak{M}^B$ such that $\|\boldsymbol{\mu}(\boldsymbol{X}) = 0\| = 1$, and

$$(38.3) \qquad \|(\forall Y \in \check{U})\boldsymbol{X} \cap Y \neq \varnothing\| = b \neq 0$$

Let $\varepsilon = m(b)$. To get a contradiction with (38.3), it suffices to find $Y \in U$ such that $m(\|\boldsymbol{X} \cap \check{Y} \neq \varnothing\|) < \varepsilon$. Since $\|\boldsymbol{X} \cap \check{Y} \neq \varnothing\| = \sum\{\|\check{n} \in \boldsymbol{X}\| : n \in Y\}$, it suffices to find $Y \in U$ such that

$$(38.4) \qquad \sum_{n \in Y} \|\check{n} \in \boldsymbol{X}\| < \varepsilon$$

If $\|\boldsymbol{\mu}(\boldsymbol{X}) = 0\| = 1$, then we note that by (34.9), $\mu_a(A) = 0$ for all $a \in B$. Hence

$$\lim_U \|\check{n} \in \boldsymbol{X}\| = 0$$

and (38.4) follows from this lemma:

Lemma 38.4. *If U is a Ramsey ultrafilter and $\{a_n\}_{n=0}^{\infty}$ is a sequence of nonnegative real numbers such that* $\lim_U a_n = 0$, *then for every $\varepsilon > 0$ there is $Y \in U$ such that $\sum_{n \in Y} a_n < \varepsilon$.*

Proof. For each $k = 1, 2, 3, \ldots$ let $X_i = \{n : \varepsilon/2^{k+1} \le a_n < \varepsilon/2^k\}$. If $\{n : a_n = 0\} \in U$, then there is nothing to prove; otherwise, $\bigcup_{k=1}^{\infty} X_k \in U$, and since $\lim_U a_n = 0$, we have $X_k \notin U$ for all k. Let $Y \in U$ be such that $|Y \cap X_k| \le 1$ for all k. Clearly, $\sum_{n \in Y} a_n < \varepsilon$. ∎

This completes the proof of Theorem 91. ∎

Regularity of Ultrafilters

Let U be a nonprincipal ultrafilter over S. We say that U is *regular* if there exists a family $\{M_x : x \in S\}$ of subsets of S such that

(38.5) (a) each M_x is a finite set;
(b) for every $a \in S$, $\{x : a \in M_x\} \in U$.

[More precisely, we say that U is $(\aleph_0, |S|)$-regular and, in general, call U (κ, λ)-regular if there exists a family $\{M_x : x \in S\}$ of subsets of λ such that

$|M_x| < \kappa$ for all x, and $\{x : \gamma \in M_x\} \in U$ for all $\gamma < \lambda$. (Compare with (33.24).) Many of the results presented below can be generalized.]

Let me recall that an ultrafilter U over S is *uniform* if $|X| = |S|$ for every $X \in U$. It is evident from the definition that every regular ultrafilter is uniform (if $X \in U$ and $|X| < |S|$, then $\bigcup \{M_x : x \in X\} \neq S$).

Exercise 38.5. U is regular if and only if there exists a collection $\{X_a : a \in S\} \subseteq U$ such that for any infinite $Z \subseteq S$, $\bigcap_{a \in Z} X_a = \emptyset$.

[Let $x \in X_a$ iff $a \in M_x$.]

It follows that if U is a countably complete ultrafilter, then U is not regular. Furthermore, it is easy to see that every nonprincipal ultrafilter over a countable set is regular.

Lemma 38.5. *For every infinite cardinal κ there exists a set S of size κ and a regular ultrafilter U over S.*

Proof. Let S be the set of all finite subsets of κ. For each $a \in S$, let $\hat{a} = \{x \in S : x \supseteq a\}$. Let U be an ultrafilter such that $\hat{a} \in U$ for all $a \in S$.

For each $x \in S$, let $M_x = P(x)$. For every $a \in S$, we have $a \in M_x$ if and only if $x \in \hat{a}$. Hence U is regular. ■

Regular ultrafilters have been studied in model theory, in connection with the cardinality problem for ultrapowers. If A is an infinite set, then clearly

$$|A| \leq |Ult_U(A)| \leq |A|^{|S|}$$

for any ultrafilter over S. If U is $|A|^+$-complete, then $|Ult_U(A)| = |A|$, and the next lemma states that if U is regular, then $|Ult_U(A)| = |A|^{|S|}$. There are various results on the size (or order-type) of the ultrapower, and we present some of them below; most of the natural questions about the size of ultrapowers remain open however.

Exercise 38.6. If U is a uniform ultrafilter over S, then $|Ult_U(S)| > |S|$.

[There is a family F of functions $f: S \to S$ such that $|F| = |S|^+$ and that $|\{x : f(x) = g(x)\}| < |S|$ whenever $f \neq g \in F$; compare with Lemma 24.7.]

Lemma 38.6. *Let U be a regular ultrafilter over S and let A be an infinite set. Then*

$$|Ult_U(A)| = |A|^{|S|}$$

Proof. We may assume that $S = \kappa$ is an infinite cardinal. Let B be the set of all finite sequences in A; since A is infinite, we have $|B| = |A|$. We will show that

$$|A|^\kappa \leq |Ult_U(B)| \tag{38.6}$$

Let M_ξ, $\xi < \kappa$, be a family of finite subsets of κ such that each $\alpha < \kappa$ belongs to almost all (mod U) M_ξ. For each $g \in {}^\kappa A$, let $f = F(g)$ be the following function from κ into B:

$$f(\xi) = \langle g(\alpha_0), g(\alpha_1), \ldots, g(\alpha_{k-1})\rangle$$

where $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1}$ enumerates M_ξ. Now if $g \neq h$, then there is $\alpha < \kappa$ such that $g(\alpha) \neq h(\alpha)$, and for almost all ξ, α is in M_ξ and hence $F(g)$ and $F(h)$ differ almost everywhere. Thus F gives a one-to-one mapping of ${}^\kappa A$ into $Ult_U(B)$, proving (38.6). ■

One of the results on size of ultrapowers is the following:

Exercise 38.7. If U is not countably complete, if A is an infinite set, and if $\lambda = |Ult_U(A)|$, then $\lambda^{\aleph_0} = \lambda$.

[Let B be the set of all finite sequences in A; we show that $|Ult\ A|^{\aleph_0} \leq |Ult\ B|$. Let $\{X_n : n < \omega\}$ be a partition of S such that $X_n \notin U$, for all n. If $\vec{f} = \langle f_n : n < \omega\rangle$ is a sequence of functions $f_n : S \to A$, we let $F(\vec{f})$ be the function h: $S \to B$ defined as follows: If $x \in X_n$, let $h(x) = \langle f_0(x), f_1(x), \ldots, f_n(x)\rangle$. If $\vec{f} = \langle f_n : n < \omega\rangle$ and $\vec{g} = \langle g_n : n < \omega\rangle$ are such that for some $n, f_n \neq g_n$ mod U, then $F(\vec{f}) \neq F(\vec{g})$ mod U.]

If α is a limit ordinal, then $Ult_U(\alpha, <)$ is a linear ordering without a greatest element, and we may ask what its cofinality is. The following two exercises deal with this problem for $Ult_U\ \omega$.

Exercise 38.8. If U is not countably complete, then every countable subset of $Ult_U(\omega, <)$ is bounded. Hence the cofinality of $Ult\ \omega$ is at least ω_1.

[Let $\{X_n : n < \omega\}$ be a partition of S such that $X_n \notin U$ for all n. If f_k, $k < \omega$, are functions f_k: $S \to \omega$, let f: $S \to \omega$ be as follows: If $x \in X_n$, let $f(x) = \max(f_0(x), \ldots, f_n(x)) + 1$. Clearly, $f > f_k$ mod U, for all k.]

Exercise 38.9. If U is a regular ultrafilter over S, then every subset of $Ult_U(\omega, <)$ of size $|S|$ is bounded. Hence the cofinality of $Ult\ \omega$ is at least $|S|^+$.

[Let M_x, $x \in S$, be a family of finite subsets of S such that each $a \in S$ is in almost all M_x. If f_a, $a \in S$, are functions f_a: $S \to \omega$, let f: $S \to \omega$ be as follows: If $x \in S$, let $f(x) = \max\{f_a(x) : a \in M_x\} + 1$. Clearly, $f > f_a$ mod U, for all $a \in S$.]

Let us consider now a uniform ultrafilter U over an infinite cardinal κ and the ultrapower $Ult_U\ \kappa$. The *constant functions* are the functions c_γ, $\gamma < \kappa$, such that $c_\gamma(\alpha) = \gamma$ for all $\alpha < \kappa$. A function f: $\kappa \to \kappa$ is *unbounded* if $f > c_\gamma$ mod U for all $\gamma < \kappa$; f is *monotone* if $f(\alpha) \leq f(\beta)$ whenever $\alpha < \beta$. The *diagonal function* is the function $d(\alpha) = \alpha$.

Lemma 38.7. *If U is a regular ultrafilter over κ, then there is no least unbounded function in the ultrapower; i.e., if f: $\kappa \to \kappa$ is unbounded, then there is an unbounded g such that $g < f$* mod *U.*

Proof. Let M_α, $\alpha < \kappa$, be a family of finite subsets of κ such that each $\gamma \in \kappa$ is in almost all M_α; let f: $\kappa \to \kappa$ be an unbounded function. Let g be defined as follows: Let $g(\alpha)$ be the greatest element of M_α less than $f(\alpha)$, if such an element exists. Since f is unbounded, $g(\alpha)$ is defined almost everywhere. It is clear that g is unbounded because $g < c_\gamma$ mod U would imply $f < c_{\gamma+1}$ mod U. ■

The following lemma plays an important role in the investigation of regularity of ultrafilters:

Lemma 38.8. *Let κ be a regular cardinal and let U be a nonregular uniform ultrafilter over κ. Then the ultrapower has a least unbounded monotone function; i.e., there exists an unbounded monotone $f\colon \kappa \to \kappa$ such that there is no unbounded monotone g with $g < f \bmod U$.*

Proof. Let U be a uniform ultrafilter over κ and let us assume that the ultrapower has no least unbounded monotone function. We shall show that U is regular. We shall construct a κ-sequence of unbounded monotone functions f_0, $f_1, \ldots, f_\gamma, \ldots, \gamma < \kappa$, such that if $\gamma < \delta < \kappa$, then $f_\gamma(\alpha) > f_\delta(\alpha)$ for almost all $\alpha < \kappa$, and $f_\gamma(\alpha) \ge f_\delta(\alpha)$ for all α. Then it follows that U is regular, as witnessed by the family $\{M_\alpha : \alpha < \kappa\}$, where

$$M_\alpha = \{\gamma : f_\gamma(\alpha) > f_{\gamma+1}(\alpha)\}$$

Clearly, each γ is in almost all M_α, and each M_α is finite.

We construct the f_γ as follows: Given f_γ, there exists a monotone $f_{\gamma+1} < f_\gamma$ because we assume that there is no least unbounded monotone function (and we let $f_{\gamma+1}(\alpha) \le f_\gamma(\alpha)$ for all α). If γ is a limit ordinal, we define $f_\gamma(\alpha) = \min\{f_\xi(\alpha) : \xi < \gamma\}$ for $\alpha < \kappa$. It is clear that f_γ is monotone. To show that f_γ is unbounded, let $\nu < \kappa$ and let us show that $f_\gamma \ge c_\nu$. For each $\xi < \gamma$ there is α_ξ such that $f_\xi(\beta) \ge \nu$ for all $\beta \ge \alpha_\xi$. Let $\alpha = \sup_{\xi<\gamma} \alpha_\xi$; since κ is regular, we have $\alpha < \kappa$, and clearly $f_\gamma(\beta) \ge \nu$ for all $\beta \ge \alpha$. ∎

Corollary. *Let κ be a regular cardinal. If there exists a nonregular uniform ultrafilter over κ, then there exists one that extends the closed unbounded filter.*

Proof. Let U be a uniform nonregular ultrafilter over κ and let $f\colon \kappa \to \kappa$ be the least unbounded monotone function in Ult_U. Let $D = f_*(U) = \{X : f_{-1}(X) \in U\}$.

D is clearly a uniform ultrafilter over κ, and it is easy to see that the diagonal function is the least unbounded monotone function in Ult_D. Now if C is a closed unbounded subset of κ, then $C \in D$ because otherwise consider the function $g(\alpha) = \sup(C \cap \alpha)$. The function g is unbounded and monotone, and $g(\alpha) < \alpha$ whenever $0 \ne \alpha \notin C$. Hence D extends the closed unbounded filter. (Compare with Exercise 28.12.)

It remains to show that D is not regular. Thus assume, on the contrary, that M_α, $\alpha < \kappa$, is a family of finite sets such that for every $\gamma < \kappa$, the set $X_\gamma = \{\alpha : \gamma \in M_\alpha\}$ is in D. We shall show that U is regular. For each $\alpha < \kappa$, let $N_\alpha = M_{f(\alpha)}$. Each N_α is finite, and if $\gamma < \kappa$, then $\{\alpha : \gamma \in N_\alpha\} = f_{-1}(X_\gamma)$ is in U. ∎

As for existence of nonregular ultrafilters, it is not very difficult to construct one over a measurable cardinal. The problem is to show that a nonregular ultrafilter can exist over a small cardinal. The best result to date in this direction is a model with a nonregular ultrafilter over ω_2.

Nonregular Ultrafilters Over ω_1

We shall now turn our attention to this particular problem: Can there be a nonregular uniform ultrafilter over ω_1? Although the question is open, some consequences of existence of such ultrafilters are known; most significant is Theorem 92 below, which shows that existence of a nonregular ultrafilter over ω_1 implies that $0^\#$ exists.

We start with a combinatorial lemma. Let us recall that two functions f and g on ω_1 are *almost disjoint* if $\{\alpha : f(\alpha) = g(\alpha)\}$ is at most countable.

Lemma 38.9. *Let U be a uniform ultrafilter over ω_1. Each of the following conditions is sufficient for U to be regular:*

(a) *There exist $\aleph_1$ almost disjoint functions f_ξ, $\xi < \omega_1$, from ω_1 into ω, and a function $f: \omega_1 \to \omega$ such that $f_\xi < f \bmod U$ for all ξ.*
(b) *There exist $\aleph_2$ almost disjoint functions from ω_1 into ω.*
(c) *There exist $\aleph_2$ almost disjoint functions f_ξ, $\xi < \omega_2$, from ω_1 into ω_1 and a function $f: \omega_1 \to \omega_1$ such that $f_\xi < f \bmod U$ for all ξ.*
(d) *There exist $\aleph_3$ almost disjoint functions from ω_1 into ω_1.*

Proof. (a) For each $\alpha < \omega_1$, let

$$M_\alpha = \{\gamma : f_\gamma(\alpha) < f(\alpha) \text{ and } f_\gamma(\alpha) \neq f_\xi(\alpha) \text{ for all } \xi < \gamma\}$$

Since for each α, $f(\alpha)$ is a natural number, it is easy to see that each M_α is finite. Also, every γ belongs to almost all M_α because the set $\{\alpha : f_\gamma(\alpha) = f_\xi(\alpha) \text{ for some } \xi < \gamma\}$ is countable and thus $\notin U$.

(b) Let $\mathscr{F}$ be a family of $\aleph_2$ almost disjoint functions from ω_1 to ω. Distinct functions in $\mathscr{F}$ represent distinct ordinals in the ultrapower Ult_U; these ordinals thus form a linearly ordered set of size $\aleph_2$. Therefore there exists $f \in \mathscr{F}$ such that for uncountably many $g \in \mathscr{F}$ we have $g < f \bmod U$. Now we apply (a).

(c) For every $\xi < \omega_2$, let f'_ξ be the restriction of f_ξ to the set $\{\alpha : f_\xi(\alpha) < f(\alpha)\}$. Since for each α, $f(\alpha)$ is countable, it follows that there exists a family $\{g_\xi : \xi < \omega_2\}$ of functions with $\operatorname{dom}(g_\xi) \in U$ and $\operatorname{ran}(g_\xi) \subseteq \omega$ such that if $\xi \neq \eta$, then g_ξ and g_η are almost disjoint, i.e., $\{\alpha : g_\xi(\alpha) = g_\eta(\alpha)\}$ is at most countable. The same argument as in (b) shows that there is a function $g: \omega_1 \to \omega$ and a family $\{h_\xi : \xi < \omega_1\}$ of almost disjoint functions such that for each $\xi < \omega_1$, $\operatorname{dom}(h_\xi) \in U$ and $h_\xi < g \bmod U$. To prove that U is regular, we argue as in (a), using the sets $M_\alpha = \{\gamma : \alpha \in \operatorname{dom}(h_\gamma),\ h_\gamma(\alpha) < g(\alpha) \text{ and } h_\gamma(\alpha) \neq h_\xi(\alpha) \text{ for all } \xi < \gamma\}$.

(d) Let $\mathscr{F}$ be a family of $\aleph_3$ almost disjoint function from ω_1 into ω_1. The ordinals represented in Ult_U by functions in $\mathscr{F}$ form a linearly ordered set of size $\aleph_3$, and therefore there exists $f \in \mathscr{F}$ such that there are $\aleph_2$ functions $g \in \mathscr{F}$ with $g < f \bmod U$. Now we apply (c). ■

Note that (a) implies that the cofinality of $Ult_U(\omega, <)$ is ω_1 if U is a nonregular uniform ultrafilter over ω_1. Note also that by (b), if there exists a Kurepa tree, then every uniform ultrafilter over ω_1 is regular (and hence if there is a nonregular uniform ultrafilter over ω_1, then $\aleph_2$ is an inaccessible cardinal in L).

Corollary. *Assume that there exists a nonregular uniform ultrafilter over ω_1. Then:*

(a) *if $2^{\aleph_0} = \aleph_1$, then $2^{\aleph_1} = \aleph_2$;*
(b) *if $\aleph_1 < 2^{\aleph_0} < \aleph_{\omega_1}$, then $2^{\aleph_1} = 2^{\aleph_0}$.*

Proof. (a) We recall that there exist $2^{\aleph_1}$ almost disjoint functions from ω_1 into $2^{\aleph_0}$. If $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_2$, then we have $\aleph_3$ almost disjoint functions from ω_1 into ω_1 and hence U is regular.

(b) Let us assume that $\aleph_1 < 2^{\aleph_0} < \aleph_{\omega_1}$ and that $2^{\aleph_0} < 2^{\aleph_1}$. Then $2^{\aleph_1} \geq \aleph_3$ and by Lemma 35.2 there exists a family of $\aleph_3$ almost disjoint functions from ω_1 into ω_1, and hence U is regular. ■

In the proof of Lemma 38.9 we used almost disjointness of the functions f_ξ only to make sure that (1) $f_\xi \neq f_\eta$ mod U if $\xi \neq \eta$, and (2) $\{\alpha : f_\gamma(\alpha) = f_\xi(\alpha)$ for some $\xi < \gamma\} \notin U$ (if γ is countable). Thus the lemma can be generalized:

Lemma 38.10. *Let D be an ultrafilter over ω_1 extending the closed unbounded filter. Throughout* (a)–(d) *of Lemma 38.9, replace "almost disjoint" by "$\{\alpha : f_\xi(\alpha) = f_\eta(\alpha)\}$ is thin." Then each of the modified conditions* (a)–(d) *implies that D is regular.* ■

Corollary. *Assume that there exists a nonregular uniform ultrafilter over ω_1. If $\aleph_{\omega_1}$ is a strong limit limit cardinal, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$.*

Proof. By Theorem 24 (see Lemma 8.7), $2^{\aleph_{\omega_1}} < \aleph_\theta$ where θ is the norm of the constant function c_{ω_1} on ω_1 (or, in other words, θ is the length of the well-founded relation $f \prec g$ ($f, g : \omega_1 \to \omega_1$) defined by $f \prec g$ iff $\{\alpha : f(\alpha) < g(\alpha)\}$ contains a closed unbounded set). Thus it suffices to show that there is no function f: $\omega_1 \to \omega_1$ such that $\|f\| = \omega_2$.

By exercise 35.13, if there is f: $\omega_1 \to \omega_1$ that has norm ω_2, then there exist $\aleph_2$ functions f_γ: $\omega_1 \to \omega$ such that any two differ on a closed unbounded set. By Lemma 38.10b, any ultrafilter over ω_1 extending the closed unbounded filter must be regular. Hence by the corollary of Lemma 38.8, every uniform ultrafilter over ω_1 is regular. ■

The following lemma is not necessary for the forthcoming proof of Theorem 92, but shows that a nonregular ultrafilter over ω_1 can be "normalized":

Lemma 38.11. *Let U be a uniform nonregular ultrafilter over ω_1 and let f: $\omega_1 \to \omega_1$ be an unbounded function in Ult_U. Then there exists a monotone unbounded function h such that $h \leq f$* mod U.

Proof. For every $\beta < \omega_1$, let $P_\beta = \{\alpha : f(\alpha) = \beta\}$; for every $\beta_0 < \omega_1$ we have $\bigcup \{P_\beta : \beta < \beta_0\} \notin U$. We claim that there exists a function $g\colon \omega_1 \to \omega_1$ such that $g(f(\alpha)) \geq \alpha$ for almost all α.

To prove the claim, let $\{g_\xi : \xi < \omega_2\}$ be a family of almost disjoint functions from ω_1 into ω_1 (see Lemma 24.7). For each $\xi < \omega_2$, let $f_\xi = g_\xi \circ f$. We note that $f_\xi \neq f_\eta$ mod U whenever $\xi \neq \eta$, and that if $\gamma < \omega_2$ and $S \subseteq \omega_2$ is countable such that $\gamma \notin S$, then

$$\{\alpha : f_\gamma(\alpha) = f_\xi(\alpha) \text{ for some } \xi \in S\} \notin U \tag{38.7}$$

Let us assume that for all $\xi < \omega_2$, $f_\xi < d$ mod U (where d is the diagonal function). Now the property (38.7) is sufficient for the arguments in Lemma 38.9(c), (b), and (a) (see the remark preceding Lemma 38.10); and hence the existence of functions f_ξ, $\xi < \omega_2$, with property (38.7) and bounded by d implies that U is regular. Hence there is ξ such that $g_\xi \circ f \geq d$ mod U, and the claim is proved.

Now we let $A = \{\alpha : g(f(\alpha)) \geq \alpha\}$; we have $A \in U$. We shall show that for every $\beta < \omega_1$, the set $A \cap P_\beta$ is at most countable. If $f(\alpha) = \beta$ and $\alpha \in A$, then $g(\beta) \geq \alpha$; hence $A \cap P_\beta \subseteq g(\beta)$. Thus the function $f \mid A$ has the property that each value β is taken at most countably many times. It is now easy to construct an unbounded monotone function $h\colon \omega_1 \to \omega_1$ such that $h(\alpha) \leq f(\alpha)$ for all $\alpha \in A$. ■

Corollary. *If U is a uniform nonregular ultrafilter over ω_1, then the ultrapower has a least unbounded function. Consequently, if there is a nonregular uniform ultrafilter over ω_1, then there is one that is normal in the following sense: Every regressive function is bounded in the ultrapower by some constant function c_γ, $\gamma < \omega_1$.*

Proof. By Lemma 38.8, there is a least monotone unbounded function f; and by Lemma 38.11, f is the least unbounded function. Now $D = f_*(U)$ is a nonregular ultrafilter and has the property that the diagonal is the least unbounded function. ■

Theorem 92 (Ketonen). *If there is a uniform nonregular ultrafilter over ω_1, then $0^\#$ exists.*

Proof. Let us assume that there exists a uniform nonregular ultrafilter over ω_1. By the corollary of Lemma 38.8, there exists a nonregular ultrafilter over ω_1 that extends the closed unbounded filter. Let D be such ultrafilter.

To show that $0^\#$ exists it suffices to produce a nontrivial elementary embedding of L. We shall show that there is a nonprincipal L-ultrafilter U over a countable ordinal α such that $Ult_U(L)$ is well-founded; then the natural embedding j_U is a nontrivial embedding of L into L.

We start the proof by constructing functions $f_\gamma\colon \omega_1 \to \omega_1$, for $\aleph_2$ ordinals γ. Let Γ be the set of all limit ordinals γ such that $\omega_1 < \gamma < \omega_2$ and that L_γ is an elementary submodel of $\langle L_{\omega_2}, \in\rangle$. Let $\gamma \in \Gamma$. We consider a relation E_γ on ω_1 such that the model $\langle \omega_1, E_\gamma\rangle$ is isomorphic to $\langle L_\gamma, \in\rangle$. Let $\pi_\gamma\colon L_\gamma \to \omega_1$ be the unique isomorphism. Let C_γ be the set of all $\alpha < \omega_1$ such that:

(38.8) (i) $\langle \alpha, E_\gamma\rangle$ is an elementary submodel of $\langle \omega_1, E_\gamma\rangle$ and $\pi_\gamma(\omega_1) \in \alpha$;
(ii) for all $\xi < \alpha$, $\pi_\gamma(\xi) \in \alpha$;
(iii) for all $\xi < \omega_1$, if $\alpha \le \xi$, then $\pi_\gamma(\xi) \notin \alpha$.

It is rather easy to verify that C_γ is a closed unbounded subset of ω_1. Moreover, if $\alpha \in C_\gamma$, then $\langle \alpha, E_\gamma\rangle$ is isomorphic to some (unique) L_β; let $f_\gamma(\alpha)$ be such that $\langle \alpha, E_\gamma\rangle$ is isomorphic to $\langle L_{f_\gamma(\alpha)}, \in\rangle$, and let $\rho_\gamma^\alpha\colon L_{f_\gamma(\alpha)} \to \alpha$ be the unique isomorphism. Let j_γ^α be the elementary embedding

$$j_\gamma^\alpha\colon L_{f_\gamma(\alpha)} \to L_\gamma$$

defined by $j_\gamma^\alpha(x) = \pi_\gamma^{-1}(\rho_\gamma^\alpha(x))$. If $\xi < \alpha$, then (38.8) (ii) implies that $j_\gamma^\alpha(\xi) = \xi$. By (iii), $j_\gamma^\alpha(\alpha) \ge \omega_1$; and since $\pi_\gamma(\omega_1)$ is in $\langle \alpha, E_\gamma\rangle$, we have $j_\gamma^\alpha(\alpha) = \omega_1$. Thus α is the least ordinal moved by j_γ^α, and $j_\gamma^\alpha(\alpha) = \omega_1$.

We have defined, for each $\gamma \in \Gamma$, a function f_γ from C_γ into ω_1.

Lemma 38.12. If γ, $\delta \in \Gamma$ *and* $\gamma < \delta$, *there exists a closed unbounded set* $C_{\gamma\delta}$ *such that* $f_\gamma(\alpha) < f_\delta(\alpha)$ *for all* $\alpha \in C_{\gamma\delta}$.

Proof. Let $\gamma < \delta \in \Gamma$. Let $i\colon \omega_1 \to \omega_1$ be the embedding of $\langle \omega_1, E_\gamma\rangle$ into $\langle \omega_1, E_\delta\rangle$ corresponding to the inclusion of L_γ in L_δ (see the diagram below):

$$\begin{tikzcd}
L_{f_\gamma(\alpha)} \arrow[r, "\rho_\gamma^\alpha"'] \arrow[rr, bend left, "j_\gamma^\alpha"] & \langle \alpha, E_\gamma\rangle \prec \langle \omega_1, E_\gamma\rangle \arrow[d, "i"'] & L_\gamma \arrow[l, "\pi_\gamma"] \arrow[d, "\text{inclusion}"] \\
L_{f_\delta(\alpha)} \arrow[r, "\rho_\delta^\alpha"'] \arrow[rr, bend right, "j_\delta^\alpha"'] & \langle \alpha, E_\delta\rangle \prec \langle \omega_1, E_\delta\rangle & L_\delta \arrow[l, "\pi_\delta"]
\end{tikzcd}$$

FIGURE 38.1

We let

$$C_{\gamma\delta} = \{\alpha \in C_\gamma \cap C_\delta : \pi_\delta(L_\gamma) \in \alpha \text{ and } (\forall \xi \in \alpha) i(\xi) \in \alpha\} \tag{38.9}$$

Clearly, $C_{\gamma\delta}$ is closed unbounded. Let $\alpha \in C_{\gamma\delta}$; we shall show that $f_\gamma(\alpha) < f_\delta(\alpha)$. Let $\beta = f_\gamma(\alpha)$; we shall find a transitive set $z \in L_{f_\delta(\alpha)}$ such that $\beta \subseteq z$.

We follow the diagram in Fig. 38.1 from the upper left corner to the lower left corner. For each $\nu < \beta$, consider $i(\rho_\gamma^\alpha(\nu))$. This element of $\langle \omega_1, E_\delta\rangle$ is in $\langle \alpha, E_\delta\rangle$ (because of 38.9); also, it is in $\operatorname{ext}_{E_\delta}(\pi_\delta(L_\gamma))$. Thus $\pi_\delta(L_\gamma)$ has in $\langle \alpha, E_\delta\rangle$ a subset of E_δ-order-type β. So we let $z = (\rho_\delta^\alpha)^{-1}\pi_\delta(L_\gamma)$. Now it follows that $f_\delta(\alpha) > \beta = f_\gamma(\alpha)$. ■

Thus f_γ, $\gamma \in \Gamma$, are $\aleph_2$ functions from ω_1 into ω_1 such that any two differ on a closed unbounded set. Since D is a nonregular ultrafilter extending the closed unbounded filter, it is necessary, by Lemma 38.10c that for every function $f\colon \omega_1 \to \omega_1$ there is $\gamma \in \Gamma$ such that $f < f_\gamma$ mod D. This is the key element of the proof of Theorem 92.

Lemma 38.13. *ω_1 is inaccessible in L.*

Proof. Assume that ω_1 is not inaccessible in L. Then there is α_0 such that for all α such that $\alpha_0 \le \alpha < \omega_1$, α is not a cardinal in L. For each $\alpha \ge \alpha_0$, let $f(\alpha)$ be a countable ordinal such that $L_{f(\alpha)} \vDash \alpha$ is not a cardinal.

There exists $\gamma \in \Gamma$ such that $f < f_\gamma$ mod D. Hence for almost all α(mod D), $L_{f_\gamma(\alpha)} \vDash \alpha$ is not a cardinal. However, for almost all α there is an elementary embedding j_γ^α of $L_{f_\gamma(\alpha)}$ into L_γ such that $j_\gamma^\alpha(\alpha) = \omega_1$, and L_γ is an elementary submodel of L_{ω_2}. This is a contradiction because $L_{\omega_2} \vDash \omega_1$ is a cardinal. ■

Now consider the function $f(\alpha) = (\alpha^+)^L$; since ω_1 is inaccessible in L, f is a function from ω_1 into ω_1. Thus there exists $\gamma_0 \in \Gamma$ such that $f < f_{\gamma_0}$ mod D; let $A \in D$ be such that for all $\alpha \in A$, $f_{\gamma_0}(\alpha) > (\alpha^+)^L$.

If $\gamma \in \Gamma$ is such that $\gamma > \gamma_0$, then for every $\alpha \in A \cap C_{\gamma_0\gamma}$, $f_\gamma(\alpha) > (\alpha^+)^L$; hence $L_{f_\gamma(\alpha)}$ contains all constructible subsets of α. Therefore, the elementary embedding $j_\gamma^\alpha\colon L_{f_\gamma(\alpha)} \to L_\gamma$ defines an L-ultrafilter U_γ^α over α:

$$x \in U_\gamma^\alpha \quad \text{iff} \quad \alpha \in j_\gamma^\alpha(x) \qquad (x \subseteq \alpha,\quad x \in L) \tag{38.10}$$

Lemma 38.14. *If $\gamma < \delta$ are in Γ and $\gamma_0 < \gamma$, then for almost all α, $U_\gamma^\alpha = U_\delta^\alpha$.*

Proof. Let $\alpha \in A \cap C_{\gamma_0\gamma} \cap C_{\gamma_0\delta} \cap C_{\gamma\delta}$; we shall show that $U_\gamma^\alpha = U_\delta^\alpha$. Let us use again the diagram in Fig. 38.1. Let $x \subseteq \alpha$ be in L. We have

$$x \in U_\gamma^\alpha \quad \text{iff} \quad \alpha \in j_\gamma^\alpha(x), \qquad x \in U_\delta^\alpha \quad \text{iff} \quad \alpha \in j_\delta^\alpha(x)$$

Thus it suffices to show that $j_\gamma^\alpha(x) = j_\delta^\alpha(x)$ for all $x \subseteq \alpha$, $x \in L$.

We follow the diagram in Fig. 38.1 from the upper left corner to the lower left corner. Let $x \subseteq \alpha$, $x \in L$, and let $z = j_\gamma^\alpha(x)$. We denote $u = \rho_\gamma^\alpha(x) = \pi_\gamma(z)$, and $v = i(u) = \pi_\delta(z)$. By (38.9), we have $v \in \alpha$, and therefore $v = \rho_\delta^\alpha(y)$ for some $y \in L_{f_\delta(\alpha)}$. Hence $z = j_\delta^\alpha(y)$.

Now both j_γ^α and j_δ^α have the property that α is the least ordinal moved. Therefore $j_\gamma^\alpha(x) \cap \alpha = x$, and $j_\delta^\alpha(y) \cap \alpha = y$. Thus we have $y = z \cap \alpha = x$, and hence $j_\gamma^\alpha(x) = j_\delta^\alpha(x)$. ■

We are now ready to finish the proof of Theorem 92. It suffices to show that for some γ and α for which U_γ^α is defined, the ultrapower $Ult_{U_\gamma^\alpha}(L)$ is well-founded. Thus let us assume that the ultrapower is not well-founded for any γ and α.

Let $\gamma > \gamma_0$ be a fixed ordinal in Γ. For each $\alpha \in A \cap C_{\gamma_0\gamma}$ there is a sequence of constructible functions $g_0, g_1, \ldots, g_n, \ldots$ on α such that for each n, $\{\xi < \alpha : g_n(\xi) > g_{n+1}(\xi)\} \in U_\gamma^\alpha$. As in the proof of Lemma 32.12, we observe that if there are such functions, then there are such functions with values $< \omega_1$, and hence we fix for each α a descending (mod U_γ^α) sequence $\{g_n^\alpha : n < \omega\}$ of functions from α into ω_1.

For each α there exists a countable ordinal β such that $g_n^\alpha \in L_\beta$, for every n. Let $f(\alpha)$ denote such β. The function f is defined on $A \cap C_{\gamma_0\gamma}$ with values $<\omega_1$. Therefore there exists $\delta > \gamma$ such that $f < f_\delta$ mod D. Hence for almost all α, $g_n^\alpha \in L_{f_\delta(\alpha)}$ for all n. By Lemma 38.14, $U_\gamma^\alpha = U_\delta^\alpha$ for almost all α; and therefore there exists α such that U_δ^α is defined, that $g_n^\alpha \in L_\delta(\alpha)$ for all α, and that $g_n^\alpha > g_{n+1}^\alpha$ mod U_δ^α for all n.

Now we are reaching a contradiction. For each n, g_n^α is in $L_{f_\delta(\alpha)}$ and hence $j_\delta^\alpha(g_n^\alpha)$ is defined, and moreover, $g_n^\alpha > g_{n+1}^\alpha$ mod U_δ^α means that

$$(j_\delta^\alpha(g_n^\alpha))(\alpha) > (j_\delta^\alpha(g_{n+1}^\alpha))(\alpha).$$

This gives a descending sequence of ordinals

$$(j_\delta^\alpha g_0^\alpha)(\alpha) > (j_\delta^\alpha g_1^\alpha)(\alpha) > \cdots,$$

a contradiction. ■

PART IV

Sets of Reals

CHAPTER 7

DESCRIPTIVE SET THEORY

39. BOREL AND ANALYTIC SETS

Descriptive set theory deals with sets of reals that are described in some simple way: sets that have a simple topological structure (e.g., continuous images of closed sets) or are definable in a simple way. The main theme is that questions that are difficult to answer if asked for arbitrary sets of reals, become much easier when asked for sets that have a simple description. An example of that is the Cantor–Bendixson theorem (Theorem 14): Every closed set of reals is either at most countable or has size $2^{\aleph_0}$; compare this with the independence of the continuum hypothesis.

Since properties of definable sets can usually be established effectively, without use of the axiom of choice, we shall work in set theory *without the axiom choice*. When some statement depends on the axiom of choice, I shall explicitly say so. However, we shall assume a weak form of the axiom of choice. The reason is that in descriptive set theory one frequently considers unions and intersections of countably many sets of reals, and we shall often use facts like "the union of countably many countable sets is countable." Thus we shall work, throughout this chapter, in set theory ZF + DC, where DC is the *principle of dependent choices* (cf. (5.4)):

(39.1) If ρ is a binary relation over a set A and if for every $a \in A$ there exists $b \in A$ such that $b \mathrel{\rho} a$, then there is a sequence $a_0, a_1, \ldots, a_n, \ldots$ in A such that $a_{n+1} \mathrel{\rho} a_n$ for all $n \in N$.

Recall some consequences of the principle of dependent choices:

(39.2) (a) The countable axiom of choice (Exercise 5.7).
- (b) Every infinite set has a countable subset (Exercise 5.4).
- (c) The union of countably many countable sets is countable.
- (d) ω_1 is a regular cardinal.

(e) A linear ordering without infinite descending chains is a well-ordering, and a binary relation without infinite descending chains is well-founded (Lemma 5.2).

Most of the arguments in descriptive set theory apply not only to subsets of $\boldsymbol{R}$ (the real line), but to a number of other topological spaces with a similar structure; the n-dimensional Euclidean space $\boldsymbol{R}^n$, the closed unit interval $[0, 1]$, the Cantor space $\boldsymbol{C}$, or the Baire space $\mathcal{N} = {}^{N}N$.

A topological space $\mathcal{X}$ is a *Polish space* if it is homeomorphic to a complete separable metric space. [A set $D \subseteq \mathcal{X}$ is *dense* if it intersects every nonempty open set; $\mathcal{X}$ is *separable* if it has a countable dense subset. If $d(x, y)$ is a metric on $\mathcal{X}$, then $\{a_n\}_{n=0}^{\infty}$ is a *Cauchy sequence* if $\lim_{m,n\to\infty} d(a_n, a_m) = 0$; a metric space $\mathcal{X}$ is *complete* if every Cauchy sequence in $\mathcal{X}$ converges to some point $a \in \mathcal{X}$.]

The spaces mentioned above are Polish spaces. For instance, the Baire space $\mathcal{N}$ can be endowed with the following metric: If $a, b \in {}^{N}N$ and $a \neq b$, let

$$(39.3) \qquad d(a, b) = \frac{1}{2^n} \qquad \text{where} \quad n \text{ is the least } n \text{ such that } a(n) \neq b(n)$$

As a matter of fact, the Baire space is somewhat easier to work with than the real line, and we shall therefore, starting in the next section, work almost exclusively with sets in the Baire space.

Exercise 39.1. For each n, the product space $\mathcal{N}^n$ is homeomorphic to $\mathcal{N}$. The product space ${}^{\omega}\mathcal{N}$ is homeomorphic to $\mathcal{N}$.

Note that the Baire category theorem (Theorem 13) holds for any Polish space $\mathcal{X}$: Replace the rational intervals by a countable basis for the topology and use completeness of the metric (and the countable axiom of choice). The facts on compact sets given in Exercise 4.5 hold for any topological space. [A *compact set* is closed set C with the property that whenever C is covered by a collection of open sets, $C \subseteq \bigcup_{i \in I} G_i$, then there are *finitely many* $i_1, \ldots, i_k \in I$ such that $C \subseteq G_{i_1} \cup \cdots \cup G_{i_k}$.]

Exercise 39.2. A closed set $F \subseteq \mathcal{N}$ is compact if and only if the tree T_F has the following property: Every node $s \in T_F$ has only finitely many immediate successors $s^\frown n_1, \ldots, s^\frown n_k$ in T_F.

Borel Sets

A *σ-algebra of sets* is an algebra of sets (see Section 17) closed under countable unions and intersections: $\mathcal{B} \subseteq P(S)$ is a σ-algebra (of subsets of S) if:

(39.4) (i) $S \in \mathcal{B}$;

(ii) if $X \in \mathcal{B}$, then $S - X \in \mathcal{B}$;

(iii) if $X_n \in \mathcal{B}$, $n = 0, 1, \ldots$, then $\bigcup_{n=0}^{\infty} X_n \in \mathcal{B}$ and $\bigcap_{n=0}^{\infty} X_n \in \mathcal{B}$.

Let $\mathscr{X}$ be a Polish space. A set $A \subseteq \mathscr{X}$ is a ***Borel set*** if it belongs to the smallest σ-algebra of subsets of $\mathscr{X}$ containing all closed sets.

Let me first give a more explicit description of Borel sets. For each $\alpha < \omega_1$, let us define the collections $\mathbf{\Sigma}_\alpha^0$ and $\mathbf{\Pi}_\alpha^0$ of subsets of $\mathscr{X}$:

(39.5)
$$\begin{aligned}
\mathbf{\Sigma}_1^0 &= \text{the collection of all open sets}\\
\mathbf{\Pi}_1^0 &= \text{the collection of all closed sets}\\
\mathbf{\Sigma}_\alpha^0 &= \text{the collection of all sets } A = \bigcup_{n=0}^{\infty} A_n, \text{ where each } A_n\\
&\qquad \text{belongs to } \mathbf{\Pi}_\beta^0 \text{ for some } \beta < \alpha\\
\mathbf{\Pi}_\alpha^0 &= \text{the collection of all complements of sets in } \mathbf{\Sigma}_\alpha^0\\
&= \text{the collection of all sets } A = \bigcap_{n=0}^{\infty} A_n, \text{ where each } A_n\\
&\qquad \text{belongs to } \mathbf{\Sigma}_\beta^0 \text{ for some } \beta < \alpha
\end{aligned}$$

It is clear (by induction on α) that the elements of each $\mathbf{\Sigma}_\alpha^0$ and each $\mathbf{\Pi}_\alpha^0$ are Borel sets. Since every open set is the union of countably many closed sets, we have $\mathbf{\Sigma}_1^0 \subseteq \mathbf{\Sigma}_2^0$, and consequently

$$\text{if} \quad \alpha < \beta, \qquad \text{then} \quad \mathbf{\Sigma}_\alpha^0 \subseteq \mathbf{\Sigma}_\beta^0, \quad \mathbf{\Sigma}_\alpha^0 \subseteq \mathbf{\Pi}_\beta^0, \quad \mathbf{\Pi}_\alpha^0 \subseteq \mathbf{\Pi}_\beta^0, \quad \mathbf{\Pi}_\alpha^0 \subseteq \mathbf{\Sigma}_\beta^0$$

Hence

(39.6)
$$\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}_\alpha^0 = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}_\alpha^0$$

and it is easy to verify that the collection (39.6) is a σ-algebra (here we use the countable axiom of choice). Hence every Borel set is in some $\mathbf{\Sigma}_\alpha^0$, $\alpha < \omega_1$.

Note that each $\mathbf{\Sigma}_\alpha^0$ (and each $\mathbf{\Pi}_\alpha^0$) is closed under finite unions, finite intersections, and inverse images by continuous functions (i.e., if $A \in \mathbf{\Sigma}_\alpha^0$ in $\mathscr{Y}$, then $f_{-1}(A) \in \mathbf{\Sigma}_\alpha^0$ in $\mathscr{X}$ whenever $f\colon \mathscr{X} \to \mathscr{Y}$ is a continuous function).

The sets in $\mathbf{\Sigma}_2^0$ are customarily called F_σ-sets; the elements of $\mathbf{\Pi}_2^0$ are the G_δ-sets.

Exercise 39.3. Let f be a real-valued function on $\boldsymbol{R}$. Show that the set of all points at which f is continuous is a G_δ-set.

If the Polish space $\mathscr{X}$ is countable, then of course every $A \subseteq \mathscr{X}$ is a Borel set, in fact an F_σ-set. Uncountable Polish spaces are however more interesting: Not all sets are Borel, and the collections $\mathbf{\Sigma}_\alpha^0$ form a hierarchy. We shall show that for each α, $\mathbf{\Sigma}_\alpha^0 \nsubseteq \mathbf{\Pi}_\alpha^0$, and hence

$$\mathbf{\Sigma}_1^0 \subset \mathbf{\Sigma}_2^0 \subset \cdots \subset \mathbf{\Sigma}_\alpha^0 \subset \cdots$$

(where $\subset$ denotes proper inclusion).

The following lemma holds for any uncountable Polish space; I shall however prove it only for the Baire space $\mathcal{N}$.

Lemma 39.1. *For each* $\alpha \geq 1$ *there exists a set* $U \subseteq \mathcal{N}^2$ *such that* U *is* Σ^0_α *(in* $\mathcal{N}^2$*), and that for every* Σ^0_α *set* A *in* $\mathcal{N}$ *there exists* $a \in \mathcal{N}$ *such that*

$$A = \{x : (x, a) \in U\} \tag{39.7}$$

(U is a *universal* Σ^0_α set.) See Fig. 39.1.

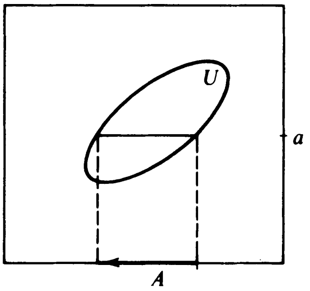

FIGURE 39.1

Proof. By induction on α. To construct a universal open set in $\mathcal{N}^2$, let $G_1, \ldots, G_k, \ldots$ be an enumeration of all basic open sets in $\mathcal{N}$, and let $G_0 = \emptyset$. Let

$$(x, y) \in U \quad \text{iff} \quad x \in G_{y(n)} \quad \text{for some } n \tag{39.8}$$

Since $U = \bigcup_{n=0}^{\infty} H_n$ where each $H_n = \{(x, y) : x \in G_{y(n)}\}$ is an open set in $\mathcal{N}^2$, we see that U is open. Now if G is an open set in $\mathcal{N}$, we let $a \in {}^N N$ be such that $G = \bigcup_{n=0}^{\infty} G_{a(n)}$; then $G = \{x : (x, a) \in U\}$.

Next let U be a universal Σ^0_α set, and let us construct a universal $\Sigma^0_{\alpha+1}$ set V. Let us consider some continuous mapping of $\mathcal{N}$ onto the product space ${}^\omega\mathcal{N}$; for each $a \in \mathcal{N}$ and each n, let $a_{(n)}$ be the nth coordinate of the image of a. [For instance, let us define $a_{(n)}$ as follows: $a_{(n)}(k) = a(\Gamma(n, k))$, where Γ is the canonical one-to-one correspondence Γ: $N \times N \to N$.] Now let

$$(x, y) \in V \quad \text{iff} \quad \text{for some } n, \quad (x, y_{(n)}) \notin U \tag{39.9}$$

Since $V = \bigcup_{n=0}^{\infty} H_n$ where each $H_n = \{(x, y) : (x, y_{(n)}) \notin U\}$ is a Π^0_α-set, we see that V is $\Sigma^0_{\alpha+1}$. If A is a $\Sigma^0_{\alpha+1}$ set in $\mathcal{N}$, then $A = \bigcup_{n=0}^{\infty} A_n$ where each A_n is Π^0_α. For each n, let a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U\}$, and let a be such that $a_{(n)} = a_n$ for all n. Then $A = \{x : (x, a) \in V\}$.

Finally, let α be a limit ordinal, and let U_β, $1 \leq \beta < \alpha$, be universal Σ^0_β-sets. Let $1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots$ be an increasing sequence of ordinals such that $\lim_{n\to\infty} \alpha_n = \alpha$. We notice that if A is a Σ^0_α-set, then there exist sets A_n, $n < \omega$, such that $A = \bigcup_{n=0}^{\infty} A_n$ and that $A_n \in \Sigma^0_{\alpha_n}$ for all n. Thus if we let

$$(x, y) \in U \quad \text{iff} \quad \text{for some } n, \quad (x, y_{(n)}) \notin U_{\alpha_n} \tag{39.10}$$

the same argument as above shows that U is a universal Σ^0_α set. ■

Corollary. *For every $\alpha \geq 1$, there is a set $A \subseteq \mathcal{N}$ that is Σ^0_α but not Π^0_α.*

Proof. Let $U \subseteq \mathcal{N}^2$ be a universal Σ^0_α-set. Let us consider the set

$$A = \{x : (x, x) \in U\} \tag{39.11}$$

Clearly, A is a Σ^0_α-set. If A were Π^0_α, then its complement would be Σ^0_α and there would be some a such that

$$A = \{x : (x, a) \notin U\}$$

But this contradicts (39.11): Simply let $x = a$. ■

While the collection of Borel sets of reals is closed under Boolean operations, and countable unions and intersections, it is not closed under continuous images: As we shall learn presently, the image of a Borel set by a continuous function need not be a Borel set. We shall now investigate the continuous images of Borel sets.

A subset of A of a Polish space $\mathcal{X}$ is called *analytic* if there exists a continuous function $f\colon \mathcal{N} \to \mathcal{X}$ such that $A = f[\mathcal{N}]$.

The *projection* of a set $S \subseteq \mathcal{X} \times \mathcal{Y}$ (into $\mathcal{X}$) is the set

$$P = \{x \in \mathcal{X} : \exists y\, (x, y) \in S\} \tag{39.12}$$

See Fig. 39.2.

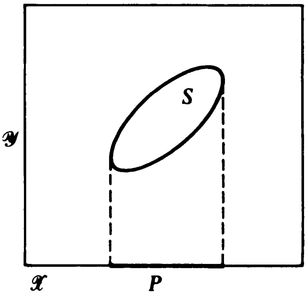

FIGURE 39.2

Lemma 39.2. *The following are equivalent, for any set A in a Polish space $\mathcal{X}$:*

(a) *A is analytic.*
(b) *A is the continuous image of a Borel set B (in some Polish space $\mathcal{Y}$).*
(c) *A is the projection of a Borel set in $\mathcal{X} \times \mathcal{Y}$, for some $\mathcal{Y}$.*
(d) *A is the projection of a closed set in $\mathcal{X} \times \mathcal{N}$.*

Proof. We shall prove that:

(1) every closed set (in any Polish space) is analytic; and
(2) every Borel set is the projection of a closed set in $\mathcal{X} \times \mathcal{N}$.

Then the lemma follows: Since the projection map $\pi\colon \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ defined by $\pi(x, y) = x$ is continuous, it follows that every Borel set is analytic and that the continuous image of a Borel set is analytic. Conversely, if $A \subseteq \mathcal{X}$ is an analytic set, $A = f[\mathcal{N}]$, then A is the projection of the set

$$\{(f(x), x) : x \in \mathcal{N}\}$$

which is a closed set in $\mathcal{X} \times \mathcal{N}$.

In order to prove (1), note that every closed set in a Polish space is itself a Polish space, and thus it suffices to show that every Polish space is a continuous image of the Baire space $\mathcal{N}$.

Let $\mathcal{X}$ be a complete separable metric space; we construct a mapping f of $\mathcal{N}$ onto $\mathcal{X}$ as follows: It is easy to construct, by induction on the length of $s \in Seq$, a collection $\{C_s : s \in Seq\}$ (where $Seq = \bigcup_{n=0}^{\infty} {}^n N$) of closed balls such that $C_\emptyset = \mathcal{X}$ and

(39.13) (i) diameter $(C_s) \le 1/n$ where $n = \text{length}(s)$;
(ii) $C_s \subseteq \bigcup_{k=0}^{\infty} C_{s^\frown k}$ (all $s \in Seq$).

For each $a \in \mathcal{N}$, let $f(a)$ be the unique point in $\bigcap \{C_s : s \subset a\}$; it is easily checked that f is continuous and that $\mathcal{X} = f[\mathcal{N}]$.

In order to prove that every Borel set in $\mathcal{X}$ is the projection of a closed set in $\mathcal{X} \times \mathcal{N}$, it suffices to show that the family P of all subsets of $\mathcal{X}$ that are such projections contains all closed sets, all open sets, and is closed under countable unions and intersections.

Clearly, the family P contains all closed sets. Moreover, every open set is a countable union of closed sets; thus it suffices to show that P is closed under $\bigcup_{n=0}^{\infty}$ and $\bigcap_{n=0}^{\infty}$.

Recall the continuous mapping $a \mapsto \langle a_{(n)} : n \in N \rangle$ of $\mathcal{N}$ onto ${}^\omega\mathcal{N}$ from Lemma 39.1, and also recall that the inverse image of a closed set under a continuous function is closed. Let A_n, $n < \omega$, be projections of closed sets in $\mathcal{X} \times \mathcal{N}$; we shall show that $\bigcup_{n=0}^{\infty} A_n$ and $\bigcap_{n=0}^{\infty} A_n$ are projections of closed sets.

For each n, let $F_n \subseteq \mathcal{X} \times \mathcal{N}$ be a closed set such that

$$A_n = \{x : \exists a\, (x, a) \in F_n\}$$

Thus

$$\begin{aligned} x \in \bigcup_{n=0}^{\infty} A_n \quad &\text{iff} \quad \exists n\, \exists a\, (x, a) \in F_n \\ &\text{iff} \quad \exists a\, \exists b\, (x, a) \in F_{b(0)} \\ &\text{iff} \quad \exists c\, (x, c_{(0)}) \in F_{c_{(1)}(0)} \end{aligned}$$

and

$$\begin{aligned} x \in \bigcap_{n=0}^{\infty} A_n \quad &\text{iff} \quad \forall n\, \exists a\, (x, a) \in F_n \\ &\text{iff} \quad \exists c\, \forall n\, (x, c_{(n)}) \in F_n \\ &\text{iff} \quad \exists c\, (x, c) \in \bigcap_{n=0}^{\infty} \{(x, c) : (x, c_{(n)}) \in F_n\} \end{aligned}$$

Hence $\bigcup_{n=0}^{\infty} A_n$ is the projection of the closed set

$$\{(x, c) : (x, c_{(0)}) \in F_{c_{(1)}(0)}\}$$

and $\bigcap_{n=0}^{\infty} A_n$ is the projection of an intersection of closed sets. ■

The Suslin Operation $\mathscr{A}$

First some notation: We recall that *Seq* is the set of all finite sequences of integers: $Seq = \bigcup_{n=0}^{\infty} {}^nN$. If $a \in {}^NN$ and $n \in N$, let $\bar{a}(n)$ denote the finite sequence

$$\bar{a}(n) = \langle a(0), \ldots, a(n-1)\rangle \tag{39.14}$$

(hence $\bar{a}(n) = a | n \in Seq$). For each $s \in Seq$, U_s is the basic open set $\{a \in \mathscr{N} : \bar{a}(n) = s\}$ of $\mathscr{N}$, where $n = \text{length}(s)$; U_s is also closed. $\bar{A}$ denotes the closure of $A \subseteq \mathscr{X}$.

Let $\{A_s : s \in Seq\}$ be a collection of sets indexed by elements of *Seq*. We define

$$\mathscr{A}\{A_s : s \in Seq\} = \bigcup_{a \in {}^NN} \bigcap_{n=0}^{\infty} A_{\bar{a}(n)} \tag{39.15}$$

Exercise 39.4. The operations $\bigcup_{n=0}^{\infty}$ and $\bigcap_{n=0}^{\infty}$ are special cases of the operation $\mathscr{A}$.

Note that if $\{B_s : s \in Seq\}$ is arbitrary, then

$$\bigcup_{a \in {}^NN} \bigcap_{n=0} B_{\bar{a}(n)} = \bigcup_{a \in {}^NN} \bigcap_{n=0} (B_{\bar{a}(0)} \cap B_{\bar{a}(1)} \cap \cdots \cap B_{\bar{a}(n)})$$

and hence $\mathscr{A}\{B_s : s \in Seq\} = \mathscr{A}\{A_s : s \in Seq\}$ where the sets A_s are finite intersections of the sets B_s and satisfy the following *normality* condition:

$$\text{if} \quad s \subseteq t, \qquad \text{then} \quad A_s \supseteq A_t \tag{39.16}$$

Thus we shall restrict our use of $\mathscr{A}$ to families that satisfy the condition (39.16). The operation $\mathscr{A}$ is called the *Suslin operation.*

Lemma 39.3. *A set $A \subseteq \mathscr{X}$ is analytic if and only if A is the result of the operation $\mathscr{A}$ applied to a family of closed sets.*

Proof. First we show that if F_s, $s \in Seq$, are closed sets in $\mathscr{X}$, then $A = \mathscr{A}\{F_s : s \in Seq\}$ is analytic. We have

$$\begin{aligned} x \in A \quad &\text{iff} \quad \exists a \in \mathscr{N}\ x \in \bigcap_{n=0}^{\infty} F_{\bar{a}(n)} \\ &\text{iff} \quad \exists a\ (x, a) \in \bigcap_{n=0}^{\infty} B_n \end{aligned}$$

where $B_n = \{(x, a) : x \in F_{\bar{a}(n)}\}$. Clearly, each B_n is a Borel set in $\mathscr{X} \times \mathscr{N}$ and hence A is analytic.

Conversely, let $A \subseteq \mathscr{X}$ be analytic. There is a continuous function $f: \mathscr{N} \to \mathscr{X}$ such that $A = f[\mathscr{N}]$. Notice that for every $a \in \mathscr{N}$,

$$(39.17) \qquad \bigcap_{n=0}^{\infty} f[U_{\bar{a}(n)}] = \bigcap_{n=0}^{\infty} \overline{f[U_{\bar{a}(n)}]} = \{f(a)\}$$

(this is easy to verify; here $\bar{X}$ denotes the closure of a set X). Thus

$$A = f[\mathscr{N}] = \bigcup_{a \in \mathscr{N}} \bigcap_{n=0}^{\infty} \overline{f[U_{\bar{a}(n)}]}$$

and hence A is the result of the operation $\mathscr{A}$ applied to the closed sets $\overline{f[U_s]}$ (which satisfy the normality condition (39.16)). ■

Exercise 39.5. Let A_s, $s \in Seq$, be Borel sets satisfying (39.16) and the additional condition: For each $s \in Seq$ and all $n \neq m$, $A_{sn} \cap A_{sm} = \emptyset$. Then $\mathscr{A}\{A_s : s \in Seq\}$ is a Borel set.

$$\left[\bigcup_{a \in \mathscr{N}} \bigcap_{n=0}^{\infty} A_{\bar{a}(n)} = \bigcap_{n=0}^{\infty} \bigcup \{A_s : \text{length}(s) = n\}\right]$$

It follows from the preceding lemmas that the collection of all analytic sets in a Polish space is closed under countable unions and intersections, continuous images, and inverse images, and the Suslin operation (the last statement is proved just like the first part of Lemma 39.3). It is however not the case that the complement of an analytic set is analytic (if $\mathscr{X}$ is an uncountable Polish space). In the next lemma, we establish exactly that; we show that there exists an analytic set (in $\mathscr{N}$) whose complement is not analytic.

The Hierarchy of Projective Sets

For each $n \geq 1$, we define the collections Σ_n^1, Π_n^1, and Δ_n^1 of subsets of a Polish space $\mathscr{X}$ as follows:

$$(39.18) \qquad \begin{aligned} \Sigma_1^1 &= \text{the collection of all analytic sets} \\ \Pi_1^1 &= \text{the complements of analytic sets} \\ \Sigma_{n+1}^1 &= \text{the collection of the projections of all} \\ & \Pi_n^1 \text{ sets in } \mathscr{X} \times \mathscr{N} \\ \Pi_n^1 &= \text{the complements of the } \Sigma_n^1 \text{ sets in } \mathscr{X} \\ \Delta_n^1 &= \Sigma_n^1 \cap \Pi_n^1 \end{aligned}$$

The sets belonging to one of the collections Σ_n^1 or Π_n^1 are called *projective* sets. It is easily seen that the following inclusions hold:

$$\begin{tikzcd}
 & \Sigma^1_1 \arrow[dr, phantom, sloped, "\supseteq"] & & \Sigma^1_2 \arrow[dr, phantom, sloped, "\supseteq"] & & \Sigma^1_3 \\
\text{Borel} \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^1_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^1_3 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \cdots \\
 & \Pi^1_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^1_2 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^1_3
\end{tikzcd}$$

FIGURE 39.3

We shall show that for each n there is a Σ^1_n set in $\mathcal{N}$ that is not Π^1_n; thus the inclusions in Fig. 39.3 are proper inclusions.

Lemma 39.4. *For each $n \geq 1$, there exists a universal Σ^1_n set in $\mathcal{N}^2$; i.e., a set $U \subseteq \mathcal{N}^2$ such that U is Σ^1_n and that for every Σ^1_n set A in $\mathcal{N}$ there exists $v \in \mathcal{N}$ such that*

$$A = \{x : (x, v) \in U\}$$

Proof. Let h be a homeomorphism of $\mathcal{N} \times \mathcal{N}$ onto $\mathcal{N}$. If $n = 1$, let V be a universal Σ^0_1 set; if $n > 1$, let V be, by the induction hypothesis, a universal Σ^1_{n-1} set. Let

$$(x, y) \in U \quad \text{iff} \quad \exists a \in \mathcal{N}\ (h(x, a), y) \notin V \tag{39.19}$$

Since the set $\{(x, y, a) : (h(x, a), y) \notin V\}$ is closed (if $n = 1$) or Π^1_{n-1} (if $n > 1$), U is Σ^1_n.

If $A \subseteq \mathcal{N}$ is Σ^1_n, there is a closed (or Π^1_{n-1}) set B such that

$$x \in A \quad \text{iff} \quad \exists a \in \mathcal{N}\ (x, a) \in B \tag{39.20}$$

If we let $C = h[\mathcal{N}^2 - B]$, we have

$$(x, a) \in B \quad \text{iff} \quad h(x, a) \notin C \tag{39.21}$$

The set C is open (or Σ^1_{n-1}) in $\mathcal{N}$ and since V is universal, there exists v such that

$$C = \{u : (u, v) \in V\}$$

By (39.21), we have

$$(x, a) \in B \quad \text{iff} \quad (h(x, y), v) \notin V$$

then (39.20) gives

$$x \in A \quad \text{iff} \quad \exists a \in \mathcal{N}(h(x, y), v) \notin V$$

and by (39.19),

$$A = \{x : (x, v) \in U\}$$

Hence U is a universal Σ^1_n set. ■

Corollary. *For each $n \geq 1$, there is a set $A \subseteq \mathcal{N}$ that is Σ^1_n but not Π^1_n.*

Proof. Let $U \subseteq \mathcal{N}^2$ be a universal Σ_n^1 set and let

$$A = \{x : (x, x) \in U\} \quad \blacksquare$$

The collection of all Δ_1^1 sets in a Polish space is a σ-algebra and contains all Borel sets. It turns out that Δ_1^1 is exactly the collection of all Borel sets.

Theorem 93 (Suslin). *Every analytic set whose complement is also analytic is a Borel set. Thus Δ_1^1 is the collection of all Borel sets.*

Let $\mathscr{X}$ be a Polish space and let A and B be two disjoint analytic sets in $\mathscr{X}$. We say that A and B are *separable by a Borel set* if there exists a Borel set D such that $A \subseteq D$ and $B \subseteq \mathscr{X} - D$.

Lemma 39.5. *Any two disjoint analytic sets are separable by a Borel set.*

This lemma is often called "the Σ_1^1-separation principle." It clearly implies Suslin's theorem since if A is an analytic set such that $B = \mathscr{X} - A$ is also analytic, A and B are separable by a Borel set D and we clearly have $D = A$.

Proof. First we make the following observation: If $A = \bigcup_{n=0}^{\infty} A_n$ and $B = \bigcup_{m=0}^{\infty} B_m$ are such that for all n and m, A_n and B_m are separable, then A and B are separable. This is proved as follows: For each n and each m, let D_{nm} be a Borel set such that $A_n \subseteq D_{nm} \subseteq \mathscr{X} - B_m$. Then A and B are separable by the Borel set $D = \bigcup_{n=0}^{\infty} \bigcap_{m=0}^{\infty} D_{nm}$.

Let A and B be two disjoint analytic sets in $\mathscr{X}$. Let f and g be continuous functions such that $A = f[\mathcal{N}]$ and $B = g[\mathcal{N}]$. For each $s \in Seq$, let $A_s = f[U_s]$ and $B_s = g[U_s]$; the sets A_s and B_s are all analytic sets. For each s we have $A_s = \bigcup_{n=0}^{\infty} A_{s^\frown n}$ and $B_s = \bigcup_{m=0}^{\infty} B_{s^\frown m}$. If $a \in {}^N N$, then

$$\{f(a)\} = \bigcap_{n=0}^{\infty} f[U_{\bar{a}(n)}] = \bigcap_{n=0}^{\infty} A_{\bar{a}(n)}$$

and similarly for the sets B_s.

Let $a, b \in {}^N N$ be arbitrary. Since $f[\mathcal{N}]$ and $g[\mathcal{N}]$ are disjoint, we have $f(a) \neq g(b)$. Let G_a and G_b be two disjoint open neighborhoods of $f(a)$ and $g(b)$, respectively. By the continuity of f and g there exists $n \in N$ such that $A_{\bar{a}(n)} \subseteq G_a$ and $B_{\bar{b}(n)} \subseteq G_b$. It follows that the sets $A_{\bar{a}(n)}$ and $B_{\bar{b}(n)}$ are separable by a Borel set.

We shall now show, by contradiction, that the sets A and B are separable by a Borel set. If A and B are not separable, then because $A = \bigcup_{n=0}^{\infty} A_{\langle n \rangle}$ and $B = \bigcup_{m=0}^{\infty} A_{\langle m \rangle}$. there exist n_0 and m_0 such that the sets $A_{\langle n_0 \rangle}$ and $B_{\langle m_0 \rangle}$ are not separable. Then similarly there exist n_1 and m_1 such that the sets $A_{\langle n_0 n_1 \rangle}$ and $B_{\langle m_0 m_1 \rangle}$ are not separable, and so on. In other words, there exist $a = \langle n_0 n_1 n_2 \cdots \rangle$ and $b = \langle m_0 m_1 m_2 \cdots \rangle$ such that for every k, the sets $A_{\langle n_0 \cdots n_k \rangle}$ and $B_{\langle m_0 \cdots m_k \rangle}$ are not separable. This is a contradiction since in the preceding paragraph we proved exactly the opposite: There is k such that $A_{\bar{a}(k)}$ and $B_{\bar{b}(k)}$ are separable. $\blacksquare$

Exercise 39.6. Let A_n, $n = 0, 1, 2, \ldots$, be pairwise disjoint analytic sets. Then there exist pairwise disjoint Borel sets D_n such that $A_n \subseteq D_n$ for all n.

[Modify the proof of Lemma 39.5.]

Exercise 39.7. If $B \subseteq \mathcal{N}$ is a Borel set and if f is a one-to-one continuous function, then $A = f[B]$ is a Borel set.

[For each $s \in Seq$, let $B_s = B \cap U_s$ and $A_s = f[B_s]$. The sets A_s are analytic and for each n, the sets A_s such that length$(s) = n$ are pairwise disjoint. Let D_s be Borel sets such that for each n, $\{D_s : \text{length}(s) = n\}$ are pairwise disjoint and such that $A_s \subseteq D_s \subseteq \bar{A}_s$. By (39.17), $A = \mathscr{A}\{A_s : s \in Seq\} = \mathscr{A}\{\bar{A}_s : s \in Seq\}$, and hence $A = \mathscr{A}\{D_s : s \in Seq\}$. By Exercise 39.5, A is Borel.]

Measure and Category

We shall now turn our attention to two properties that are often used as a measure of good behavior of sets of real numbers: *Lebesgue measurability* and the *property of Baire*. Our goal is to show that all analytic sets of reals are Lebesgue measurable and have the Baire property. Since most readers are probably more familiar with Lebesgue measurability, let me first give a review of the theory of Lebesgue measure (in the n-dimensional Euclidean space $\boldsymbol{R}^n$).

The Lebesgue Measure

The standard way of defining Lebesgue measure is to define first the *outer measure* $\mu^*(X)$ of a set $X \subseteq \boldsymbol{R}^n$ as the infimum of all possible sums $\sum\{v(I_k) : k \in \mathcal{N}\}$ where $\{I_k : k \in \mathcal{N}\}$ is a collection of n-dimensional intervals such that $X \subseteq \bigcup_{k=0}^{\infty} I_k$, and $v(I)$ denotes the volume of I. For each X, $\mu^*(X) \geq 0$ and possibly $= \infty$. A set X is *null* if $\mu^*(X) = 0$.

A set $A \subseteq \boldsymbol{R}^n$ is *Lebesgue measurable* if for each $X \subseteq \boldsymbol{R}^n$,

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X - A)$$

For a measurable set A, we write $\mu(A)$ instead of $\mu^*(A)$ and call $\mu(A)$ the *Lebesgue measure of A*.

The standard development of the theory of Lebesgue measure gives the following facts:

(39.22) (a) Every interval is Lebesgue measurable, and its measure is equal to its volume.

(b) The Lebesgue measurable sets form a σ-algebra; hence every Borel set is measurable.

(c) μ is σ-additive: If A_n, $n < \omega$, are pairwise disjoint and measurable, then

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n)$$

(d) μ is σ-finite: If A is measurable, then there exist measurable sets A_n, $n < \omega$, such that $A = \bigcup_{n=0}^{\infty} A_n$, and $\mu(A_n) < \infty$ for each n.
(e) Every null set is measurable. The null sets form a σ-ideal and contain all singletons.

From the other basic facts on Lebesgue measure let me mention these two:

(f) If A is measurable, then

$$\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ is compact}\}$$

and

(g) If A is measurable, then there is an F_σ-set F and a G_δ-set G such that $F \subseteq A \subseteq G$ and that $G - F$ is null.

This last property gives this characterization of Lebesgue measurable sets: A set $A \subseteq \boldsymbol{R}^n$ is measurable if and only if there is a Borel set B such that the symmetric difference $A \mathbin{\triangle} B = (A - B) \cup (B - A)$ is null.

One consequence of this is that if we denote by $\mathscr{B}$ the σ-algebra of Borel sets and by $\mathscr{M}$ the σ-algebra of measurable sets, and if I_μ is the ideal of all null sets, then $\mathscr{B}/I_\mu = \mathscr{M}/I_\mu$. The Boolean algebra $\mathscr{B}/I_\mu$ is σ-complete; and since a familiar argument shows that I_μ is (as an ideal in $\mathscr{M}$) σ-saturated, we conclude that $\mathscr{B}/I_\mu$ is a complete Boolean algebra. We shall return to it later.

Exercise 39.8. If X is a null set and $a_0 \geq a_1 \geq \cdots \geq a_n \geq \cdots$ is a sequence of positive numbers with $\lim a_n = 0$, then there exists a sequence G_n, $n = 0, 1, \ldots$, of finite unions of open intervals such that $X \subseteq \bigcup_{n=0}^{\infty} G_n$ and $\mu(G_n) < a_n$ for each n. Moreover, the intervals can be required to have rational endpoints.

[First find an open $G \supseteq X$ of measure $\leq a_0$ and express G as a disjoint union of intervals.]

Assuming the axiom of choice one can show that there exists a set of reals that is not Lebesgue measurable. One such example is the Vitali set in Exercise 27.1. As another example there exists a set $X \subseteq \boldsymbol{R}^n$ such that neither X nor its complement has a perfect subset (see Exercise 5.1 for a construction of such a set). The set X is not measurable: Otherwise, e.g., $\mu(X) > 0$ and by (39.22f) there is a closed $K \subseteq X$ such that $\mu(K) > 0$; thus K is uncountable and hence contains a perfect subset, a contradiction.

However, we shall show in Section 42 that it is consistent (with ZF + DC) that all sets of reals are Lebesgue measurable.

I shall conclude this review of Lebesgue measurability with two lemmas. One is the well-known Fubini theorem, and I state it here, without proof, for the sake of completeness (we shall actually use it in Section 41). The other lemma will be used in the proof of Theorem 94.

If A is a subset of the plane $\boldsymbol{R}^2$ and $x \in \boldsymbol{R}$, let A_x denote the set $\{y : (x, y) \in A\}$.

Lemma 39.6. *Let $A \subseteq \boldsymbol{R}^2$ be a measurable set. Then A is null if and only if for almost all x, A_x is null (i.e., the set $\{x : A_x \text{ is not null}\}$ is null).* ■

Lemma 39.7. *For any set $X \subseteq \boldsymbol{R}^n$ there exists a measurable set $A \supseteq X$ with the property that whenever $Z \subseteq A - X$ is measurable, then Z is null.*

Proof. If $\mu^*(X) < \infty$, then because $\mu^*(X) = \inf\{\mu(A) : A \text{ is measurable and } A \supseteq X\}$, there is a measurable $A \supseteq X$ such that $\mu(A) = \mu^*(X)$; clearly such an A will do. If $\mu^*(X) = \infty$, there exist pairwise disjoint X_n such that $X = \bigcup_{n=0}^{\infty} X_n$ and that for each n, $\mu^*(X_n) < \infty$. Let $A_n \supseteq X_n$, $n < \omega$, be measurable sets such that $\mu(A_n) = \mu^*(X_n)$, and let $A = \bigcup_{n=0}^{\infty} A_n$. ■

It should be mentioned that the main results of descriptive set theory on Lebesgue measure (e.g., the forthcoming Theorem 94) can be proved in a more general context, namely for reasonable σ-additive measures in Polish spaces. An example of such a measure is the product measure in the Cantor space ${}^{\omega}\{0, 1\}$ (see (34.3) for details).

The Property of Baire

In Section 14, we have proved the Baire category theorem (Theorem 13): The intersection of countably many dense open sets of reals is nonempty. It is fairly easy to see that the proof works not only for the real line $\boldsymbol{R}$ but for any Polish space.

Let us work in an arbitrary Polish space $\mathscr{X}$. Let us call a set $A \subseteq \mathscr{X}$ *rare* (or *nowhere dense*) if the complement of A contains a dense open set. Note that A is rare just in case for every nonempty open set G, there is a nonempty open set $H \subseteq G$ such that $A \cap H = \emptyset$. A set A is rare if and only if its closure $\bar{A}$ is rare.

A set $A \subseteq \mathscr{X}$ is *meager* (or of *first category*) if A is the union of countably many rare sets. A nonmeager set is called a set of *second category*.

The Baire category theorem states in effect that in a Polish space every nonempty open set is of second category.

The meager sets form a σ-ideal. Moreover, in case of $\boldsymbol{R}^n$, $\mathcal{N}$, or the Cantor space, every singleton $\{x\}$ is rare and so the ideal of meager sets contains all countable sets.

A set A has the *Baire property* if there exists an open set G such that $A \triangle G$ is meager. Clearly, every meager set has the Baire property. Note that if G is open, then $\bar{G} - G$ is rare. Hence if $A \triangle G$ is meager then $(\mathscr{X} - A) \triangle (\mathscr{X} - \bar{G}) = A \triangle \bar{G}$ is meager, and it follows that the complement of a set with the Baire property also has the Baire property. It is also easy to see that the union of countably many sets with the Baire property has the Baire property and we have:

(39.23) (a) The sets having the Baire property form a σ-algebra; hence every Borel set has the Baire property.

Exercise 39.9. For every set A with the Baire property, there is a G_δ-set G and an F_σ-set F such that $G \subseteq A \subseteq F$ and such that $F - G$ is meager.

[Note that every meager set is included in a meager F_σ-set.]

If $\mathscr{B}$ denotes again the σ-algebra of Borel sets, and if we denote by $\mathscr{C}$ the σ-algebra of sets with the Baire property, and if I is the σ-ideal of meager sets, we have $\mathscr{B}/I = \mathscr{C}/I$. Note that the algebra $\mathscr{B}/I$ is σ-saturated: Let $\mathscr{G}$ be a countable topology base for $\mathscr{X}$. For each nonmeager set X with the Baire property there exists $G \in \mathscr{G}$ such that $G - X$ is meager. Thus the set $D = \{[G] : G \in \mathscr{G}\}$ of equivalence classes is a dense set in $\mathscr{B}/I$. Hence $\mathscr{B}/I$ is σ-saturated and is a complete Boolean algebra.

Exercise 39.10. For every set A with the Baire property, there exists a unique regular open set U such that $A \triangle U$ is meager. The algebra $\mathscr{B}/I$ is isomorphic to the algebra of regular open sets (with the operations defined in (17.5)).

As is the case with Lebesgue measurability, the axiom of choice implies that there exists a set of reals without the Baire property.

Exercise 39.11. The Vitali set M from Exercise 27.1 does not have the Baire property.

[" Meager " and " Baire property " are invariant under translation. If M has the Baire property, then there is an interval (a, b) such that $(a, b) - M$ is meager. Then $(a, b) \cap M_q$ is meager for all rational $q \neq 0$, hence each $M \cap (a - q, b - q)$ is meager, hence M is meager, hence each M_q is meager; a contradiction since $\boldsymbol{R} = \bigcup_{q \in Q} M_q$.]

If $X \subseteq \boldsymbol{R}^n$ is such that neither X nor its complement has a perfect subset, then X does not have the Baire property: Otherwise, e.g., X is of second category and hence X contains a G_δ-subset G of second category. Now G is uncountable, and this is a contradiction since as we shall prove in Theorem 94, every uncountable Borel set (even analytic) has a perfect subset.

The following two lemmas are analogs of Lemmas 39.6 and 39.7. The first one, although not very difficult to prove, is again stated without proof.

Lemma 39.8. *Let $A \subseteq \boldsymbol{R}^2$ have the property of Baire. Then A is meager if and only if A_x is meager for all x except a meager set.* ■

Lemma 39.9. *For any set X in a Polish space $\mathscr{X}$, there exists a set $A \supseteq X$ that has the Baire property and such that whenever $Z \subseteq A - X$ has the Baire property, then Z is meager.*

Proof. Let us consider a fixed countable topology basis $\mathscr{G}$ for $\mathscr{X}$. Let $X \subseteq \mathscr{X}$. Let

$$D(X) = \{x \in \mathscr{X} : \text{for every } U \in \mathscr{G} \text{ such that } x \in U,\ U \cap X \text{ is not meager}\}$$

Note that the complement of $D(X)$ is the union of open sets and hence open; thus $D(X)$ is closed.

The set $X - D(X)$ is the union of all $X \cap U$ where $U \in \mathscr{G}$ and $X \cap U$ is meager; since G is countable, $X - D(X)$ is meager. Let

$$A = X \cup D(X)$$

Since $A = (X - D(X)) \cup D(X)$ is the union of a meager and a closed set, A has the Baire property.

Let $Z \subseteq A - X$ have the Baire property; we shall show that Z is meager. Otherwise there is $U \in \mathscr{G}$ such that $U - Z$ is meager; hence $U \cap X$ is meager. Since $U \cap Z \neq \emptyset$ and $Z \subseteq D(X)$, there is $x \in U$ such that $x \in D(X)$, and hence $U \cap X$ is not meager, a contradiction. ■

Although both "null" and "meager" mean in a sense "negligible," the following exercise shows that the real line can be decomposed into a null set and a meager set:

Exercise 39.12. There is a null set of reals whose complement is meager.

[Let $q_1, q_2, \ldots$ be an enumeration of the rationals. For each $n \geq 1$ and $k \geq 1$, let I_{nk} be the open interval with center q_n and length $1/(k \cdot 2^n)$. Let $D_k = \bigcup_{n=1}^{\infty} I_{nk}$, and $A = \bigcap_{k=1}^{\infty} D_k$. Each D_k is open and dense, and $\mu(D_k) \leq 1/k$. Hence A is null and $\boldsymbol{R} - A$ is meager.]

I shall conclude this section with the following theorem.

Theorem 94.

(a) *Every analytic set of reals is Lebesgue measurable.*
(b) *Every analytic set has the Baire property.*
(c) *Every uncountable analytic set contains a perfect subset.*

Corollary 1. *Every $\boldsymbol{\Pi}_1^1$ set of reals is Lebesgue measurable and has the Baire property.*

Corollary 2. *Every analytic (and in particular every Borel) set is either at most countable or has cardinality $2^{\aleph_0}$.*

We prove (b) and (c) for an arbitrary Polish space. The proof of (a) is general enough to work for other measures (in Polish spaces) than just Lebesgue measure in $\boldsymbol{R}^n$.

Proof. The proof of (a) and (b) is exactly the same and uses either Lemma 39.7 or Lemma 39.9 (and basic facts on Lebesgue measure and the Baire property). Let me give the proof of (a) and leave (b) to the reader.

Let A be an analytic set of reals (or a subset of $\boldsymbol{R}^n$). Let $f\colon \mathcal{N} \to \boldsymbol{R}$ be a continuous function such that $A = f[\mathcal{N}]$. For each $s \in Seq$, let $A_s = f[U_s]$. We have

$$(39.24) \qquad A = \mathscr{A}\{A_s : s \in Seq\} = \mathscr{A}\{\bar{A}_s : s \in Seq\}$$

and for every $s \in Seq$,

$$A_s = \bigcup_{n=0}^{\infty} A_{s^\frown n} \tag{39.25}$$

By Lemma 39.7, there exists for each $s \in Seq$ a measurable set $B_s \supseteq A_s$ such that every measurable $Z \subseteq B_s - A_s$ is null. Since $\bar{A}_s$ is measurable, we may actually find B_s such that $A_s \subseteq B_s \subseteq \bar{A}_s$.

Let $B = B_\varnothing$. Since B is measurable, it suffices to show that $B - A$ is a null set. Notice that because $A_s \subseteq B_s \subseteq \bar{A}_s$, and because (39.24) holds, we have

$$A = \mathscr{A}\{B_s : s \in Seq\}$$

Thus

$$B - A = B - \bigcup_{a \in {}^N N} \bigcap_{n=0}^{\infty} B_{\bar{a}(n)}$$

I claim that

$$B - \bigcup_{a \in {}^N N} \bigcap_{n=0}^{\infty} B_{\bar{a}(n)} \subseteq \bigcup_{s \in Seq} \left(B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k} \right) \tag{39.26}$$

To prove (39.26), assume that $x \in B$ is such that x is not a member of the right-hand side. Then for every s, if $x \in B_s$, then $x \in B_{s^\frown k}$ for some k. Hence there is k_0 such that $x \in B_{\langle k_0 \rangle}$; then there is k_1 such that $x \in B_{\langle k_0 k_1 \rangle}$, etc. Let $a = \langle k_0 k_1 k_2 \cdots \rangle$; we have $x \in \bigcap_{n=0}^{\infty} B_{\bar{a}(n)}$ and hence x is not a member of the left-hand side.

Thus we have

$$B - A \subseteq \bigcup_{s \in Seq} \left(B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k} \right)$$

Since *seq* is a countable set, it suffices to show that each $B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}$ is null. Let $s \in Seq$, and let $Z = B_s - \bigcup_{k=1}^{\infty} B_{s^\frown k}$. We have

$$Z = B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k} \subseteq B_s - \bigcup_{k=0}^{\infty} A_{s^\frown k} = B_s - A_s$$

Now because $Z \subseteq B_s - A_s$ and because Z is measurable, Z must be null.

(c) The proof is a variant of the Cantor–Bendixson argument for closed sets in the Baire space. Recall that every closed set F in $\mathcal{N}$ is of the form $F = [T] = \{a : \forall n\ \bar{a}(n) \in T\}$, where T is a tree, $T \subseteq Seq$. For each tree $T \subseteq Seq$ and each $s \in Seq$, let T_s denote the tree $\{t \in T : t \subseteq s \text{ or } s \subseteq t\}$; note that $[T_s] = [T] \cap U_s$.

Let A be an analytic set (in a Polish space $\mathscr{X}$), and let f be a continuous function such that $A = f[\mathcal{N}]$. For each tree $T \subseteq Seq$, we define

$$T' = \{s \in T : f[[T_s]] \text{ is uncountable}\}$$

($f[[T_s]]$ is the image of the set $[T_s]$ by the function f). For each $\alpha < \omega_1$, we define $T^{(\alpha)}$ as follows:

$$T^{(0)} = Seq$$
$$T^{(\alpha+1)} = (T^{(\alpha)})'$$
$$T^{(\alpha)} = \bigcap_{\beta<\alpha} T^{(\beta)} \qquad \text{if} \quad \alpha \text{ is a limit}$$

Let $\alpha < \omega_1$ be the least ordinal such that $T^{(\alpha+1)} = T^{(\alpha)}$ and let $T = T^{(\alpha)}$. If $T = \emptyset$, then

$$A = \bigcup_{\beta<\alpha} \{f[[T_s^{(\beta)}]] : s \in T^{(\beta)} - T^{(\beta+1)}\}$$

and hence A is countable. Thus if A is uncountable, T is nonempty and for every $s \in T, f[[T_s]]$ is uncountable. In this case, we shall find a perfect subset of A.

Let $s \in T$ be arbitrary. Since $f[[T_s]]$ has at least two elements, there exist $s_0 \supset s$ and $s_1 \supset s$ (in T) such that $f[[T_{s_0}]]$ and $f[[T_{s_1}]]$ are disjoint. Then there are $s_{00} \supset s_0$ and $s_{01} \supset s_0$, and $s_{10} \supset s_1$, $s_{11} \supset s_1$ such that the four sets $f[[T_{s_{ij}}]]$, $i, j = 0, 1$, are pairwise disjoint. In this fashion we construct $s_t \in T$ for each finite 0–1-sequence t. These elements s_t generate a subtree $U = \{s : s \subseteq s_t \text{ for some } t\}$ of T such that (1) U is perfect, (2) every s has at most two immediate successors in U (hence $[U]$ is a compact set in $\mathcal{N}$), and (3) f is one-to-one on $[U]$.

Let P be the image of $[U]$ under the function f. Since $[U]$ is compact and f is continuous, P is also compact, and hence closed. Moreover, P has no isolated points because $[U]$ is perfect and f is continuous. Thus P is a perfect subset of A. ■

40. Σ_n^1 AND Π_n^1 SETS AND RELATIONS IN THE BAIRE SPACE

In Section 39 we introduced analytic subsets of a Polish space and showed that a set $A \subseteq \mathcal{X}$ is analytic if and only if there is a closed set $F \subseteq \mathcal{X} \times \mathcal{N}$ such that

(40.1) $$x \in A \quad \leftrightarrow \quad \exists a\, (x, a) \in F$$

Let us consider the product space $\mathcal{X} = \mathcal{N}^r$, for an arbitrary integer $r \geq 1$. As in the case $r = 1$, the closed subsets of $\mathcal{N}^r$ can be represented by trees: Let Seq_r denote the set of all r-tuples $(s_1, \ldots, s_r) \in Seq^r$ such that $\text{length}(s_1) = \cdots = \text{length}(s_r)$. A set $T \subseteq Seq_r$ is an *(r-dimensional sequential) tree* if for every $(s_1, \ldots, s_r) \in T$ and each $n \leq \text{length}(s_1)$, $(s_1 | n, \ldots, s_r | n)$ is also in T. Let

(40.2) $$[T] = \{(a_1, \ldots, a_r) \in \mathcal{N}^r : \forall n (\bar{a}_1(n), \ldots, \bar{a}_r(n)) \in T\}$$

The set $[T]$ is closed, and every closed set in $\mathcal{N}^r$ has the form (40.2), for some tree T.

It follows that a set $A \subseteq \mathcal{N}^r$ is analytic if and only if there is a tree $T \subseteq Seq_{r+1}$ such that

(40.3) $(x_1, \ldots, x_r) \in A \leftrightarrow (\exists z \in \mathcal{N})\,(\forall n \in N)\,(\bar{x}_1(n), \ldots, \bar{x}_r(n), \bar{z}(n)) \in T$

The formula (40.3) provides a more detailed analysis of Σ^1_1 sets in $\mathcal{N}^r$. Let $\mathrm{HF} = V_\omega$ be the collection of all hereditarily finite sets. Let us call a subset $X \subseteq \mathrm{HF}$ *arithmetical* if X is a definable set in the model $\langle \mathrm{HF}, \in \rangle$. More generally, if $Y_1, \ldots, Y_k$ are subsets of HF, then $X \subseteq \mathrm{HF}$ is *arithmetical in* $Y_1, \ldots, Y_k$ if X is definable in the model $\langle \mathrm{HF}, \in, Y_1, \ldots, Y_k \rangle$ where $Y_1, \ldots, Y_k$ are considered one-place predicates.

We shall now define the lightface Σ^1_n hierarchy of relations over $\mathcal{N}$. Let a be an element of $\mathcal{N}$, and let $A \subseteq \mathcal{N}^r$. We say that the r-ary relation A over $\mathcal{N}$ is Σ^1_1 in a ($A \in \Sigma^1_1(a)$) if there is a tree $T \subseteq Seq_{r+1}$ that is arithmetical in a and such that (40.3) holds. If A is Σ^1_1 in $\emptyset$, we say that A is Σ^1_1. Clearly,

$$\boldsymbol{\Sigma}^1_1 = \bigcup_{a \in \mathcal{N}} \Sigma^1_1(a)$$

We call (40.3) the *normal form* for Σ^1_1 relations.

For each $n \geq 1$, we define the collections $\Sigma^1_n(a)$, and $\Pi^1_n(a)$, and $\Delta^1_n(a)$ as follows (compare with (39.18)):

(40.4)
$$\begin{aligned} &A \in \Pi^1_1(a) && \text{if } \mathcal{N}^r - A \in \Sigma^1_1(a) \\ &A \in \Sigma^1_{n+1}(a) && \text{if } A = \{(x_1, \ldots, x_r) : (\exists z \in \mathcal{N})\,(x_1, \ldots, x_r, z) \in B\} \\ &&& \qquad \text{for some } (r + 1\text{-ary})\ B \in \Pi^1_n(a) \\ &A \in \Pi^1_n(a) && \text{if } \mathcal{N}^r - A \in \Sigma^1_n(a) \\ &\Delta^1_n(a) = \Sigma^1_n(a) \cap \Pi^1_n(a) \end{aligned}$$

Using the definition (40.4) and the normal form (40.3) for Σ^1_1 relations, we obtain the following characterization of Σ^1_n and Π^1_n relations: For instance, $A \subseteq \mathcal{N}^r$ is Σ^1_2 in a if there is a tree $T \subseteq Seq_{r+2}$, arithmetical in a, such that

$$(x_1, \ldots, x_r) \in A \quad \leftrightarrow \quad \exists z_2\, \forall z_1\, \exists n(\bar{x}_1(n), \ldots, \bar{x}_r(n), \bar{z}_1(n), \bar{z}_2(n)) \notin T$$

(the variables z_1, z_2 range over $\mathcal{N}$, n ranges over N). We shall now present a slightly different normal form.

First, let us give a new definition of $\Sigma^1_1(a)$ relations, and let me use this opportunity to be somewhat more general. Let A be an $(r + l)$-ary relation, $A \subseteq \mathcal{N}^r \times N^l$. We say that A is Σ^1_1 *in* $a \in \mathcal{N}$ if there is an $(r + l + 2)$-ary relation $R \subseteq \mathrm{HF}$, arithmetical in a, such that

(40.5)

$$(x_1, \ldots, x_r, k_1, \ldots, k_l) \in A \quad \leftrightarrow \quad \exists z\, \forall n(\bar{x}_1(n), \ldots, \bar{x}_r(n), k_1, \ldots, k_l, \bar{z}(n), n) \in R$$

(z ranges over $\mathcal{N}$, n ranges over N). Either r or l can be 0; if $r = l = 0$, then the right-hand side of (40.5), i.e., $\exists z\, \forall n(\bar{z}(n), n) \in R$ is either true or false; we call it a $\Sigma_1^1(a)$ *sentence*. Accordingly, we say that A in (40.5) is a $\Sigma_1^1(a)$ *property* of $(x_1, \ldots, x_r, k_1, \ldots, k_l)$ or a $\Sigma_1^1(a)$ *formula*.

The new definition of $\Sigma_1^1(a)$ agrees with the old one: If $T \subseteq Seq_{r+1}$ is such that (40.3) holds, then let R be the set of all $(s_1, \ldots, s_{r+1}, n)$ such that $(s_1, \ldots, s_{r+1}) \in T$ and $n = \text{length}(s_1)$; clearly R is arithmetical in T and (40.5) holds. Conversely, if R is an $(r + 2)$-ary relation such that (40.5) holds (with $l = 0$), then let $T \subseteq Seq_{r+1}$ be such that

$$(s_1, \ldots, s_{r+1}) \in T \quad \text{iff} \quad \forall n \leq \text{length}(s_1)(s_1 | n_1, \ldots, s_{r+1} | n, n) \in R$$

T is arithmetical in R and (40.3) holds.

This leads to the following normal form for $\Sigma_n^1(a)$ (and $\Pi_n^1(a)$) relations over $\mathcal{N}^r \times N^l$: If n is odd, then A is $\Sigma_n^1(a)$ if

$$(40.6) \qquad (x_1, \ldots, x_r, k_1, \ldots, k_1) \in A \leftrightarrow$$

$$\exists z_1\, \forall z_2 \cdots \exists z_n\, \forall m(\bar{x}_1(m), \ldots, \bar{x}_r(m), k_1, \ldots, k_l, \bar{z}_1(m), \ldots, \bar{z}_n(m), m) \in R$$

for some R arithmetical in a, and A is $\Pi_n^1(a)$ if

$$(40.7) \qquad (x_1, \ldots, x_r, k_1, \ldots, k_l) \in A \leftrightarrow$$

$$\forall z_1\, \exists z_2 \cdots \forall z_n\, \exists m(\bar{x}_1(m), \ldots, \bar{x}_r(m), k_1, \ldots, k_l, \bar{z}_1(m), \ldots, \bar{z}_n(m), m) \in R$$

for some R arithmetical in a. (The variables $z_1, \ldots, z_n$ range over $\mathcal{N}$, m ranges over N.) If n is even, then the quantifier $\forall m$ in (40.6) is replaced by $\exists m$, and $\exists m$ in (40.7) is replaced by $\forall m$.

[A student of recursion theory knows that we could obtain an even stronger normal form for $\Sigma_n^1(a)$ relations: The R in (40.6) can be required to be recursive in a, not just arithmetical.]

We shall now use the normal form to prove some closure properties for Σ_n^1 relations. Since a relation over $\mathcal{N}^r \times N^l$ can be considered as an $(r + l)$-ary predicate, we can use the following notation

$$A \vee B, \qquad A \wedge B, \qquad \neg A, \qquad A \rightarrow B, \qquad A \leftrightarrow B$$

for Boolean operations (in fact, $A \vee B$ is $A \cup B$ etc), and

$$\exists x\, A, \qquad \forall x\, A, \qquad \exists m\, A, \qquad \forall m\, A:$$

$$(x_1, \ldots, x_r, k_1, \ldots, k_l) \in \exists x\, A \quad \text{iff} \quad \exists x \in \mathcal{N}(x, x_1, \ldots, x_r, k_1, \ldots, k_l) \in A$$

$$(x_1, \ldots, x_r, k_1, \ldots, k_l) \in \exists m\, A \quad \text{iff} \quad \exists m \in N(x_1, \ldots, x_r, m, k_1, \ldots, k_l) \in A$$

Lemma 40.1. *Let* $n \geq 1$.

(i) *If* A, B *are* $\Sigma_n^1(a)$ *relations, then so are*

$$\exists x\, A, \qquad A \wedge B, \qquad A \vee B, \qquad \exists m\, A, \qquad \forall m\, A$$

(ii) *If A, B are $\Pi^1_n(a)$ relations, then so are*

$$\forall x\, A, \qquad A \wedge B, \qquad A \vee B, \qquad \forall m\, A, \qquad \exists m\, A$$

(iii) *If A is $\Sigma^1_n(a)$, then $\neg A$ is $\Pi^1_n(a)$; if A is $\Pi^1_n(a)$, then $\neg A$ is $\Sigma^1_n(a)$*

(iv) *If A is $\Pi^1_n(a)$ and B is $\Sigma^1_n(a)$, then $A \rightarrow B$ is $\Sigma^1_n(a)$;*
if A is $\Sigma^1_n(a)$ and B is $\Pi^1_n(a)$, then $A \rightarrow B$ is $\Pi^1_n(a)$.

(v) *If A and B are $\Delta^1_n(a)$, then so are $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \leftrightarrow B$, $\forall m\, A$, $\exists m\, A$.*

[Compare this with Lemma 14.2.]

Proof. It suffices to prove the lemma for $n = 1$; the general case follows by induction. Moreover, the clauses (ii)–(v) follow fairly easily from (i).

First, let $A \in \Sigma^1_1(a)$ and let us show that $\exists x\, A$ is $\Sigma^1_1(a)$. For simplicity, let $r = 2$ and $l = 0$:

$$(x, y) \in A \quad \leftrightarrow \quad \exists z\, \forall n(\bar{x}(n), \bar{y}(n), \bar{z}(n), n) \in R \tag{40.8}$$

where R is arithmetical in a. Thus

$$y \in \exists x\, A \quad \leftrightarrow \quad \exists x\, \exists z\, \forall n(\bar{x}(n), \bar{y}(n), \bar{z}(n), n) \in R$$

We want to contract the two quantifiers $\exists x\, \exists z$ into one. Let us consider some nice homeomorphism between $\mathcal{N}$ and $\mathcal{N}^2$, e.g., for $u \in \mathcal{N}$ let u^+ and u^- be

$$u^+(n) = u(2n), \qquad u^-(n) = u(2n+1) \qquad (n \in N)$$

There exists a relation $R' \subseteq Seq^2 \times N$, arithmetical in R, such that for all $u, y \in \mathcal{N}$,

$$\forall n(\bar{u}(n), \bar{y}(n), n) \in R' \qquad \text{iff} \qquad \forall k\, (\overline{u^+}(k), \bar{y}(k), \overline{u^-}(k), k) \in R \tag{40.9}$$

Namely, if $n = 2k$ (or $n = 2k + 1$), we let $(s, t, n) \in R'$ just in case length(s) = length$(t) = n$ and

$$(\langle s(0), \ldots, s(2k-2)\rangle, \langle t(0), \ldots, t(k-1)\rangle, \langle s(1), \ldots, s(2k-1)\rangle, k) \in R$$

Now (40.9) implies that

$$y \in \exists x\, A \quad \leftrightarrow \quad \exists u\, \forall n\, (\bar{u}(n), \bar{y}(n), n) \in R'$$

and hence $\exists x\, A$ is $\Sigma^1_1(a)$.

Next let A and B be $\Sigma^1_1(a)$:

$$x \in A \quad \leftrightarrow \quad \exists z\, \forall n\, (\bar{x}(n), \bar{z}(n), n) \in R_1$$

$$x \in B \quad \leftrightarrow \quad \exists z\, \forall n\, (\bar{x}(n), \bar{z}(n), n) \in R_2$$

where both R_1 and R_2 are arithmetical in a. Note that

$$x \in A \wedge B \quad \leftrightarrow \quad \exists z_1\, \exists z_2\, \forall n[(\bar{x}(n), \bar{z}_1(n), n) \in R_1 \wedge (\bar{x}(n), \bar{z}_2(n), n) \in R_2]$$

and hence, by contraction of $\exists z_1\ \exists z_2$, there is R, arithmetical in R_1 and R_2 such that

$$x \in A \wedge B \quad \leftrightarrow \quad \exists z\ \forall n(\bar{x}(n), \bar{z}(n), n) \in R$$

Thus $A \wedge B$ is $\Sigma_1^1(a)$.

The following argument shows that the case $A \vee B$ can be reduced to the case $\exists m\ C$. Let us define R as follows ($s, t \in Seq$, $m, n \in N$):

$$\begin{aligned}(s, m, t, n) \in R \quad \leftrightarrow \quad &\text{either} \quad m = 1 \text{ and } (s, t, n) \in R_1 \\ &\text{or} \quad m = 2 \text{ and } (s, t, n) \in R_2\end{aligned}$$

R is arithmetical in R_1 and R_2, and

$$\begin{aligned}x \in A \vee B \quad &\leftrightarrow \quad \exists z\ \forall n\ (\bar{x}(n), \bar{z}(n), n) \in R_1 \vee \exists z\ \forall n\ (\bar{x}(n), \bar{z}(n), n) \in R_2 \\ &\leftrightarrow \quad \exists m\ \exists z\ \forall n\ (\bar{x}(n), m, \bar{z}(n), n) \in R \\ &\leftrightarrow \quad x \in \exists m\ C\end{aligned}$$

where C is $\Sigma_1^1(a)$.

The contraction of quantifiers $\exists m\ \exists z$ is easier than the contraction $\exists x\ \exists z$ above. We employ the following homeomorphism between $\mathcal{N}$ and $N \times \mathcal{N}$: $h(u) = (u(0), u')$, where

$$u'(n) = u(n + 1) \qquad (n \in N)$$

If

$$(x, m) \in A \quad \leftrightarrow \quad \exists z\ \forall n\ (\bar{x}(n), m, \bar{z}(n), n) \in R$$

then we leave it to the reader to find a relation R', arithmetical in R, such that for all $u, x \in \mathcal{N}$,

$$\forall n\ (\bar{x}(n), \bar{u}(n), n) \in R' \qquad \text{iff} \qquad \forall k\ (\bar{x}(k), u(0), \bar{u}'(k), k) \in R$$

Then

$$x \in \exists m\ A \quad \leftrightarrow \quad \exists u\ \forall n\ (\bar{x}(n), \bar{u}(n), n) \in R'$$

It remains to show that if A is $\Sigma_1^1(a)$, then $\forall m\ A$ is $\Sigma_1^1(a)$. Let

$$(x, m) \in A \quad \leftrightarrow \quad \exists z\ \forall n\ (\bar{x}(n), m, \bar{z}(n), n) \in R \tag{40.10}$$

where R is arithmetical in a. Thus

$$x \in \forall m\ A \quad \leftrightarrow \quad \forall m\ \exists z\ \forall n\ (\bar{x}(n), m, \bar{z}(n), n) \in R \tag{40.11}$$

We want to replace the quantifiers $\forall m\ \exists z$ by $\exists u\ \forall m$ and then contract the two quantifiers $\forall m\ \forall n$ into one. Let us consider the one-to-one function Γ: $N \times N \to N$ and the following homeomorphism between $\mathcal{N}$ and ${}^{\omega}\mathcal{N}$: For each $u \in \mathcal{N}$, let u_m, $m \in N$, be

$$u_m(n) = u(\Gamma(m, n)) \qquad (m, n \in N)$$

Now we can replace $\forall m\ \exists z$ in (40.11) by $\exists u\ \forall m$ (note that in the implication $\rightarrow$ we use the countable axiom of choice):

$$(40.12)\quad \forall m\ \exists z\ \forall n\ (\bar{x}(n), m, \bar{z}(n), n) \in R \quad\leftrightarrow\quad \exists u\ \forall m\ \forall n\ (\bar{x}(n), m, \bar{u}_m(n), n) \in R$$

Let $\alpha\colon N \rightarrow N$ and $\beta\colon N \rightarrow N$ be the inverses of the function Γ: if $\Gamma(m, n) = k$, then $m = \alpha(k)$ and $n = \beta(k)$. From (40.11) and (40.12) we get

$$(40.13)\quad x \in \forall m\ A \quad\leftrightarrow\quad \exists u\ \forall k\ (\bar{x}(\beta(k)), \alpha(k), \overline{u_{\alpha(k)}}(\beta(k)), \beta(k)) \in R$$

Now it suffices to show that there exists a relation $R' \subseteq Seq^2 \times N$, arithmetical in R, such that for all $u, x \in \mathcal{N}$,

$$(40.14)\quad \forall k\ (\bar{x}(k), \bar{u}(k), k) \in R' \quad\text{iff}\quad \forall k\ (\bar{x}(\beta(k), \alpha(k), \overline{u_{\alpha(k)}}(\beta(k)), \beta(k)) \in R$$

The relation R' is found in a way similar to the relation R' in (40.9), and we leave the details as an exercise.

Hence $\forall m\ A$ is $\Sigma^1_1(a)$ because by (40.13) and (40.14),

$$x \in \forall m\ A \quad\leftrightarrow\quad \exists u\ \forall k\ (\bar{x}(k), \bar{u}(k), k) \in R' \quad \blacksquare$$

One consequence of the closure properties proved above is that the Σ^1_n relations form a hierarchy as in Fig. 40.1. The inclusions in Fig. 40.1 are proper inclusions; the analog of Lemma 39.4 holds for the lightface Σ^1_n too. In fact, for each n there is a Σ^1_n set (in $\mathcal{N}^{r+1}$) universal for Σ^1_n sets (in $\mathcal{N}^r$).

$$\begin{tikzcd}
 & \Sigma^1_1(a) \arrow[dr, phantom, "\subset" sloped] & & \Sigma^1_2(a) & \\
\Delta^1_1(a) \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \Delta^1_2(a) \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \cdots \\
 & \Pi^1_1(a) \arrow[ur, phantom, "\subset" sloped] & & \Pi^1_2(a) &
\end{tikzcd}$$

FIGURE 40.1

The closure properties in Lemma 40.1 are very useful in practical computations of complexity of projective sets in $\mathcal{N}^r$, and we shall use them extensively throughout this chapter. Before we start deeper investigations of the Σ^1_n hierarchy, let us make one more preliminary remark.

If $R \subseteq Seq \times N$ is an arithmetical relation, then the relation $A \subseteq \mathcal{N} \times N$

$$(x, n) \in A \quad\leftrightarrow\quad (\bar{x}(n), n) \in R$$

is both Σ^1_1 and Π^1_1. More generally, if

$$(40.15)\quad (x_1, \ldots, x_r, k_1, \ldots, k_l) \in A \leftrightarrow$$
$$(\bar{x}_1(k_1), \bar{x}_1(k_2), \ldots, \bar{x}_1(k_l), \ldots, \bar{x}_r(k_1), \ldots, \bar{x}_r(k_l), k_1, \ldots, k_l) \in R$$

where R is arithmetical is a Δ^1_1 relation in $\mathcal{N}^r \times N^l$. Let us verify this for the case $r = 1$, $l = 2$:

$$(x, k_1, k_2) \in A \quad\leftrightarrow\quad (\bar{x}(k_1), \bar{x}(k_2), k_1, k_2) \in R$$

There is a relation R', arithmetical in R, such that for all x,

$$(\bar{x}(k_1), \bar{x}(k_2), k_1, k_2) \in R \quad \leftrightarrow \quad (\bar{x}(\Gamma(k_1, k_2)), \Gamma(k_1, k_2)) \in R'$$

and hence

$$\begin{aligned}(x, k_1, k_2) \in A \quad &\leftrightarrow \quad (\bar{x}(\Gamma(k_1, k_2)), \Gamma(k_1, k_2)) \in R' \\ &\leftrightarrow \quad \exists z\ \forall n[n = \Gamma(k_1, k_2) \to (\bar{x}(n), n) \in R'] \\ &\leftrightarrow \quad \forall z\ \exists n[n = \Gamma(k_1, k_2) \wedge (\bar{x}(n), n) \in R']\end{aligned}$$

Thus A is both Σ^1_1 and Π^1_1.

Let us call a relation $A \subseteq \mathcal{N}^r \times N^l$ *arithmetical* if

(40.16) $$(x_1, \ldots, x_r, k_1, \ldots, k_l) \in A$$
$$\leftrightarrow \quad \forall m_1\ \exists m_2 \cdots \forall m_n(\bar{x}_i(m_j), \bar{x}_i(k_j), m_j, k_j) \in R$$

where R is an arithmetical relation over HF.

Arithmetical relations over $\mathcal{N}^r \times N^l$ have the following closure properties: If A and B are arithmetical, then so are $\neg A$, $A \wedge B$, $A \vee B$, $A \to B$, $\exists m\ A$ and $\forall m\ A$. Moreover, since Δ^1_1 is closed under $\exists m$ and $\forall m$ and since (40.15) is arithmetical, it follows that every arithmetical relation is Δ^1_1. Also, if A is arithmetical, then $\exists z\ A$ is Σ^1_1 and $\forall z\ A$ is Π^1_1.

It goes without saying that the preceding discussion relativizes to $\Delta^1_1(a)$: A relation A is *arithmetical in a* if (40.16) holds for some R arithmetical in a. Every A arithmetical in a is Δ^1_1 in a.

The Sets WF and WO

We shall now present two important examples of Π^1_1 sets in $\mathcal{N}$. Let us associate with each $x \in \mathcal{N}$ the following binary relation E_x over N:

(40.17) $$m\ E_x\ n \quad \text{iff} \quad x(\Gamma(m, n)) = 0$$

(Γ is the canonical one-to-one mapping of $N \times N$ onto N); we say that x *codes* the relation E_x. We define

$$\mathrm{WF} = \{x \in \mathcal{N} : x \text{ codes a well-founded relation}\}$$

$$\mathrm{WO} = \{x \in \mathcal{N} : x \text{ codes a well-ordering of } N\}$$

Lemma 40.2. *The sets* WF *and* WO *are* Π^1_1.

Proof. We prove in some detail that WF is Π^1_1. By the principle of dependent choices (see (39.2e)), E_x is well-founded if and only if there is no z: $N \to N$ such that $z(k + 1)\ E_x\ z(k)$ for all k. Thus

$$x \in \mathrm{WF} \quad \leftrightarrow \quad \forall z\ \exists k\ \neg z(k + 1)\ E_x\ z(k)$$

In other words, $\mathrm{WF} = \forall z\ A$, where

$$(x, z) \in A \quad \leftrightarrow \quad \exists k\ x(\Gamma(z(k + 1), z(k))) \neq 0$$

and it suffices to show that A is arithmetical. But

$$(x, z) \in A \quad \leftrightarrow \quad \exists n, m, i, j, k\,[i = (\bar{z}(n))(k + 1) \wedge j = (\bar{z}(n))(k) \wedge$$
$$m = \Gamma(i, j) \wedge (\bar{x}(n))(m) \neq 0]$$

To show that WO is Π_1^1 it suffices to verify that the set

$$\mathrm{LO} = \{x : E_x \text{ is a linear ordering of } \mathcal{N}\}$$

is arithmetical (and we leave this easy verification to the reader). Then $\mathrm{WO} = \mathrm{WF} \wedge \mathrm{LO}$ is Π_1^1. ■

We shall show later in this section that neither WF nor WO is a Σ_1^1 set; thus neither is a Borel set.

For each $x \in \mathrm{WF}$, let

(40.18) $\quad \|x\| = $ the length of the well-founded relation E_x

(see (2.11)). For each x, $\|x\|$ is a countable ordinal (and for each $\alpha < \omega_1$ there is $x \in \mathrm{WF}$ such that $\|x\| = \alpha$). If $x \in \mathrm{WO}$, then $\|x\|$ is the order type of the well ordering E_x.

Lemma 40.3. *For each* $\alpha < \omega_1$, *the sets*

$$\mathrm{WF}_\alpha = \{x \in \mathrm{WF} : \|x\| \leq \alpha\}, \qquad \mathrm{WO}_\alpha = \{x \in \mathrm{WO} : \|x\| \leq \alpha\}$$

are Borel sets.

Proof. Note that the set $\{(x, n) : n \in \mathrm{field}(E_x)\}$ is arithmetical (and hence Borel). Let us prove the lemma first for WO_α.

For each $\alpha < \omega_1$, let

$$B_\alpha = \{(x, n) : E_x \text{ restricted to } \{m : m\ E_x\ n\}$$
$$\text{is a well ordering of order type} \leq \alpha\}$$

We prove, by induction on $\alpha < \omega_1$, that each B_α is a Borel set. It is easy to see that B_0 is arithmetical. Thus let $\alpha < \omega_1$ and assume that all B_β, $\beta < \alpha$, are Borel. Then $\bigcup_{\beta<\alpha} B_\beta$ is Borel and hence B_α is also Borel because

$$(x, n) \in B_\alpha \quad \leftrightarrow \quad \forall m \left(m\ E_x\ n \rightarrow (x, m) \in \bigcup_{\beta<\alpha} B_\beta\right)$$

It follows that each WO_α is Borel because

$$x \in \mathrm{WO}_\alpha \quad \leftrightarrow \quad \forall n \left(n \in \mathrm{field}(E_x) \rightarrow (x, n) \in \bigcup_{\beta<\alpha} B_\beta\right)$$

To handle WF_α, note that the rank function ρ_E (see (2.12)) can be defined for any binary relation E; namely:

$$\rho_E(u) = \alpha \qquad \text{iff} \qquad \forall v(v\ Eu \rightarrow \rho_E(v) \text{ is defined) and}$$
$$\alpha = \sup\{\rho_E(v) + 1 : v\ E\ u\}$$

For each $\alpha < \omega_1$, let

$$C_\alpha = \{(x, n) : \rho_{E_x}(n) \text{ is defined and } \leq \alpha\}$$

Again, C_0 is arithmetical, and if we assume that all C_β, $\beta < \alpha$, are Borel, then C_α is also Borel:

$$(x, n) \in C_\alpha \quad \leftrightarrow \quad \forall m \Big(m \; E_x \; n \to (x, m) \in \bigcup_{\beta < \alpha} C_\beta \Big)$$

Hence each C_α is Borel, and it follows that each WF_α is Borel:

$$x \in \mathrm{WF}_\alpha \quad \leftrightarrow \quad \forall n \Big(n \in \mathrm{field}(E_x) \to (x, n) \in \bigcup_{\beta < \alpha} C_\beta \Big) \quad \blacksquare$$

Corollary. *The sets* $\{x \in \mathrm{WF} : \|x\| = \alpha\}$ *and* $\{x \in \mathrm{WF} : \|x\| < \alpha\}$ *are Borel (similarly for* WO*).*

Proof. $\{x \in \mathrm{WF} : \|x\| < \alpha\} = \bigcup_{\beta < \alpha} \mathrm{WF}_\beta$. ■

Exercise 40.1. The set $A = \{(x, z) : z \notin \mathrm{WO} \vee \|x\| \leq \|z\|\}$ is Σ_1^1. Hence for each α, WO_α is $\Sigma_1^1(z)$ for each $z \in \mathrm{WO}$ such that $\|z\| = \alpha$.

[$(x, z) \in A \quad \leftrightarrow \quad z \notin \mathrm{WO} \vee (\exists h : N \to N)\forall m \; \forall n (m \; E_x \; n \to h(m) \; E_z \; h(n))$.]

The Tree Representation of Π_1^1 Sets in $\mathcal{N}$

A Σ_1^1 set in $\mathcal{N}$ is the projection of a closed set in $\mathcal{N}^2$, and every closed set in $\mathcal{N}^2$ has the representation

$$[T] = \{(a_1, a_2) \in \mathcal{N}^2 : \forall n \; (\bar{a}_1(n), \bar{a}_2(n)) \in T\}$$

where T is a tree, $T \subseteq Seq_2$. The projection of $[T]$ is the Σ_1^1 set

(40.19) $$A = \{x \in \mathcal{N} : \exists z \in \mathcal{N} \; \forall n \; (\bar{a}_1(n), \bar{a}_2(n)) \in T\}$$

Let us define, for a given $T \subseteq Seq_2$ and each $x \in \mathcal{N}$, a tree $T(x) \subseteq Seq$ as follows:

(40.20) $$t \in T(x) \quad \leftrightarrow \quad (\bar{x}(n), t) \in T \text{ where } n = \mathrm{length}(t)$$

Thus

(40.21) $$\begin{aligned} x \in A \quad &\leftrightarrow \quad \exists z \; \forall n \; \bar{z}(n) \in T(x) \\ &\leftrightarrow \quad [T(x)] \text{ is nonempty} \end{aligned}$$

Let us consider the partial ordering of *Seq* by inverse inclusion:

(40.22) $$s_1 \leq s_2 \quad \text{iff} \quad s_1 \supseteq s_2$$

Then we can reformulate (40.21) as follows:

(40.23) $$x \in A \quad \leftrightarrow \quad (T(x), \leq) \text{ is not well-founded}$$

Thus if C is a Π_1^1 set, its complement A is Σ_1^1 and has the representation (40.19), and we have

$$(40.24) \qquad x \in C \quad \leftrightarrow \quad T(x) \text{ is well-founded}$$

Moreover, if C is $\Pi_1^1(a)$, then there is $T \subseteq Seq_2$, arithmetical in a, such that (40.24) holds. Conversely, if $T \subseteq Seq_2$ is a tree arithmetical in a, then the set C defined by (40.24) is $\Pi_1^1(a)$; this is because the complement of C satisfies the $\Sigma_1^1(a)$ definition (40.19).

The formula (40.24) gives a useful representation of Π_1^1 sets. Before we proceed, let me show how this can be slightly improved. Let us extend the partial ordering $\leq$ of Seq to a linear ordering $\prec$ as follows: If s, t are incompatible finite sequences, let n be the least n such that $s(n) \neq t(n)$; then let $s \prec t$ just in case $s(n) < t(n)$. Verify that $\preceq$ is a linear ordering and note that for every tree $T \subseteq Seq$,

$$(40.25) \qquad (T, \leq) \text{ is well-founded} \quad \text{iff} \quad (T, \preceq) \text{ is well-order}$$

Thus if C is a Π_1^1 set, there is a tree $T \subseteq Seq_2$ such that

$$(40.26) \qquad x \in C \quad \leftrightarrow \quad (T(x), \preceq) \text{ is a well ordering}$$

Lemma 40.4. *If C is a Π_1^1 set, then there exists a continuous function $f\colon \mathcal{N} \to \mathcal{N}$ such that $C = f_{-1}(\mathrm{WF})$, and there exists a continuous function $g\colon \mathcal{N} \to \mathcal{N}$ such that $C = g_{-1}(\mathrm{WO})$.*

Proof. We shall give the proof for WF; the proof for WO is similar. Let $T \subseteq Seq_2$ be such that

$$x \in C \quad \leftrightarrow \quad T(x) \text{ is well-founded}$$

Let $\{t_0, t_1, \ldots, t_n, \ldots\}$ be an enumeration of the set Seq. For each $x \in \mathcal{N}$, we let $y = f(x)$ be the following element of $\mathcal{N}$:

$$y(\Gamma(m, n)) = \begin{cases} 0 & \text{if } \ t_m, t_n \in T(x), \text{ and } t_m < t_m \\ 1 & \text{otherwise} \end{cases}$$

It is clear that E_y is isomorphic to $(T(x), <)$, and hence $y \in \mathrm{WF}$ if and only if $T(x)$ is well-founded. Thus $C = f_{-1}(\mathrm{WF})$ and it remains to show only that f is continuous. But it should be obvious from the definitions of $T(x)$ and of $y = f(x)$ that for any finite sequence $s = \langle \varepsilon_0, \ldots, \varepsilon_{k-1} \rangle$, there is $\check{s} \in Seq$ such that if $x \supset \check{s}$ and $y = f(x)$, then $\bar{y}(k) = s$. Hence f is continuous. ■

Corollary 1. WF *is not* Σ_1^1; WO *is not* Σ_1^1.

Proof. Otherwise every Π_1^1 set would be the inverse image by a continuous function of an analytic set and hence analytic; however, there are Π_1^1 sets that are not analytic.

Corollary 2. (*Boundedness Lemma*). *If $B \subseteq \mathrm{WO}$ is Σ_1^1, then there is $\alpha < \omega_1$ such that $\|x\| < \alpha$ for all $x \in B$.*

Proof. Otherwise we would have

$$\mathrm{WO} = \{x \in \mathcal{N} : \exists z(z \in B \wedge \|x\| \leq \|z\|)\}$$

Here $\|x\| \leq \|z\|$ for $x, z \in \mathcal{N}$ means: either $z \notin \mathrm{WO}$ or $\|x\| \leq \|z\|$; this relation is Σ_1^1; see Exercise 40.1. This would mean that WO is Σ_1^1, a contradiction. ■

Corollary 3. *Every Π_1^1 set is the union of $\aleph_1$ Borel sets.*

Proof. If C is Π_1^1, then $C = f_{-1}(\mathrm{WF})$ for some continuous f. But $\mathrm{WF} = \bigcup_{\alpha<\omega_1} \mathrm{WF}_\alpha$, and hence

$$C = \bigcup_{\alpha<\omega_1} f_{-1}(\mathrm{WF}_\alpha)$$

Each $f_{-1}(\mathrm{WF}_\alpha)$ is the inverse image of a Borel set by a continuous function, hence Borel. ■

Corollary 4. *Assuming the axiom of choice, every Π_1^1 set is either at most countable, or has cardinality $\aleph_1$, or cardinality $2^{\aleph_0}$.*

Proof. Follows from Corollary 3. ■

Theorem 95 below improves on Corollary 3 by showing that every Σ_2^1 set is the union of $\aleph_1$ Borel sets. The following lemma is the first step toward that theorem.

Lemma 40.5. *Every Σ_1^1 set is the union of $\aleph_1$ Borel sets.*

Proof. Let A be a Σ_1^1 set. Let $T \subseteq Seq_2$ be a tree such that

$$x \in A \quad \leftrightarrow \quad T(x) \text{ is not well-founded}$$

If $U \subseteq Seq$ is a tree and $t \in Seq$, let U_t denote the tree $\{s \in Seq : t^\frown s \in U\}$ (hence $U_\emptyset = U$). If $(U, \supseteq)$ is well-founded, let $\|U\|$ denote the length of the well-founded relation $\supseteq$ on U.

As in Lemma 40.3, we prove by induction on α that for each $t \in Seq$ and every $\alpha < \omega_1$, the set

$$\{x \in \mathcal{N} : \|(T(x))_t\| \leq \alpha\} \tag{40.27}$$

is Borel. Namely, $\{x : \|T(x)_t\| \leq 0\} = \{x : (\bar{x}(n), t) \notin T\}$ and if $\alpha > 0$, then $\|T(x)_t\| \leq \alpha$ if and only if $\forall n(\exists \beta < \alpha)\ \|T(x)_{tn}\| \leq \beta$.

Let us define, for each α, the set B_α as follows:

$$x \in B_\alpha \quad \leftrightarrow \quad \neg(\|T(x)\| < \alpha) \wedge \forall t(\neg \|(T(x))_t\| = \alpha) \tag{40.28}$$

Since the sets in (40.27) are Borel, it follows that each B_α is Borel. We shall prove that $A = \bigcup_{\alpha<\omega_1} B_\alpha$. First let $x \in A$. Thus $T(x)$ is not well-founded; hence $\|T(x)\| \not< \alpha$ for any α, and it suffices to show that there is α such that $\|T(x)_t\| \neq \alpha$ for all t. If there is no such α, then for every α there is t such that $\|T(x)_t\| = \alpha$, but there are $\aleph_1$ α's and only $\aleph_0$ t's; a contradiction.

Next let $x \notin A$, and let us show that $x \notin B_\alpha$, for all α. Let $\alpha < \omega_1$ be arbitrary. Since $T(x)$ is well-founded, then either $\|T(x)\| < \alpha$ and $x \notin B_\alpha$, or $\|T(x)\| \geq \alpha$ and there exists $t \in T(x)$ such that $\|(T(x))_t\| = \alpha$, and again $x \notin B_\alpha$. ■

Theorem 95 (Sierpinski). *Every Σ^1_2 set is the union of $\aleph_1$ Borel sets.*

We postpone the proof until we have obtained a representation of Σ^1_2 sets by trees over $\omega \times \omega_1$ (see below). If we want to use the axiom of choice, then we can give a quick proof of Theorem 95: A Σ^1_2 set A is the projection of a Π^1_1 set in $\mathcal{N}^2$, hence the continuous image of a Π^1_1 set C in $\mathcal{N}$, $A = f[C]$. The set C is the union of $\aleph_1$ Borel sets B_α, $\alpha < \omega_1$, and hence A is the union of $\aleph_1$ analytic sets $A_\alpha = f[B_\alpha]$. By Lemma 40.5, each A_α is the union of $\aleph_1$ Borel sets, and hence A is the union of $\aleph_1$ Borel sets.

Corollary. *Assuming the axiom of choice, every Σ^1_2 set is either at most countable, or has cardinality $\aleph_1$, or cardinality $2^{\aleph_0}$.* ■

Uniformization of Π^1_1 Relations

We have mentioned above (40.27) that if $T \subseteq Seq_2$ is a tree, then for every $\alpha < \omega_1$, the set

$$\{x \in \mathcal{N} : \|T(x)\| \leq \alpha\}$$

is Borel. This fact is also a consequence of the following general result:

Lemma 40.6. *Let T be a tree, $T \subseteq Seq_2$, arithmetical in $a \in \mathcal{N}$. The following relations $P, Q \subseteq \mathcal{N}^2$ are $\Sigma^1_1(a)$:*

$$\begin{aligned} &(x, y) \in P \quad \leftrightarrow \quad T(y) \text{ is not well-founded or } \|T(x)\| \leq \|T(y)\| \\ (40.29)\quad &(x, y) \in Q \quad \leftrightarrow \quad T(y) \text{ is not well-founded or } \|T(x)\| < \|T(y)\| \end{aligned}$$

Proof. First we observe that if T and S are two well-founded trees $\subseteq Seq$ and if $\|T\| \leq \|S\|$, then there is a mapping h: $T \to S$ such that $s \subset t \in T(x)$ implies $h(s) \subset h(t)$: This is proved by induction on $\|S\|$ as follows. For every $\langle n\rangle \in T$ let s_n be such that $\|T_{\langle n\rangle}\| \leq \|S_{s_n}\|$ and let h_n be an order-preserving mapping of $T_{\langle n\rangle}$ into S_{s_n}. Then let $h(\emptyset) = \emptyset$ and $h(\langle n\rangle^\frown t) = s_n{}^\frown h_n(t)$.

Conversely, if there is an order-preserving h: $T \to S$, then either $\|T\| \leq \|S\|$ or S is not well-founded. Thus we have

(40.30) $(x, y) \in P \leftrightarrow T(y)$ is not well founded or there is an order-preserving h: $T(x) \to T(y)$

and by the same argument

(40.31) $(x, y) \in Q \leftrightarrow T(y)$ is not well-founded or there is n and an order-preserving h: $T(x) \to (T(y))_{\langle n\rangle}$.

To verify that (40.30) and (40.31) are $\Sigma_1^1(a)$, we first note that "$T(y)$ is not well founded" is $\Sigma_1^1(a)$. Secondly, we replace the quantifier "there is h: $Seq \to Seq$" by the quantifier "there is z: $N \to N$" by using an arithmetical enumeration $\{s_n : n \in N\}$ of the set Seq: For each $z \in \mathcal{N}$, let h_z: $Seq \to Seq$ be defined by $h_z(s_n) = s_{z(n)}$. Then we can express

"there is an order-preserving h: $T(x) \to T(y)$"

as $\exists z \in \mathcal{N}\ R(x, y, z)$ where R is arithmetical in T; hence (40.30) is $\Sigma_1^1(a)$. The same argument shows that (40.31) is $\Sigma_1^1(a)$. ■

Remark. Note that $(x, y) \in P \wedge (y, x) \in P$ is equivalent to: Either $\|T(x)\| = \|T(y)\|$, or neither $T(x)$ nor $T(y)$ are well-founded.

Lemma 40.6 can be slightly generalized. Again, let $T \subseteq Seq_2$ be a tree arithmetical in $a \in \mathcal{N}$, and let $\{t_n : n \in N\}$ be an arithmetical enumeration of the set Seq. Then the following relations A, $B \subseteq \mathcal{N}^2 \times N$ are $\Sigma_1^1(a)$:

$$(40.32) \qquad (x, y, n) \in A \;\leftrightarrow\; (T(y))_{t_n} \text{ is not well-founded or } \|(T(x))_{t_n}\| \le \|(T(y))_{t_n}\|$$

$$(x, y, n) \in B \;\leftrightarrow\; (T(y))_{t_n} \text{ is not well-founded or } \|(T(x))_{t_n}\| < \|(T(y))_{t_n}\|$$

The proof of (40.32) is essentially the same as the proof of the lemma.

The following theorem is a typical application of the representation (40.24) of Π_1^1 sets. Let $P \subseteq \mathcal{N} \times \mathcal{N}$ be a relation, and let F be a function such that $F \subseteq \mathcal{N} \times \mathcal{N}$, i.e., $\mathrm{dom}(F) \subseteq \mathcal{N}$, and $\mathrm{ran}(F) \subseteq \mathcal{N}$. We say that F *uniformizes* P if $\mathrm{dom}(F) = \mathrm{dom}(P)$, i.e., $\mathrm{dom}(F) = \{x : \exists y\ (x, y) \in P\}$.

Theorem 96 (*Kondô's Π_1^1-uniformization Theorem*). *Every $\Pi_1^1(a)$ relation $P \subseteq \mathcal{N} \times \mathcal{N}$ can be uniformized by a $\Pi_1^1(a)$ function $f \subseteq \mathcal{N} \times \mathcal{N}$.*

["f is Π_1^1" means that $\{(x, y) : y = f(x)\}$ is a Π_1^1 relation.]

To prove the theorem, we have to construct, for each $x \in \mathcal{N}$, a subset Q_x of the set $P_x = \{(y : (x, y) \in P\}$ such that if P_x is nonempty, then Q_x has exactly one element, and that $Q = \{(x, y) : y \in Q_x\}$ is $\Pi_1^1(a)$. This we shall do uniformly for all $x \in \mathcal{N}$. The following lemma shows how the sets $Q_x \subseteq P_x$ are obtained.

Lemma 40.7. *Let $P \subseteq \mathcal{N}$ be a Π_1^1 set. Then there is a Π_1^1 set $Q \subseteq P$ such that Q has at most one element; and if $P \neq \emptyset$, then $Q \neq \emptyset$.*

Proof. Let $T \subseteq Seq_2$ be an arithmetical tree such that

$$y \in P \;\leftrightarrow\; T(y) \text{ is well-founded}$$

Let $\{t_n : n \in N\}$ be an arithmetical enumeration of the set Seq and let us stipulate that $t_0 = \emptyset$.

We shall construct, by induction on n, a sequence

$$P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_n \supseteq \cdots$$

and let $Q = \bigcap_{n=0}^{\infty} P_n$. It will be clear that Q has at most one element; we shall then show that if $P \neq \emptyset$, then $Q \neq \emptyset$, and that Q is Π^1_1. Let $P_0 = P$. For each n, let

(40.33)

$$\begin{array}{lll} y \in P_{n+1} & \text{iff} & \text{(a)}\ \ y \in P_n \\ & & \text{(b)}\ \ y(n) \leq z(n) \qquad \text{for all} \quad z \in P_n \\ & & \text{(c)}\ \ \|(T(y))_{t_n}\| \leq \|(T(z))_{t_n}\| \qquad \text{for all} \quad z \in P_n \\ & & \qquad \text{such that } z(n) = y(n) \end{array}$$

In other words, we let P_{n+1} be a subset of P_n such that all $y \in P_{n+1}$ have the same $y(n)$ and the same $\|(T(y))_{t_n}\|$. It is therefore clear that there is at most one $y \in \mathcal{N}$ that belongs to all P_n, $n \in N$.

Let P be nonempty. Then each P_n is nonempty, and let y be the unique candidate for membership in Q; i.e., for each n, $y(n)$ is the unique k_n such that $x(n) = k_n$ for all $x \in P_{n+1}$. In order to show that Q is nonempty, we have to verify that $y \in P$. In other words, we have to prove that $T(y)$ is well-founded.

Let $t \in T(y)$, and let $n \in N$ be such that $t = t_n$. For all $x \in P_{n+1}$, the ordinal $\|(T(x))_{t_n}\|$ is the same; let $\alpha(t)$ denote this unique ordinal. We shall show that if $s \subset t \in T(y)$, then $\alpha(s) > \alpha(t)$. This implies immediately that $T(y)$ is well-founded.

Let $s, t \in T(y)$ be such that $s \subset t$, and let m and n be such that $s = t_m$ and $t = t_n$. Let k be such that $k \geq m$ and $k \geq n$, and let $x \in P_{k+1}$. Then $x \in P_{m+1}$ and hence $\alpha(s) = \|(T(x))_s\|$; and similarly $\alpha(t) = \|(T(x))_t\|$. But because $s \subset t$, we have $\alpha(s) = \|(T(x))_s\| > \|(T(x))_t\| = \alpha(t)$.

It remains to show that Q is Π^1_1. The following equivalences verify that $\neg Q$ is Σ^1_1; A and B are the relations from (40.32):

$$y \notin Q \qquad \text{iff} \qquad y \notin P \vee \exists n (y \in P_n \wedge y \notin P_{n+1}) \qquad \text{iff} \qquad y \notin P \vee y \in R$$

where

$$\begin{aligned} y \in R \quad \leftrightarrow \quad & \exists n\, \exists z [z \in P \wedge (\forall k < n)(z(k) = y(k) \wedge \|(T(z))_{t_k}\| = \|(T(y))_{t_k}\|) \\ & \wedge (z(n) < y(n) \vee (z(n) = y(n) \wedge \|T(z)_{t_n}\| < \|T(y)_{t_n}\|))] \end{aligned}$$

Now, if $y \in P$, $T(y)$ is well-founded; and it is easy to see that for all $y \in P$, $y \in R$ iff $y \in S$ where

$$\begin{aligned} y \in S \quad \leftrightarrow \quad & \exists n\, \exists z [(\forall k < n)(z(k) = y(k) \wedge (y, z, k) \in A \wedge (z, y, n) \in A) \\ & \wedge (z(n) < y(n) \vee (z(n) = y(n) \wedge (z, y, n) \in B))] \end{aligned}$$

(Note that if z satisfies the right-hand side, then $T(z) = (T(z))_{t_0}$ is well-founded and hence $z \in P$.) Thus

$$y \notin Q \quad \text{iff} \quad y \notin P \vee y \in S$$

Since T is arithmetical, the relations A and B are Σ^1_1 and hence Q is Π^1_1. ∎

Proof of Theorem 96. Although we stated the tree representation (40.24) explicitly only for Π^1_1 subsets of $\mathcal{N}$, it is obvious that r-ary relations can be characterized in an analogous way: If $C \subseteq \mathcal{N}^r$ is $\Pi^1_1(a)$, then there is $T \subseteq Seq_{r+1}$, arithmetical in a, such that

$$(x_1, \ldots, x_r) \in C \quad \text{iff} \quad T(x_1, \ldots, x_r) \text{ is well-founded}$$

where

$$t \in T(x_1, \ldots, x_r) \;\leftrightarrow\; (\bar{x}_1(n), \ldots, \bar{x}_r(n), t) \in T \qquad \text{where} \quad n = \text{length}(t)$$

Thus let $P \subseteq \mathcal{N} \times \mathcal{N}$ be $\Pi^1_1(a)$. Let $T \subseteq Seq_3$ be a tree arithmetical in a, such that

$$(x, y) \in P \quad \text{iff} \quad T(x, y) \text{ is well-founded}$$

For each $x \in \mathcal{N}$, we let $P^x = \{y : (x, y) \in P\}$. We define $P^x_0 \supseteq P^x_1 \supseteq \cdots \supseteq P^x_n \supseteq \cdots$ exactly as in (40.33) (with $T(y)$ replaced by $T(x, y)$), and then $Q^x = \bigcap_{n=0}^{\infty} P^x$. If P^x is nonempty, then Q^x has a single element which we denote $f(x)$. Thus $f = \{(x, y) : y \in Q^x\}$.

It is obvious that the proof of Lemma 40.7 is uniform for all x; and in particular, the proof of Π^1_1-ness provides a proof that the set f is Π^1_1 in a. ∎

Exercise 40.2. Every $\Sigma^1_2(a)$ relation $R \subseteq \mathcal{N} \times \mathcal{N}$ can be uniformized by a $\Sigma^1_2(a)$ function.

[Let $P \subseteq \mathcal{N}^3$ be such that $(x, y) \in R \leftrightarrow \exists z (x, y, z) \in P$, and that P is $\Pi^1_1(a)$. There is a $\Pi^1_1(a)$ relation $Q \subseteq P$ such that for every x, if there is (y, z) such that $(x, y, z) \in P$, then there is a unique (y, z) such that $(x, y, z) \in Q$. Let $f = \{(x, y) : \exists z (x, y, z) \in Q\}$; f uniformizes P.]

A collection $\mathscr{C}$ of subsets of $\mathcal{N}^r$ satisfies the *reduction principle* if for every pair $A, B \in \mathscr{C}$ there are disjoint $A', B' \in \mathscr{C}$ such that $A' \subseteq A$, $B' \subseteq B$, and $A' \cup B' = A \cup B$. $\mathscr{C}$ satisfies the *separation principle* if for every pair of disjoint set $A, B \in \mathscr{C}$ there is a set E such that both E and $\neg E$ are in $\mathscr{C}$, and that $A \subseteq E$ and $B \subseteq \neg E$. Lemma 39.5 proves that the collection of all analytic sets satisfies the separation principle.

Exercise 40.3. The collection of Π^1_1 sets satisfies the reduction principle.

[Let A and B be Π^1_1 and let T_1 and T_2 be such that $x \in A \leftrightarrow T_1(x)$ is well-founded, and $x \in B \leftrightarrow T_2(x)$ is well-founded. Let

$$x \in A' \leftrightarrow x \in A \wedge \neg \|T_2(x)\| < \|T_1(x)\|, \qquad x \in B' \leftrightarrow x \in B \wedge \neg \|T_1(x)\| \leq \|T_2(x)\|.]$$

Exercise 40.4. The collection of Σ^1_2 sets satisfies the reduction principle.

[Let A and B be Σ_2^1, and let P and Q be Π_1^1 such that $x \in A \leftrightarrow \exists z\ (x, z) \in P$, $x \in B \leftrightarrow \exists z\ (x, z) \in Q$. Let f be a Π_1^1 function that uniformizes $P \cup Q$. Let $x \in A' \leftrightarrow \exists z((x, z) \in P \wedge f(x) = z)$, $x \in B' \leftrightarrow \exists z((x, z) \in Q \wedge (x, z) \notin P \wedge f(x) = z)$.]

The two exercises above hold also for $\Pi_1^1(a)$ and $\Sigma_2^1(a)$.

Exercise 40.5. If a collection $\mathscr{C}$ satisfies the reduction principle then the collection $\mathscr{C}^* = \{A : \neg A \in \mathscr{C}\}$ satisfies the separation principle.

[If $A, B \in \mathscr{C}^*$ are disjoint, then $\neg A \cup \neg B = \mathcal{N}^r$ and so if $A', B' \in \mathscr{C}$ are disjoint such that $A' \subseteq \neg A$, $B' \subseteq \neg B$ and $A' \cup B' = \neg A \cup \neg B$, then $B' = \neg A'$ and both A' and B' are in $\mathscr{C}^*$.]

Hence the separation principle holds for Σ_1^1 and for Π_2^1 (and $\Sigma_1^1(a)$ and $\Pi_2^1(a)$).

Exercise 40.6. There is no universal Δ_n^1 set, for any $n \in N$, i.e. no $D \subseteq \mathcal{N}^2$ such that D is Δ_n^1 and that for every Δ_n^1 set $A \subseteq \mathcal{N}$ there is $v \in \mathcal{N}$ such that $A = \{x : (x, v) \in D\}$.

[Assume there is such a D and let $A = \{x : (x, x) \in D\}$.]

Exercise 40.7. The collection of Π_1^1 sets (or Σ_2^1 sets) does not satisfy the separation principle.

[The reason is that Π_1^1 satisfies the reduction principle (Σ_2^1 is similar). Let h be a homeomorphism of $\mathcal{N} \times \mathcal{N}$ onto $\mathcal{N}$, and let $U \subseteq \mathcal{N}^2$ be a universal Π_1^1 set. Let $(x, h(u, v)) \in A$ iff $(x, u) \in U$, $(x, h(u, v)) \in B$ iff $(x, v) \in P$, and let A', B' be disjoint Π_1^1 sets such that $A' \subseteq A$, $B' \subseteq B$, and $A' \cup B' = A \cup B$. If there existed $E \in \Delta_1^1$ such that $A' \subseteq E$ and $B' \subseteq \neg E$, then E would be a universal Δ_1^1 set.]

The Tree Representation of Σ_2^1 Sets

Let us generalize the notion of a sequential tree. Let K be a set. We denote $Seq(K)$ the set of all finite sequences in K. A *tree over* $\omega \times K$ is a set of pairs $(s, h) \in Seq \times Seq(K)$ such that length(s) = length(h) and such that for each $n \leq$ length(s), $(\bar{s}(n), \bar{h}(n))$ is also in T.

(For any sequence f of length $\leq \omega$, $\bar{f}(n)$ denotes $\langle f(0), \ldots, f(n-1)\rangle$.)

Let T be a tree over $\omega \times K$. For each $s \in Seq$, we define $T_s \subseteq Seq(K)$ as follows: Let n = length(s), and let

(40.34) $h \in T_s \quad \leftrightarrow \quad \text{length}(h) \leq n$ and $(\bar{s}(k), h) \in T$ where $k = \text{length}(h)$

If $h \in T_s$, then $\bar{h}(k) \in T_s$ for all $k \leq$ length(h). If $s \subseteq t$, then $T_s \subseteq T_t$; and in fact T_t is an end extension of T_s because if length(h) = length(s), then $h \in T_t$ iff $h \in T_s$. Thus for every $x \in \mathcal{N}$ we let

(40.35) $$T(x) = \bigcup \{T_s : s \subset x\}$$

$T(x)$ is a tree over K (i.e., $T \subseteq Seq(K)$). Notice that (40.35) is a generalization of (40.20) (where $K = \omega$).

If T is a tree over $\omega \times K$, we define

$$(40.36) \qquad \begin{aligned} p[T] &= \{x \in \mathcal{N} : (\exists f \in {}^{\omega}K)\, \forall n\, (\bar{x}(n), \bar{f}(n)) \in T)\} \\ &= \{x \in \mathcal{N} : \exists f\, \forall n\, \bar{f}(n) \in T(x)\} \\ &= \{x \in \mathcal{N} : T(x) \text{ is not well-founded}\} \end{aligned}$$

("$T(x)$ is well-founded" means the relation $\supset$ over $T(x)$ is well-founded) and

$$[T] = \{(x, f) \in \mathcal{N} \times {}^{\omega}K : \forall n\, (\bar{x}(n), \bar{f}(n)) \in T\}$$

Trees over $\omega^r \times K$ are defined analogously; if T is a tree over $\omega^r \times K$, then $p_r[T]$ denotes the set

$$\{(x_1, \ldots, x_r) \in \mathcal{N}^r : \exists f\, \forall n\, (\bar{x}_1(n), \ldots, \bar{x}_r(n), \bar{f}(n)) \in T\}$$

The normal form for Σ_1^1 sets states that a set $A \subseteq \mathcal{N}$ is $\Sigma_1^1(a)$ if and only if there is a tree over $\omega \times \omega$, arithmetical in a, such that $A = p[T]$. We shall now show that Σ_2^1 sets admit a tree representation. (The representation generalizes easily from $\mathcal{N}$ to $\mathcal{N}^r$.)

Lemma 40.8. *For every Σ_2^1 set $A \subseteq \mathcal{N}$ there is a tree T over $\omega \times \omega_1$ such that $A = p[T]$. Moreover, if A is $\Sigma_2^1(a)$, then T is constructible from a.*

Proof. Let A be a $\Sigma_2^1(a)$ subset of $\mathcal{N}$. There is a tree $U \subseteq Seq_3$, arithmetical in a, such that

$$x \in A \quad \leftrightarrow \quad \exists y\, \forall z\, \exists n\, (\bar{x}(n), \bar{y}(n), \bar{z}(n)) \notin U$$

In other words,

$$x \in A \quad \leftrightarrow \quad \exists y\; U(x, y) \text{ is well-founded}$$

A necessary and sufficient condition for a countable relation to be well-founded is that it admits an order-preserving mapping into ω_1. Thus

$$\begin{aligned} x \in A \quad &\leftrightarrow \quad \exists y\, (\exists f : U(x, y) \to \omega_1)[\text{if } u \subset v, \text{ then } f(u) > f(v)] \\ &\leftrightarrow \quad \exists y\, (\exists f : Seq \to \omega_1)[f \restriction U(x, y) \text{ is order-preserving}] \end{aligned}$$

Let $\{u_n : n \in N\}$ be an arithmetical enumeration of the set Seq such that for every n, length$(u_n) \leq n$. If f is a function on (a subset of) N, let f^* be the function on (a subset of) Seq defined by $f^*(u_n) = f(n)$. Thus

$$(40.37) \quad x \in A \quad \leftrightarrow \quad \exists y\, (\exists f : \omega \to \omega_1)[f^* \restriction U(x, y) \text{ is order-preserving}]$$

Now we define a tree T' over $\omega^2 \times \omega_1$ as follows: If $s, t \in Seq$ and $h \in Seq(\omega_1)$ are all of length n, we let

$$(40.38) \qquad (s, t, h) \in T' \quad \leftrightarrow \quad h^* \restriction U_{s,t} \text{ is order-preserving}$$

Clearly, T' is a tree over $\omega^2 \times \omega_1$.

Let $x, y \in \mathcal{N}$. I claim that if $(\bar{x}(n), \bar{y}(n), h) \in T'$, then $h^* \mid U(x, y)$ is order-preserving. This is because if $u, v \in \text{dom}(h) \cap U(x, y)$, then $u = u_i$, $v = u_j$ for some $i, j < n$, hence length(u), length$(v) < n$ and hence $u, v \in U_{st}$, where $s = \bar{x}(n)$, $t = \bar{y}(n)$. Thus

$$(40.39) \qquad f \in T'(x, y) \quad \leftrightarrow \quad \forall n[(\bar{f}(n))^* \mid U(x, y) \text{ is order-preserving}]$$

But clearly a mapping $f\colon \omega \to \omega_1$ satisfies the right-hand side if and only if $f^* \mid U(x, y)$ is order-preserving. Hence (40.37) and (40.38) give

$$\begin{aligned} x \in A \quad &\leftrightarrow \quad \exists y\ \exists f\colon \omega \to \omega_1\ f \in T'(x, y) \\ &\leftrightarrow \quad \exists y\ \exists f\colon \omega \to \omega_1\ \forall n\ (\bar{x}(n), \bar{y}(n), \bar{f}(n)) \in T' \end{aligned}$$

Now we transform T' (over $\omega^2 \times \omega_1$) into a tree T'' (over $\omega \times K$ where $K = \omega \times \omega_1$) such that we replace triples

$$(\langle s(0), \ldots, s(n-1)\rangle, \langle t(0), \ldots, t(n-1)\rangle, \langle h(0), \ldots, h(n-1)\rangle)$$

by pairs

$$(\langle s(0), \ldots, s(n-1)\rangle, \langle (t(0), h(0)), \ldots, (t(n-1), h(n-1))\rangle)$$

and we get

$$x \in A \quad \leftrightarrow \quad (\exists g\colon \omega \to K)\ \forall n\ (\bar{x}(n), \bar{g}(n)) \in T''$$

Since $K = \omega \times \omega_1$ is in an obvious one-to-one correspondence with ω_1, it is clear that we can find a tree T over $\omega \times \omega_1$ such that

$$(40.40) \qquad x \in A \quad \leftrightarrow \quad (\exists g\colon \omega \to \omega_1)\ \forall n\ (\bar{x}(n), \bar{g}(n)) \in T$$

that is $A = p[T]$. As for the "moreover" clause, it is obvious that the tree T so obtained is constructible from the tree U, which in turn is constructible from a. ■

The tree representation of Σ^1_2 sets is very useful in applications. As a first application, we shall finish the proof of Theorem 95.

Proof of Theorem 95. Let A be a Σ^1_2 set. By Lemma 40.8 there is a tree T over $\omega \times \omega_1$ such that $A = p[T]$. For each $\gamma < \omega_1$ let $T^\gamma = \{(s, h) \in T : h \in Seq(\gamma)\}$. Since every $f\colon \omega \to \omega_1$ has the range included in some $\gamma < \omega_1$, it is clear that

$$A = \bigcup_{\gamma < \omega_1} p[T^\gamma]$$

For each $\gamma < \omega_1$, the set $p[T^\gamma]$ is analytic (because $p[T^\gamma] = p[\tilde{T}]$ for some $\tilde{T} \subseteq Seq_2$) and is the union of $\aleph_1$ Borel sets. In fact, Lemma 40.5 gives a uniform decomposition into Borel sets for any $p[U]$ where U is a tree over $\omega \times S$ with S countable. If we let

$$x \in B^\gamma_\alpha \quad \to \quad \neg(\|T^\gamma(x)\| < \alpha) \wedge (\forall t \in Seq(\gamma))\ (\neg \|(T^\gamma(x))_t\| = \alpha)$$

(cf. 40.28), then $A = \bigcup_{\alpha < \omega_1} \bigcup_{\gamma < \omega_1} B^\gamma_\alpha$. ■

41. PROJECTIVE SETS IN THE CONSTRUCTIBLE UNIVERSE

We have seen in Chapters 2 and 3 that many problems in abstract set theory cannot be decided on the basis of the axioms of ZFC alone. The same is true for many problems in descriptive set theory. However, various questions about projective sets (e.g., which projective sets are Lebesgue measurable) can be settled in the constructible universe. In this section we shall investigate the relationship between constructibility and complexity of projective sets.

All results hold interchangeably for projective sets (and relations) in the real line $\boldsymbol{R}$, the Baire space $\mathcal{N}$, or the Cantor space $\boldsymbol{C}$. I shall usually consider the space for which the proof is technically easiest.

Theorem 97 (Gödel). *The set of all constructible reals is a Σ_2^1 set. The relation $<_L$ (the canonical well-ordering of constructible sets) is a Σ_2^1 relation over $\boldsymbol{R}$.*

The field of $<_L$ is $\boldsymbol{R} \cap L$. If all reals are constructible, then $<_L$ is also Π_2^1 (because $x \not<_L y$ iff $y <_L x$) and hence $<_L$ is a Δ_2^1 relation over $\boldsymbol{R}$.

Theorem 97 easily generalizes to $L[a]$: If $a \in \boldsymbol{R}$ (or $a \subseteq \omega$ or $a \in \mathcal{N}$), then the set $\boldsymbol{R} \cap L[a]$ is $\Sigma_2^1(a)$; also, the relation "x is constructible from y" is a Σ_2^1 relation over $\boldsymbol{R}$.

We proved in Section 14 (see Theorem 35 and Exercises 14.16 and 14.17) that "x is constructible" and "$x <_L y$" are Σ_1 relations over the model HC of all hereditarily countable sets. Thus Theorem 97 follows from this lemma:

Lemma 41.1. *If $A \subseteq \mathcal{N}$ is such that A is Σ_1 over $\langle \mathrm{HC}, \in \rangle$, then A is a Σ_2^1 set in $\mathcal{N}$.*

Proof. If A is Σ_1 over HC, there exists a Σ_0 formula φ such that

$$\begin{aligned} x \in A \quad &\leftrightarrow \quad \mathrm{HC} \vDash \exists u\, \varphi(u, x) \\ &\leftrightarrow \quad (\exists u \in \mathrm{HC})\ \mathrm{HC} \vDash \varphi[u, x] \end{aligned}$$

Since φ is Σ_0, it is absolute for transitive models and we have

$$x \in A \quad \leftrightarrow \quad (\exists \text{ transitive set } M \in \mathrm{HC})(\exists u \in M)\ M \vDash \varphi[u, x]$$

(e.g. $M = \mathrm{TC}(\{u, x\})$). By the principle of dependent choices every $M \in \mathrm{HC}$ is countable and we have

$$\begin{aligned} x \in A \quad &\leftrightarrow \quad (\exists \text{ countable transitive set } M)\ (\exists u \in M)\ M \vDash \varphi[u, x] \\ &\leftrightarrow \quad (\exists \text{ well founded extensional relation } E \text{ over } \omega) \\ &\qquad \exists n\ \exists m\ [\pi_E(m) = x \text{ and } \langle \omega, E \rangle \vDash \varphi[n, m]] \end{aligned}$$

where π_E is the transitive collapse of $\langle \omega, E \rangle$ onto $\langle M, \in \rangle$. Recall the definition

(40.17) of E_z for $z \in \mathcal{N}$; thus

$$(41.1) \quad x \in A \quad \leftrightarrow \quad (\exists z \in \mathcal{N})[z \in \mathrm{WF} \text{ and } \langle \omega, E_z \rangle \vDash \text{Extensionality}, \\ \text{and } \exists n\ \exists m\ (\pi_{E_z}(m) = x \text{ and } \langle \omega, E_z \rangle \vDash \varphi[n, m])]$$

We shall verify that (41.1) gives a Σ^1_2 definition of A. Since WF is Π^1_1, it suffices to show that the relations "$\langle \omega, E \rangle \vDash \varphi[n_1, \ldots, n_k]$" and "$\pi_E(m) = x$" are arithmetical in E. It is easy to see that $\langle \omega, E \rangle \vDash \varphi$ is a property arithmetical in E. As for the transitive collapse, we notice first that if $k \in N$, then

$$\pi_E(m) = k \quad \leftrightarrow \quad \exists \langle r_0, \ldots, r_k \rangle \text{ such that } m = r_k \text{ and } \langle \omega, E \rangle \vDash r_0 = \varnothing \\ \text{and } (\forall i < k)\ \langle \omega, E \rangle \vDash (r_{i+1} = r_i \cup \{r_i\})$$

Then for $x \subseteq \omega$ we have

$$\pi_E(m) = x \quad \leftrightarrow \quad \forall n[n\ E\ m \quad \leftrightarrow \quad \pi_E(n) \in x]$$

and a similar formula, arithmetical in E, defines $\pi_E(m) = x$ for $x \in \mathcal{N}$.

Hence A is Σ^1_2. ∎

The set $A = \mathcal{N} \cap L$ is a Σ^1_2 set in $\mathcal{N}$. In case that $\omega_1^L = \omega_1$ and $2^{\aleph_0} > \aleph_1$, A is an example of a Σ^1_2 set of size $\aleph_1$ although $2^{\aleph_0} > \aleph_1$.

We can obtain an even better example. Since A is Σ^1_2, A is the projection of a Π^1_1 set B in $\mathcal{N}^2$. By the uniformization theorem, B has a Π^1_1 subset f that is a function and has the same projection A. Since $|A| = \aleph_1$ (under our assumption that $\omega_1^L = \omega_1$), the set f has also size $\aleph_1$. Thus it is consistent that $2^{\aleph_0} > \aleph_1$ and there is a Π^1_1 set of size $\aleph_1$.

Theorem 97 provides examples concerning Lebesgue measure, the property of Baire, and perfect sets. It is particularly useful that the order-type of $<_L$ on L_{ω_1} is ω_1, and hence for every constructible $x \in \mathrm{HC}$, the set $\{y : y <_L x\}$ is at most countable.

Corollary 1. *If $V = L$, then there exists a Δ^1_2 set that is not Lebesgue measurable and does not have the Baire property.*

Proof. Let $A = \{(x, y) : x <_L y\}$. For every y, the set $\{x : (x, y) \in A\}$ is countable, hence null and meager, and by Lemmas 39.6 and 39.8, if A is measurable, then it is null; and if it has the Baire property, then it is meager.

Let B be the complement of A in $\boldsymbol{R}^2$, $B = \{(x, y) : y \leq_L x\}$. Again, for every x, the set $\{y : (x, y) \in B\}$ is countable, and hence null if measurable, and meager if it has the Baire property.

It clearly follows that A neither is Lebesgue measurable nor has the property of Baire. ∎

The following lemma tells even more than that $<_L$ is a Σ^1_2 relation. For any $z \in \mathcal{N}$, let z_m, $m \in N$, be defined by $z_m(n) = z(\Gamma(m, n))$ (the canonical homeomorphism between $\mathcal{N}$ and ${}^{\omega}\mathcal{N}$).

Lemma 41.2. *The following relation R over $\mathcal{N}$ is Σ_2^1:*

$$(z, x) \in R \quad \text{iff} \quad \{z_n : n \in N\} = \{y : y <_L x\}$$

Proof. Since the relation $\{z_n : n \in N\} \subseteq \{y : y <_L x\}$ is clearly Σ_2^1, it suffices to show that

$$\forall y <_L x \, \exists n(y = z_n) \tag{41.2}$$

is Σ_2^1. There is a sentence Θ such that if M is a transitive model of Θ, then $<_L$ is absolute for M; and if $x \in M$ is constructible, then every $y <_L x$ is in M. Thus (41.2) is equivalent to

$\exists$ countable transitive model M that contains x, z, and all z_n,
and $M \vDash [\Theta \text{ and } \forall y <_L x \, \exists n(y = z_n)]$.

This last property is Σ_2^1 by a proof similar to Lemma 41.1. ■

Corollary 2. *If $V = L$, then there exists an uncountable Π_1^1 set without a perfect subset.*

Proof. First we construct an uncountable Σ_2^1 set without a perfect subset. Let

$$x \in A \quad \leftrightarrow \quad x \in \mathrm{WO} \wedge \forall y <_L x(\neg \|y\| = \|x\|)$$

The set is uncountable: A is a subset of WO and for every $\alpha < \omega_1$ there is exactly one x in A such that $\|x\| = \alpha$. Let me show that A is Σ_2^1: Let R be the Σ_2^1 relation from Lemma 41.2; thus

$$x \in A \quad \leftrightarrow \quad x \in \mathrm{WO} \wedge \exists z[R(z, x) \wedge \forall n(\neg \|z_n\| = \|x\|)]$$

and since $\neg \|z_n\| = \|x\|$ is Π_1^1, A is Σ_2^1.

The set A does not have a perfect subset; in fact, it does not have an uncountable analytic subset. This follows from the boundedness lemma: For every analytic set $X \subseteq A$, the set $\{\|x\| : x \in X\}$ is bounded, and hence countable (because of the definition of A).

Now A is Σ_2^1 and hence is the projection of some Π_1^1 set $B \subseteq \mathcal{N}^2$. By the uniformization theorem, B has a Π_1^1 subset f that is a function and has the same projection A. The set f is uncountable, and I claim that f does not have a perfect subset. Assume that $P \subseteq f$ is perfect. The projection of P is an analytic subset of A. Since $P \subseteq f$, P is itself a function and because P is uncountable, the projection $A = \mathrm{dom}(P)$ is also uncountable. This is a contradiction since we proved that every analytic subset of A is countable. ■

Corollary 3. *Assume that there is $a \subseteq \omega$ such that $\aleph_1^{L[a]} = \aleph_1$. Then there is an uncountable Π_1^1 set without a perfect subset.*

Proof. This is just a generalization of Corollary 2. Let

$$x \in A \quad \leftrightarrow \quad x \in L[a] \wedge x \in \mathrm{WO} \wedge \forall y <_{L[a]} x \, (\neg \|y\| = \|x\|)$$

The same argument as before shows that A is a $\Sigma^1_2(a)$ subset of WO and that for all $\alpha < \omega_1$, A has exactly one element x such that $\|x\| = \alpha$. The rest of the proof proceeds as before, and we obtain a $\Pi^1_1(a)$ set of cardinality $\aleph_1$ without a perfect subset. ■

Exercise 41.1. Let σ be a theorem of ZF + DC such that σ implies Extensionality, and let $P(x)$ be a property (of reals) upward absolute for transitive models of σ. Then P is a Σ^1_2 property.

[Use Exercise 14.18 and Lemma 41.1, or prove directly, as Lemma 41.1.]

Exercise 41.2. If A is a Σ^1_2 set in $\mathcal{N}$, then $x \in A$ is Σ_1 over HC.

[Let $T \subseteq Seq_3$ be arithmetical such that $x \in A$ iff $(\exists y \in \mathcal{N})\ T(x, y)$ is well-founded. Show that $\{(T, x, y) : T(x, y)$ is well-founded$\}$ is Δ^1_1 over HC (cf. Lemma 14.3) and use the fact that T is arithmetical.]

Absoluteness of Σ^1_2 Relations

Let $A \subseteq \mathcal{N}^r \times N^l$ be a Σ^1_n (or Π^1_n) relation and let us assume that A is presented in a normal form (see (40.6), (40.7) or (40.3), and (40.4)). Let $\mathfrak{M}$ be a transitive model and assume that $\mathfrak{M}$ satisfies enough axioms of ZF + DC.

We denote by $A^{\mathfrak{M}}$ the relation (in $\mathfrak{M}$) which is defined in $\mathfrak{M}$ by the same normal form formula as A is in the universe. (Thus $A^{\mathfrak{M}}$ depends on the normal form for A, and we shall always implicitly consider such a normal form.) For instance, if $A = \{x : \exists y\ \forall n\ (\bar{x}(n), \bar{y}(n), n) \in R\}$ then

$$A^{\mathfrak{M}} = \{x \in \mathfrak{M} : (\exists y \in \mathfrak{M})\ \forall n\ (\bar{x}(n), \bar{y}(n), n) \in R\}$$

Note that we also assume implicitly that if $R \subseteq \mathrm{HF}$ is arithmetical, then $R \in \mathfrak{M}$. But that is subsumed in the assumption that $\mathfrak{M}$ satisfies enough axioms of ZF.

More generally, let A be $\Sigma^1_n(a)$ (or $\Pi^1_n(a)$) and let $\mathfrak{M}$ be such that $a \in \mathfrak{M}$. Then $A^{\mathfrak{M}}$ denotes the corresponding $\Sigma^1_n(a)$ (or $\Pi^1_n(a)$) relation in $\mathfrak{M}$.

We say that a relation A is *absolute* for $\mathfrak{M}$ if $A^{\mathfrak{M}} = A \cap \mathfrak{M}$; i.e., if for every $x_1, \ldots, x_r \in \mathcal{N} \cap \mathfrak{M}$,

$$(x_1, \ldots, x_r, k_1, \ldots, k_l) \in A \quad \leftrightarrow \quad \mathfrak{M} \vDash (x_1, \ldots, x_r, k_1, \ldots, k_l) \in A$$

where the formula to the right of $\vDash$ is the defining normal form for A. For instance, if

$$x \in A \quad \leftrightarrow \quad \forall y\ \exists z\ \forall n\ (\bar{x}(n), \bar{y}(n), \bar{z}(n)) \in T$$

then "A is absolute" means that for every $x \in \mathcal{N} \cap \mathfrak{M}$,

$$x \in A \quad \leftrightarrow \quad (\forall y \in \mathcal{N} \cap \mathfrak{M})\ (\exists z \in \mathcal{N} \cap \mathfrak{M})\ \forall n\ (\bar{x}(n), \bar{y}(n), \bar{z}(n)) \in T$$

Theorem 98. (*Shoenfield's Absoluteness Theorem*). *Every* $\Sigma^1_2(a)$ *relation and every* $\Pi^1_2(a)$ *relation is absolute for all transitive models* $\mathfrak{M}$ *of* ZF + DC *containing all ordinals and such that* $a \in \mathfrak{M}$. *In particular,* Σ^1_2 *and* Π^1_2 *relations are absolute for* L.

Proof. It is enough to prove absoluteness of Σ_2^1 relations since a relation is absolute if and only if its negation is. Let us restrict ourselves to one dimension: Let $a \in \mathcal{N}$ and let A be a $\Sigma_2^1(a)$ subset of $\mathcal{N}$. Let $\mathfrak{M}$ be a transitive model of ZF + DC containing all ordinals such that $a \in \mathfrak{M}$. We shall prove that $A^{\mathfrak{M}} = A$.

Let $U \subseteq Seq_3$ be a tree, arithmetical in a, such that for all $x \in \mathcal{N}$,

$$x \in A \quad \leftrightarrow \quad \exists y \; U(x, y) \text{ is well-founded}$$

Thus for all $x \in \mathcal{N} \cap \mathfrak{M}$,

$$x \in A^{\mathfrak{M}} \quad \leftrightarrow \quad (\exists y \in \mathfrak{M}) \; \mathfrak{M} \models U(x, y) \text{ is well-founded}$$

However, for all $x, y \in \mathfrak{M}$, $U(x, y)$ is obviously the same tree in $\mathfrak{M}$ as in the universe; and since well-foundedness is absolute, we have

$$x \in A^{\mathfrak{M}} \quad \leftrightarrow \quad (\exists y \in \mathfrak{M}) \; U(x, y) \text{ is well-founded}$$

Thus, if $x \in A^{\mathfrak{M}}$, then $x \in A$, and it suffices to prove that if $x \in A \cap \mathfrak{M}$, then $x \in A^{\mathfrak{M}}$.

We shall now use the tree representation of Σ_2^1 sets. Let T be the tree over $\omega \times \omega_1$ constructed in the proof of Lemma 40.8. Hence $T \in L[a]$ and for every $x \in \mathcal{N}$,

$$x \in A \quad \leftrightarrow \quad T(x) \text{ is not well-founded} \tag{41.3}$$

Now if $x \in \mathfrak{M}$ is such that $x \in A$, then $T(x)$ is not well-founded, and by absoluteness of well-foundedness,

$$\mathfrak{M} \models T(x) \text{ is not well-founded}$$

In other words, there exists a function $g \in \mathfrak{M}$ from $\mathcal{N}$ into the ordinals such that $\forall n \; (\bar{x}(n), \bar{g}(n)) \in T$. Now following the proof of Lemma 40.8 backward, from (40.40) to the beginning, and working inside $\mathfrak{M}$, one finds $y \in \mathfrak{M}$ such that

$$\mathfrak{M} \models U(x, y) \text{ is well-founded}$$

Hence if $x \in A \cap \mathfrak{M}$, then $x \in A^{\mathfrak{M}}$ and we are done. ∎

Exercise 41.3. Show that $\Sigma_2^1(a)$ relations are absolute for all transitive models $\mathfrak{M}$ of ZF + DC containing all *countable* ordinals and such that $a \in \mathfrak{M}$.

[Go through the proof of Theorem 98 and notice that we needed only that $T \subseteq \mathfrak{M}$ is a class in $\mathfrak{M}$.]

Exercise 41.4. Prove Theorem 98 using the Lévy–Shoenfield absoluteness lemma (Theorem 36).

[Use Exercises 41.2 and 15.15.]

Exercise 41.5. Prove Theorem 36 in ZF + DC, using Shoenfield's absoluteness theorem.

[Use Lemma 41.1.]

Exercise 41.6. There is a theorem σ of ZF + DC such that every $\Sigma_1^1(a)$ and every $\Pi_1^1(a)$ relation is absolute for all transitive models $\mathfrak{M}$ of σ such that $a \in \mathfrak{M}$.

[If A is $\Pi_1^1(a)$, then $x \in A \leftrightarrow T(x)$ is well-founded; use absoluteness of well-foundedness.]

Corollary. *Every Σ_2^1 subset of ω is constructible.*

Proof. If $A \subseteq \omega$ is Σ_2^1, then $A^L = A$ and hence $A \in L$. ■

In contrast to the corollary, it is consistent that there is a Δ_3^1 subset of ω that is not constructible; an example will be presented in Section 44. Also, we have

Lemma 41.3. *If $0^\#$ exists, then $0^\#$ is a Δ_3^1 subset of ω.*

Proof. We of course identify the set $0^\#$ of sentences with the set of the Gödel numbers of the sentences, and so $0^\# \subseteq \omega$. Let us also identify $0^\#$ with its characteristic function, i.e., $z \in \mathcal{N}$ such that $z(n) = 1$ if $n \in 0^\#$ and $z(n) = 0$ if $n \notin 0^\#$.

By Exercise 30.1, the property "Σ is a well-founded remarkable E.M. set" is Π_1 over HC. Hence by Lemma 41.1, the set $\{0^\#\}$ is a Π_2^1 subset of $\mathcal{N}$. Thus

$$\begin{aligned} n \in 0^\# &\leftrightarrow \exists z(z \in \{0^\#\} \text{ and } z(n) = 1) \\ &\leftrightarrow \forall z(z \in \{0^\#\} \to z(n) = 1) \end{aligned}$$

shows that $0^\#$ is a Δ_3^1 subset of ω. ■

The following lemma is an interesting application of Theorem 98:

Lemma 41.4. *Let S be a set of countable ordinals such that the set $A = \{x \in \mathrm{WO} : \|x\| \in S\}$ is Σ_2^1. Then S is constructible. (And more generally, if A is $\Sigma_2^1(a)$, then $S \in L[a]$.)*

Proof. Let $A(x)$ be the Σ_2^1 property such that $A = \{x : A(x)\}$. For each countable ordinal α, let P_α be the notion of forcing that collapses α; i.e., the elements of P_α are finite sequences of ordinals less than α. Each P_α is constructible; let us consider, in L, the forcing languages associated with the P_α, and the corresponding Boolean valued models L^{P_α}.

We shall show that for every $\alpha < \omega_1$, α belongs to S if and only if

(41.4) $$L \vDash \text{every } p \in P_\alpha \text{ forces } \exists x(A(x) \wedge \|x\| = \alpha)$$

This will certainly show that S is constructible.

In order to prove that $\alpha \in S$ is equivalent to (41.4), let us consider a generic extension $\mathfrak{N}$ of the universe V such that in $\mathfrak{N}$, ω_1^V is countable. Let us argue in $\mathfrak{N}$.

The notion of forcing P_α has only countably many constructible dense subsets, and hence for every $p \in P_\alpha$ there exists $G \subseteq P_\alpha$ such that G is L-generic and $p \in G$. It follows that for every α, every φ and every $p \in P_\alpha$,

(41.5) $$L \vDash (p \Vdash \varphi) \quad \text{iff} \quad \text{for every } L\text{-generic } G \ni p,\ L[G] \vDash \varphi$$

Let $\alpha < \omega_1^V$, and let $z \in V$ be such that $\|z\| = \alpha$. Clearly, α belongs to S if and only if V satisfies

$$\exists x(A(x) \wedge \|x\| = \|z\|) \tag{41.6}$$

The property (41.6) is Σ_2^1; and by absoluteness, (41.6) holds in V if and only if it holds in $\mathfrak{N}$, and therefore $\mathfrak{N}$ satisfies (41.6).

Let G be an arbitrary L-generic filter on P_α, and let $u \in L[G]$ be such that $\|u\| = \alpha$. Since $\mathfrak{N}$ satisfies (41.6) if and only if it satisfies

$$\exists x(A(x) \wedge \|x\| = \|u\|) \tag{41.7}$$

and (41.7) is Σ_2^1, it follows that $\alpha \in S$ if and only if $L[G]$ satisfies (41.7). Since an L-generic filter on P_α exists (in $\mathfrak{N}$), we conclude (still in $\mathfrak{N}$), that $\alpha \in S$ is equivalent to

$$\text{for every } L\text{-generic } G \subseteq P_\alpha, \quad L[G] \models \exists x(A(x) \wedge \|x\| = \alpha) \tag{41.8}$$

But in view of (41.5) this last statement is equivalent to (41.4). ■

Perfect Subsets of Σ_2^1 Sets

Theorem 99 (Mansfield, Solovay). *Let A be a $\Sigma_2^1(a)$ set in $\mathcal{N}$. If A contains an element that is not in $L[a]$, then A has a perfect subset.*

The proof uses the tree representation of Σ_2^1 sets (Lemma 40.8). The theorem clearly follows from this more general lemma:

Lemma 41.5. *Let T be a tree over $\omega \times K$ and let $A = p[T]$. Either $A \subseteq L[T]$, or A contains a perfect subset; moreover, in the latter case there is a perfect tree $U \in L[T]$ over ω such that $[U] \subseteq A$.*

Proof. The proof follows very closely the Cantor-Bendixson argument; compare also with the proof of Theorem 94c.

For each tree T over $\omega \times K$, we define

$$T' = \{(s,h) \in T : \text{there exist } (s_0, h_0), (s_1, h_1) \in T \text{ such that } s_0 \supset s,\ s_1 \supset s,\ h_0 \supset h,\ h_1 \supset h, \text{ and that } s_0 \text{ and } s_1 \text{ are incompatible}\} \tag{41.9}$$

Let T be a tree over $\omega \times K$. Let

$$T^{(0)} = T, \qquad T^{(\alpha+1)} = (T^{(\alpha)})'$$

$$T^{(\alpha)} = \bigcap_{\beta<\alpha} T^{(\beta)} \qquad \text{if } \alpha \text{ is a limit}$$

It is obvious that the definition (41.9) is absolute for all models that contain T, and hence $T^{(\alpha)} \in L[T]$ for all α. Let α be the least ordinal such that $T^{(\alpha+1)} = T^{(\alpha)}$.

Let us assume first that $T^{(\alpha)} = \emptyset$; we shall show that $A \subseteq L[T]$. Let $x \in A$ be arbitrary. There exists $f \in {}^\omega K$ such that $(x, f) \in [T]$. It is obvious that there exists $\gamma < \alpha$ such that $(x, f) \in [T^{(\gamma)}]$ but $(x, f) \notin [T^{(\gamma+1)}]$. Thus there is $(s, h) \in T^{(\gamma)}$ such that $s \subset x$, $h \subset f$, and $(s, h) \notin T^{(\gamma+1)}$; this means that for any $(s', h') \in T^{(\gamma)}$, if $s' \supset s$ and $h' \supset h$, then $s' \subset x$. Now it follows that $x \in L[T]$; in $L[T]$, x is the unique $x = \bigcup \{s' \supseteq s : (s', h') \in T^{(\gamma)}$ for some $h \supseteq h'\}$.

Now let us assume that $T^{(\alpha)} \neq \varnothing$. The tree $T^{(\gamma)}$ has the property that for every $(s, h) \in T^{(\alpha)}$ there exist two extensions (s_0, h_0) and (s_1, h_1) of (s, h) that are incompatible in the first coordinate. Let us work in $L[T]$. Let (s_0, h_0) and (s_1, h_1) be some elements of $T^{(\alpha)}$ such that s_0 and s_1 are incompatible. Then let (s_{00}, h_{00}), (s_{01}, h_{01}), (s_{10}, h_{10}), and (s_{11}, h_{11}) be elements of $T^{(\alpha)}$ such that $s_{ij} \supset s_i$, $h_{ij} \supset h_i$ and that the s_{ij} are incompatible. In this fashion we construct $(s_t, h_t) \in T^{(\alpha)}$ for each finite 0,1-sequence t. The s_t generate a tree $U = \{s : s \subseteq s_t$ for some $t\}$. It is clear that U is a perfect tree, that $U \in L[T]$, and that $[U] \subseteq p[T] = A$. ■

Corollary. *The following are equivalent:*

(i) *For every $a \subseteq \omega$, $\aleph_1^{L[a]}$ is countable.*
(ii) *Every uncountable Π_1^1 set contains a perfect subset.*
(iii) *Every uncountable Σ_2^1 set contains a perfect subset.*

Proof. Obviously, (iii) implies (ii). By Corollary 3 of Theorem 97 (ii) implies (i). In order to show that (i) implies (iii), let us assume (i) and let A be an uncountable Σ_2^1 set. Let $a \in \mathcal{N}$ be such that $A \in \Sigma_2^1(a)$. Since $\aleph_1^{L[a]}$ is countable, there are only countably many reals in $L[a]$, and hence A has an element that is not in $L[a]$. Thus A contains a perfect subset. ■

Projective Well-Orderings of Sets of Reals

By Theorem 97, if all real numbers are constructible, then there exists a Δ_2^1 well-ordering of the set of all reals. We shall now show that a converse is true: If a Σ_2^1 relation well-orders the set of all reals, then all reals are constructible. We shall also show that there is no Σ_1^1 well-ordering of $\boldsymbol{R}$, and that every well founded Σ_2^1 relation over $\boldsymbol{R}$ has length less than ω_2. As usual, it is more convenient to prove these statements for the Baire space $\mathcal{N}$.

Theorem 100.

(a) *If $E \subseteq \mathcal{N} \times \mathcal{N}$ is a Σ_1^1 well-founded relation, then* length $(E) < \omega_1$.
(b) *If $E \subseteq \mathcal{N} \times \mathcal{N}$ is a Σ_2^1 well-founded relation, then* length$(E) < \omega_2$.
(c) *If $<$ is a Σ_2^1 well-ordering of $\mathcal{N}$, then every $x \in \mathcal{N}$ is constructible.* (*More generally, if $<$ is $\Sigma_2^1(a)$, then $\mathcal{N} \subseteq L[a]$.*)

Note that (a) implies that there is no Σ_1^1 well-ordering of $\mathcal{N}$.

Proof of (a). This is an application of the boundedness lemma, or rather of the fact that the set WO is not Σ_1^1. Assuming that a Σ_1^1 well-founded relation over $\mathcal{N}$ has length $\geq \omega_1$, we shall describe the set WO in a Σ_1^1 way, a contradiction.

Let me first do the easier case when E is a well-ordering. Thus let E be a Σ_1^1

well-ordering and let us assume that its order-type is $\geq \omega_1$. Then for every $\alpha < \omega_1$ there is an order-preserving mapping of $(\alpha, <)$ into $(\mathcal{N}, E)$. Conversely, if a countable linearly ordered set $(Q, <)$ can be embedded in $(\mathcal{N}, E)$, then $(Q, <)$ is a well-ordering. Hence let E_x be, for each $x \in \mathcal{N}$, the relation coded by x (see 40.17), and let LO be the arithmetical set of all x that code a linear ordering of N. Then

(41.10)

$$\begin{aligned} x \in \mathrm{WO} \quad &\leftrightarrow \quad x \in \mathrm{LO} \wedge (\exists f: \mathcal{N} \to \mathcal{N})\, \forall n\, \forall m (n\, E_x\, m \to (f(m), f(m)) \in E) \\ &\leftrightarrow \quad x \in \mathrm{LO} \wedge \exists z \in \mathcal{N}\, \forall n\, \forall m (n\, E_x\, m \to (z_n, z_m) \in E) \end{aligned}$$

where for each $z \in \mathcal{N}$ and each n, z_n is the element of $\mathcal{N}$ defined by $z_n(k) = z(\Gamma(n, k))$ for all $k \in N$. Now (41.10) gives a Σ_1^1 description of WO, a contradiction.

In the general case when E is a Σ_1^1 well-founded relation we observe that if α is a countable ordinal such that $\alpha < \text{length}(E)$, then there exists a countable set $S \subseteq \mathcal{N}$ and a function f of S onto α such that for every $u \in S$ and every $\beta < f(u)$ there is $v \in S$ such that $v\, E\, u$ and $\beta \leq f(v)$ (namely $f(x) = \rho_E(x)$, and the countable set S is constructed with the help of the principle of dependent choices). Conversely, if $(Q, <)$ is a linearly ordered set and if there is a function f from a subset of $\mathcal{N}$ onto Q such that for every $u \in \text{dom}(f)$ and every $q < f(u)$ there is $v \in \text{dom}(f)$ such that $v\, E\, u$ and $q \leq f(v)$, then $(Q, <)$ is a well-ordering. Thus if $\text{length}(E) \geq \omega_1$, we have

$$\begin{aligned} (41.11) \quad x \in \mathrm{WO} \quad \leftrightarrow \quad & x \in \mathrm{LO} \wedge (\exists \text{ countable } S = \{z_n : n \in N\})(\exists f: S \xrightarrow{\text{onto}} N) \\ & \forall n\, \forall k\ \ [\text{if } (k, f(z_n)) \in E_x, \text{ then } \exists m \text{ such that} \\ & (z_m, z_n) \in E \text{ and either } k = f(z_m) \\ & \text{or } (k, f(z_m)) \in E_x] \end{aligned}$$

Again, (41.11) can be written in a Σ_1^1 manner, and we get a contradiction. ■

Proof of (b). Let $E \subseteq \mathcal{N} \times \mathcal{N}$ be a Σ_2^1 relation. Let T be a tree over $\omega^2 \times \omega_1$ such that for all $x, y \in \mathcal{N}$,

$$(41.12) \qquad (x, y) \in E \quad \leftrightarrow \quad (\exists f: \omega \to \omega_1)\, \forall n\, (\bar{x}(n), \bar{y}(n), \bar{f}(n)) \in T$$

As usual, for each $z \in \mathcal{N}$ and each $n \in N$, let $z_n \in \mathcal{N}$ be such that $z_n(k) = z(\Gamma(n, k))$ for all k; similarly, for each $f: \omega \to \omega_1$ and each n, let $f_n: \omega \to \omega_1$ be such that $f_n(k) = f(\Gamma(n, k))$ for all k.

Each of the following formulas is equivalent to the statement that the relation E is not well-founded:

$$\exists x\, \forall m\, (x_{m+1}, x_m) \in E$$

$$\exists x\, \forall m\, \exists f\, \forall n\, (\bar{x}_{m+1}(n), \bar{x}_m(n), \bar{f}(n)) \in T$$

$$\exists x\, \exists f\, \forall m\, \forall n\, (\bar{x}_{m+1}(n), \bar{x}_m(n), \bar{f}_m(n)) \in T$$

It is easy to construct a tree U over $\omega \times \omega_1$ such that for all $x \in \mathcal{N}$ and all $f\colon \omega \to \omega_1$,

$$(41.13)\ \forall m\, \forall n\, (\bar{x}_{m+1}(n), \bar{x}_m(n), \bar{f}_m(n)) \in T \quad \text{iff} \quad \forall k\, (\bar{x}(k), \bar{f}(k)) \in U$$

(see (40.9) for a similar construction). It follows from (41.13) that

$$(41.14) \qquad E \text{ is well-founded} \quad \text{iff} \quad U \text{ is well-founded}$$

Now let $E \subseteq \mathcal{N} \times \mathcal{N}$ be a Σ^1_2 well-founded relation; we want to show that $\text{length}(E) < \omega_2$. Let T be a tree over $\omega^2 \times \omega_1$ such that (41.12) holds for all $x, y \in \mathcal{N}$ and let U be the tree over $\omega \times \omega_1$ constructed from T as above; since E is well-founded, U is well-founded.

Let us consider a generic extension $\mathfrak{M}[G]$ of the universe (which we henceforth call $\mathfrak{M}$) in which $\omega_1^{\mathfrak{M}}$ is countable and $\omega_2^{\mathfrak{M}} = \omega_1^{\mathfrak{M}[G]}$. Let us argue in $\mathfrak{M}[G]$.

Let E^* be the relation over $\mathcal{N}$ defined by (41.12). First we observe that $E \subseteq E^*$: If $x, y \in \mathfrak{M}$, then

$$\begin{aligned} (x, y) \in E \quad &\text{iff} \quad \mathfrak{M} \models T(x, y) \text{ is well-founded} \\ &\text{iff} \quad \mathfrak{M}[G] \models T(x, y) \text{ is well-founded} \\ &\text{iff} \quad (x, y) \in E^* \end{aligned}$$

(because well-foundedness is absolute). We notice further that E^* is well-founded: This is because by the construction of U (which is absolute) and the definition of E^*, $\mathfrak{M}[G]$ satisfies (41.14), i.e.,

$$E^* \text{ is well-founded} \quad \text{iff} \quad U \text{ is well-founded}$$

Hence E^* is well-founded, and $\text{length}(E) \leq \text{length}(E^*)$.

The tree T is a tree over $\omega \times \omega_1^{\mathfrak{M}}$ and $\omega_1^{\mathfrak{M}}$ is a countable ordinal. Since $E^* = p[T]$, it follows that E^* is Σ^1_1 relation. By part (a) of the theorem, the length of E^* is countable; i.e., $\text{length}(E^*) < \omega_1^{\mathfrak{M}[G]}$. It follows that $\text{length}(E) < \omega_1^{\mathfrak{M}[G]} = \omega_2^{\mathfrak{M}}$.

Now we can step back into the ground model and look at the result of the above argument: $\text{length}(E) < \omega_2$. ■

Proof of (c). Let $<$ be a Σ^1_2 well-ordering of $\mathcal{N}$ and let us assume that there is a nonconstructible real. Let $T_0 = Seq(\{0, 1\})$, and let $C = [T_0] = {}^{\omega}\{0, 1\}$ be the Cantor space. Let us consider trees $T \subseteq T_0$ and functions $f\colon T \to T_0$ such that (i) $s \subseteq t$ implies $f(s) \subseteq f(t)$ and (ii) for every $x \in [T]$, $\bigcup_{n=0}^{\infty} f(\bar{x}(n)) \in C$. Every such function induces a continuous function from $[T]$ into C, which we denote f^*.

Lemma 41.6. *If $T \subseteq T_0$ is a constructible perfect tree and if $f\colon T \to T_0$ is a constructible function such that f^* is one-to-one, then there exists a constructible perfect tree $U \subseteq T$ and a constructible $g\colon U \to T_0$ such that g^* is one-to-one, and $g^*(x) < f^*(x)$ for every $x \in [U]$.*

It suffices to prove this lemma because then we can construct a sequence of trees $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq \cdots$ and functions $f_0, f_1, \ldots, f_n, \ldots$ such that $f^*_{n+1}(x) < f^*_n(x)$ for all $x \in [T_{n+1}]$. Since all $[T_n]$ are compact sets, their intersection is nonempty and therefore there exists x such that $f^*_0(x) > f^*_1(x) > \cdots > f^*_n(x) > \cdots$ contrary to the assumption that $<$ is a well-ordering.

Proof of lemma. Let $T \subseteq T_0$ be a constructible tree and let $f\colon T \to T_0$ be constructible, such that f^* is one-to-one.

Since T is perfect, there exists a constructible function $h\colon T \to T_0$ such that $h^*\colon [T] \to C$ is one-to-one and onto. For each $s \in T_0$, let $\bar{s}$ be the "mirror image" of s, namely if $s = \langle s(0), \ldots, s(k)\rangle$, let $\bar{s} = \langle 1 - s(0), \ldots, 1 - s(k)\rangle$; for $x \in C$, $\bar{x}$ is defined similarly.

We claim that at least one of the sets

$$A = \{x \in [T] : f^*(x) > h^*(x)\}, \qquad B = \{x \in [T] : f^*(x) > \overline{h^*(x)}\}$$

contains a nonconstructible element. Let z be the least nonconstructible element of C, and let $x, y \in [T]$ be such that $h^*(x) = z$ and $h^*(y) = \bar{z}$. Then both x and y are nonconstructible and hence $f^*(x) \geq z$ and $f^*(y) \geq z$. Thus either $f^*(x) > z$ or $f^*(y) > z$ and so either A or B contains a nonconstructible element. For instance, assume that A does. (The other case is similar.)

Since $<$ is Σ_2, and T, f, and h are constructible subsets of HF, the set A is $\Sigma^1_2(a)$ for some $a \in L$. By Lemma 41.5, there exists a constructible perfect tree U such that $[U] \subseteq A$. If we let $g = h \mid U$, then U and g satisfy the lemma. ■

42. A MODEL WHERE ALL SETS ARE LEBESGUE MEASURABLE

This section is devoted entirely to the proof of the following theorem:

Theorem 101 (Solovay). *Assume that there exists an inaccessible cardinal.*

(a) *There is a model of* ZF + DC *in which all sets of real numbers are Lebesgue measurable and have the property of Baire, and every uncountable set of reals has a perfect subset.*

(b) *There is a model of* ZFC *in which every projective set of reals is Lebesgue measurable, has the Baire property, and if uncountable, then it contains a perfect subset.*

The proof makes substantial use of absoluteness of measure and category, and we shall deal with that first. We begin with a finer analysis of the hierarchy Σ^0_α of Borel sets.

Codes of Borel Sets

Every Borel set of reals is obtained, in fewer than ω_1 steps, from the basic open sets by taking complements and countable unions. We shall show how this procedure can be coded by a function $c \in {}^\omega\omega$. We shall define the set BC of

Borel codes and assign to each $c \in \mathrm{BC}$ a unique Borel set A_c. The code c not only describes the Borel set A_c but also describes the procedure by which the set A_c is constructed from the basic open sets.

Let $I_1, I_2, \ldots, I_k, \ldots$ be an arithmetical enumeration of open intervals with rational endpoints (i.e., the sequence of the pairs of endpoints is arithmetical). For each $c \in \mathcal{N}$, let

$$u(c) \quad \text{and} \quad v_i(c) \quad (i \in \boldsymbol{N}) \tag{42.1}$$

be elements of $\mathcal{N}$ defined as follows: If $d = u(c)$, then $d(n) = c(n+1)$ for all n; if $d = v_i(c)$, then $d(n) = c(\Gamma(i, n) + 1)$ for all n (where Γ is the canonical one-to-one correspondence between $\boldsymbol{N} \times \boldsymbol{N}$ and $\boldsymbol{N}$).

For $0 < \alpha < \omega_1$, we define sets Σ_α and $\Pi_\alpha \subseteq \mathcal{N}$ as follows:

$$\begin{array}{lll} (42.2) & c \in \Sigma_1 & \text{if } c(0) > 1 \\ & c \in \Pi_\alpha & \text{if either } c \in \Sigma_\beta \cup \Pi_\beta \text{ for some } \beta < \alpha \\ & & \quad \text{or } c(0) = 0 \text{ and } u(c) \in \Sigma_\alpha \\ (\alpha > 1) & c \in \Sigma_\alpha & \text{if either } c \in \Sigma_\beta \cup \Pi_\beta \text{ for some } \beta < \alpha \\ & & \quad \text{or } c(0) = 1 \text{ and } v_i(c) \in \bigcup_{\beta<\alpha} (\Sigma_\beta \cup \Pi_\beta) \text{ for all } i \end{array}$$

If $c \in \Sigma_\alpha$ (if $c \in \Pi_\alpha$), we call c a $\mathbf{\Sigma}^0_\alpha$*-code* (a $\mathbf{\Pi}^0_\alpha$*-code*). Let BC, the set of all *Borel codes*, be

$$\mathrm{BC} = \bigcup_{\alpha<\omega_1} \Sigma_\alpha = \bigcup_{\alpha<\omega_1} \Pi_\alpha$$

For every $c \in \mathrm{BC}$, we define a Borel set A_c as follows (we say that c *codes* A_c):

$$\begin{array}{llll} (42.3) & \text{if } c \in \Sigma_1 & \text{then} & A_c = \bigcup \{I_n : c(n) = 1\} \\ & \text{if } c \in \Pi_\alpha \text{ and } c(0) = 0 & \text{then} & A_c = \boldsymbol{R} - A_{u(c)} \\ & \text{if } c \in \Sigma_\alpha \text{ and } c(0) = 1 & \text{then} & A_c = \bigcup_{i=0}^{\infty} A_{v_i(c)} \end{array}$$

It is clear that for every $\alpha > 0$, if $c \in \Sigma_\alpha$ (if $c \in \Pi_\alpha$), then $A_c \in \mathbf{\Sigma}^0_\alpha$ ($A_c \in \mathbf{\Pi}^0_\alpha$). Conversely, if B is a $\mathbf{\Sigma}^0_\alpha$ set (a $\mathbf{\Pi}^0_\alpha$ set), then there exists $c \in \Sigma_\alpha$ ($c \in \Pi_\alpha$) such that $B = A_c$. This is proved by induction on α using facts like: If c_i, $i \in \omega$ are elements of $\bigcup_{\beta<\alpha} \Pi_\beta$, then there is $c \in \Sigma_\alpha$ such that $c_i = v_i(c)$ for all $i \in \omega$.

Thus $\{A_c : c \in \mathrm{BC}\}$ is the collection of all Borel sets.

Lemma 42.1. *The set* BC *of all Borel codes is* $\mathbf{\Pi}^1_1$.

Proof. Let us consider the following relation E over $\mathcal{N}$:

$$\begin{array}{llll} (42.4) & x \mathrel{E} y & \text{iff} & \text{either } y(0) = 0 \text{ and } x = u(y) \\ & & & \text{or } \quad y(0) = 1 \text{ and } x = v_i(y) \text{ for some } i \in \omega \end{array}$$

The relation E is arithmetical. If $y \in \Sigma_1$, then y is E-minimal (i.e. $\text{ext}_E(y) = \emptyset$) and vice versa; if $y \in \Pi_\alpha$ and $x \mathrel{E} y$, then $x \in \Sigma_\alpha$, and if $y \in \Sigma_\alpha$ $(\alpha > 1)$ and $x \mathrel{E} y$, then $x \in \bigcup_{\beta<\alpha} (\Sigma_\beta \cup \Pi_\beta)$.

We claim that

$$(42.5) \quad \begin{aligned} y \in \text{BC} \quad &\leftrightarrow \quad E \text{ is well-founded below } y \\ &\leftrightarrow \quad \text{there is no } \langle z_0, z_1, \ldots, z_n, \ldots\rangle \text{ such that } z_0 = y \\ &\qquad\quad \text{and that } \forall n(z_{n+1} \mathrel{E} z_n) \end{aligned}$$

By the remark following (42.4), if $y \in \text{BC}$, then there can be no infinite descending sequence $z_0 = y$, $z_1 \mathrel{E} z_0$, $z_2 \mathrel{E} z_1$, etc. Conversely, if E is well-founded below y, let ρ denote the rank function for E over $\text{ext}_E(y)$. By induction on $\rho(x)$, one can see that every $x \in \text{ext}_E(y)$ is a Borel code, and finally that y is itself a Borel code.

Now (42.5) gives a Π^1_1 definition of the set BC and the lemma follows. ■

Lemma 42.2. *The properties $A_c \subseteq A_d$, $A_c = A_d$, and $A_c = \emptyset$ are Π^1_1 properties of Borel codes.*

Proof. We shall show that there are properties $P, Q \subseteq \boldsymbol{R} \times \mathcal{N}$ such that P is Π^1_1 and Q is Σ^1_1 and such that for every $c \in \text{BC}$,

$$(42.6) \qquad a \in A_c \quad \leftrightarrow \quad (a, c) \in P \quad \leftrightarrow \quad (a, c) \in Q$$

(I hope that the reader does not mind that I talk about Π^1_1 relations over $\boldsymbol{R} \times \mathcal{N}$). Then

$$\begin{aligned} A_c \subseteq A_d \quad &\leftrightarrow \quad c, d \in \text{BC} \wedge \forall a((a, c) \in Q \rightarrow (a, c) \in P) \\ A_c = A_d \quad &\leftrightarrow \quad c, d \in \text{BC} \wedge A_c \subseteq A_d \wedge A_d \subseteq A_c \\ A_c = \emptyset \quad &\leftrightarrow \quad c \in \text{BC} \wedge \forall a \ (a, c) \notin Q \end{aligned}$$

To find P and Q, let $x \in \mathcal{N}$ be fixed. Let T be the smallest set $T \subseteq \mathcal{N}$ such that

$$(42.7) \qquad x \in T, \text{ and if } y \in T \text{ and } z \mathrel{E} y, \text{ then } z \in T.$$

The set T is countable. Let h: $T \rightarrow \{0, 1\}$ be a function with the following property: for all $y \in T$,

$$(42.8) \quad \begin{aligned} &\text{if } y(0) > 1, \text{ then } h(y) = 1 \text{ iff for some } n, y(n) = 1 \text{ and } a \in I_n: \\ &\text{if } y(0) = 0, \text{ then } h(y) = 1 \text{ iff } h(u(y)) = 0; \\ &\text{if } y(0) = 1, \text{ then } h(y) = 1 \text{ iff for some } i, h(v_i(y)) = 1. \end{aligned}$$

Note that if x is a Borel code then there is a unique countable set $T \subseteq \mathcal{N}$ with the property (42.7), and a unique function h with the property (42.8); moreover, for every $y \in T$ we have $h(y) = 1$ just in case $a \in A_y$. Thus we let

$$(42.9) \quad \begin{aligned} (a, x) \in P \quad \leftrightarrow \quad &(\forall \text{ countable } T \subseteq \mathcal{N})(\forall h\colon T \rightarrow \{0, 1\}) \\ &[\text{if } (42.7) \text{ and } (42.8) \text{ then } h(x) = 1] \end{aligned}$$

and

$$(42.10)\quad (a, x) \in Q \quad \leftrightarrow \quad (\exists \text{ countable } T \subseteq \mathcal{N})(\exists h\colon T \to \{0, 1\})$$
$$[(42.7) \wedge (42.8) \wedge h(x) = 1]$$

and it is clear that if $c \in \mathrm{BC}$, then $a \in A_c$ iff $(a, c) \in P$ iff $(a, c) \in Q$.

It is a routine matter to verify (with some effort) that (42.9) can be written in a Π^1_1 way and (42.10) in a Σ^1_1 way. (The quantifiers $\forall T$, $\forall h$, and $\exists T$ $\exists h$ are the only ones for which one needs quantifiers over $\mathcal{N}$; note that for instance, $\forall z(z \mathrel{E} y \to z \in T)$ in (42.7) can be written as

$$(y(0) = 0 \to u(y) \in T) \wedge (y(0) = 1 \to \forall i(v_i(y) \in T))). \quad \blacksquare$$

Absoluteness of Measure and Category

We shall now show that certain properties of Borel codes are absolute for transitive models of ZF + DC. If $\mathfrak{M}$ is a transitive model of ZF + DC and $c \in {}^\omega\omega$ is in $\mathfrak{M}$, then because the set BC is Π^1_1, c is a Borel code if and only if $\mathfrak{M} \models c$ is a Borel code. By Lemma 42.2, the properties of the codes $A_c \subseteq A_d$, $A_c = A_d$, and $A_c = \emptyset$ are Π^1_1 and therefore absolute: $A_c = A_d$ holds if and only if $A^{\mathfrak{M}}_c = A^{\mathfrak{M}}_d$, etc., where $A^{\mathfrak{M}}_c$ denotes the Borel set in $\mathfrak{M}$ coded by c. Moreover, since $a \in A_c$ is Π^1_1, it follows that $A^{\mathfrak{M}}_c = A_c \cap \mathfrak{M}$, for every Borel code $c \in \mathfrak{M}$.

Lemma 42.3. *The following properties (of codes) are absolute for all transitive models* $\mathfrak{M}$ *of* ZF + DC:

$$A_e = A_c \cup A_d, \qquad A_e = A_c \cap A_d,$$
$$A_e = \boldsymbol{R} - A_c, \qquad A_e = A_c \mathbin{\triangle} A_d, \qquad A_e = \bigcup_{n=0}^{\infty} A_{c_n}$$

(*we assume that the codes* c, d, e *are in* $\mathfrak{M}$, *as is the sequence* $\langle c_n\colon n = 0, 1, \ldots\rangle$).

We say that the *operations* $\cup$, $\cap$, $-$, $\triangle$, $\bigcup_{n=0}^{\infty}$ on Borel sets with codes in $\mathfrak{M}$ are absolute for $\mathfrak{M}$.

Proof. If $c_0, c_1, \ldots, c_n, \ldots$ is a sequence of Borel codes in $\mathfrak{M}$, let $c \in \mathcal{N}$ be such that $c(0) = 1$ and that $v_i(c) = c_i$ for all $i \in \omega$. Clearly, c is a Borel code, $c \in \mathfrak{M}$, and c codes (both in the universe and in $\mathfrak{M}$) the Borel set $\bigcup_{n=0}^{\infty} A_{c_n}$. Hence for any Borel code $e \in \mathfrak{M}$, we have

$$A^{\mathfrak{M}}_e = \bigcup_{n=0}^{\infty} A^{\mathfrak{M}}_{c_n} \quad \text{iff} \quad A^{\mathfrak{M}}_e = A^{\mathfrak{M}}_c \quad \text{iff} \quad A_e = A_c \quad \text{iff} \quad A_e = \bigcup_{n=0}^{\infty} A_{c_n}$$

because $A_e = A_c$ is absolute for $\mathfrak{M}$. Thus $A_e = \bigcup_{n=0}^{\infty} A_{c_n}$ is absolute.

An analogous argument shows that $\boldsymbol{R} - A_c$ is absolute, and the rest of the lemma follows easily because the operations $\cap$ and $\triangle$ can be defined from $\cup$ and $-$. $\blacksquare$

Lemma 42.4. *"A_c is null" and "A_c is meager" are properties absolute for all transitive models $\mathfrak{M}$ of* ZF + DC.

Proof. Let $\mathfrak{M}$ be a transitive model of ZF + DC. Let μ denote the Lebesgue measure. First we claim that if $c \in \mathfrak{M}$ is a Σ_1^0-code, then $\mu^{\mathfrak{M}}(A_c^{\mathfrak{M}}) = \mu(A_c)$. Let $k_0, k_1, \ldots, k_n, \ldots$ be all the $k \in N$ such that $c(k) = 1$; thus A_c is the union $\bigcup_{n=0}^{\infty} I_{k_n}$ of open intervals with rational endpoints. For each n, let $X_n = I_{k_n} - (I_{k_0} \cup \cdots \cup I_{k_{n-1}})$; hence $A_c = \bigcup_{n=0}^{\infty} X_n$ and $\mu(A_c) = \sum_{n=0}^{\infty} \mu(X_n)$. However, each X_n is a finite union of disjoint rational intervals, and clearly $\mu(X_n)$ is absolute. Hence $\mu^{\mathfrak{M}}(A_c^{\mathfrak{M}}) = \mu(A_c)$.

A similar argument shows that if $c \in \mathfrak{M}$ is a Π_1^0-code, then $\mu^{\mathfrak{M}}(A_c^{\mathfrak{M}}) = \mu(A_c)$.

Next we claim that if $c \in \mathfrak{M}$ is a Π_1^0-code, then A_c is rare if and only if $\mathfrak{M} \vDash A_c$ is rare. This is because $d = u(c) \in \Sigma_1$ and it is easily verified (using open rational intervals) that "A_d is dense" is absolute.

Now we are ready to prove the lemma. Let us consider first the property "A_c is null." We use the following properties of Lebesgue measure: (1) X is null if and only if for every $n \in N$, there is an open set $G \supseteq X$ of measure $\leq 1/n$, and (2) $\mu(X) > 0$ if and only if there is a closed set $F \subseteq X$ of positive measure.

If $\mathfrak{M} \vDash A_c$ is null, then $\mathfrak{M}$ satisfies

$$\forall n \, \exists e (e \in \Sigma_1 \text{ and } A_e \supseteq A_c \text{ and } \mu(A_e) \leq 1/n) \tag{42.11}$$

Since the part $(\cdots)$ of (42.11) is absolute, it is clear that (42.11) holds in V, and hence A_c is null.

If $\mathfrak{M} \vDash A_c$ is not null, then $\mathfrak{M}$ satisfies

$$\exists e (e \in \Pi_1 \text{ and } A_e \subseteq A_c \text{ and } \mu(A_e) > 0) \tag{42.12}$$

Again, $(\cdots)$ is absolute, thus (42.12) holds in V and hence A_i is not null.

Finally, we consider the property "A_c is meager." If $\mathfrak{M} \vDash (A_c$ is meager), then $\mathfrak{M}$ satisfies:

(42.13) There exist $c_n \in \Pi_1$, $n = 0, 1, \ldots,$ such that each A_{c_n} is rare, and $A_c \subseteq \bigcup_{n=0}^{\infty} A_{c_n}$.

Then (42.13) holds in V and so A_c is meager.

A Borel set B is not meager if and only if there is a nonempty open set G such that $B \triangle G$ is meager. Thus if $\mathfrak{M} \vDash A_c$ is not meager, then $\mathfrak{M}$ satisfies

$$\exists d \, \exists e (d \in \Sigma_1 \text{ and } A_d \neq \emptyset \text{ and } A_e = A_c \triangle A_d \text{ and } A_e \text{ is meager}) \tag{42.14}$$

Then (42.14) holds in V and hence A_c is not meager. ■

Exercise 42.1. "A_c is null" and "A_c is meager" are absolute for all transitive models of ZF$^-$ + DC.

[Because Π_1^1 properties are.]

Random and Generic Reals

Let us consider generic extensions of a ground model $\mathfrak{M}$, using either the algebra of Borel sets modulo the ideal of null sets or the algebra of Borel sets modulo the ideal of meager sets.

Let $\mathscr{B}$ be the σ-algebra of Borel sets of reals, let I_{m} and I_{c} (m for measure, c for category) be the σ-ideals

$$I_{\mathrm{m}} = \{B \in \mathscr{B} : \mu(B) = 0\}, \qquad I_{\mathrm{c}} = \{B \in \mathscr{B} : B \text{ is meager}\}$$

and let

$$(42.15) \quad \mathscr{B}_{\mathrm{m}} = \mathscr{B}/I_{\mathrm{m}} = \{[B]_{\mathrm{m}} : B \in \mathscr{B}\}, \qquad \mathscr{B}_{\mathrm{c}} = \mathscr{B}/I_{\mathrm{c}} = \{[B]_{\mathrm{c}} : B \in \mathscr{B}\}$$

where $[B]_{\mathrm{m}}$ and $[B]_{\mathrm{c}}$ denote equivalence classes mod I_{m} and mod I_{c}, respectively. $\mathscr{B}_{\mathrm{m}}$ and $\mathscr{B}_{\mathrm{c}}$ are complete Boolean algebras (by Exercise 17.23) and if B_n, $n \in \omega$, are Borel sets then (in either $\mathscr{B}_{\mathrm{m}}$ or $\mathscr{B}_{\mathrm{c}}$),

$$\sum_{n=0}^{\infty} [B_n] = [\bigcup_{n=0}^{\infty} B_n]$$

(Also, $-[B] = [R - B]$.)

Exercise 42.2. The algebra $\mathscr{B}_{\mathrm{c}}$ is the unique atomless complete Boolean algebra that has a countable dense subset.

[If B is a nonmeager Borel set, then there is a nonempty open set U such that $U \triangle B$ is meager; hence there is a rational interval I such that $[I]_{\mathrm{c}} \le [B]_{\mathrm{c}}$.]

Thus to form a generic extension using $\mathscr{B}_{\mathrm{c}}$ is the same as to adjoin a Cohen generic real. This observation justifies the definition below of a real number generic over $\mathfrak{M}$.

Let $\mathfrak{M}$ be a transitive model of ZFC. Let us consider the Borel sets in $\mathfrak{M}$; let $\mathscr{B}$ denote the collection of all Borel sets in $\mathfrak{M}$, and let $\mathscr{B}_{\mathrm{m}}$ and $\mathscr{B}_{\mathrm{c}}$ denote the complete Boolean algebras (42.15) in $\mathfrak{M}$.

Let B be a Borel set in $\mathfrak{M}$. B has a Borel code $c \in \mathfrak{M}$, $B = A_c$. Let us denote B^* the Borel set in the universe coded by c. This definition does not depend on the choice of $c \in \mathrm{BC}^{\mathfrak{M}}$ because by Lemma 42.2, if $A_c = A_d$, then $A_c^* = A_d^*$. We recall that $B = B^* \cap \mathfrak{M}$, for every $B \in \mathscr{B}$.

Lemma 42.5.

(a) *If G is an $\mathfrak{M}$-generic ultrafilter on $\mathscr{B}_{\mathrm{m}}$, then there is a unique real number x_G such that for all $B \in \mathscr{B}_{\mathrm{m}}$*

$$(42.16) \qquad x_G \in B^* \quad \leftrightarrow \quad [B]_{\mathrm{m}} \in G$$

The formula (42.16) *determines G and hence $\mathfrak{M}[G] = \mathfrak{M}[x_G]$*

(b) *If G is an $\mathfrak{M}$-generic ultrafilter on $\mathscr{B}_{\mathrm{c}}$, then there is a unique real number x_G such that for all $B \in \mathscr{B}$,*

$$(42.17) \qquad x_G \in B^* \quad \leftrightarrow \quad [B]_{\mathrm{c}} \in G$$

The formula (42.17) *determines G and hence $\mathfrak{M}[G] = \mathfrak{M}[x_G]$.*

If x is a real number and if $x = x_G$ for some $G \subseteq \mathscr{B}_{\mathrm{m}}$ generic over $\mathfrak{M}$, then x is *random* over $\mathfrak{M}$. If $x = x_G$ for some $G \subseteq \mathscr{B}_{\mathrm{c}}$ generic over $\mathfrak{M}$, then x is *generic* over $\mathfrak{M}$.

Proof. The same proof works for both (a) and (b); let $[B]$ denote $[B]_{\mathrm{m}}$ in case (a) and $[B]_{\mathrm{c}}$ in case (b).

To start, we claim that there is at most one real number x that satisfies

$$(42.18) \qquad x \in B^* \;\leftrightarrow\; [B] \in G \qquad (\text{for all } B \in \mathscr{B})$$

If x satisfies (42.18), then x belongs to all B^* such that $[B] \in G$. If $x < y$ are two real numbers, let r be a rational number such that $x < r < y$, and let A be the interval $(r, \infty) = \{z \in \boldsymbol{R} : z > r\}$. Either $[A]$ or $[\boldsymbol{R} - A]$ belongs to G but $x \notin A^*$ and $y \notin (\boldsymbol{R} - A)^*$.

In order to show that there exists a real number x that satisfies (42.18), let

$$(42.19) \qquad x = \sup\{r : r \text{ is a rational number and } [(r, \infty)] \in G\}$$

By the genericity of G, there exists r such that $[(r, \infty)] \notin G$, and hence the supremum (42.19) exists. Note also that $x \notin \mathfrak{M}$ (by the genericity of G). We shall show that x satisfies (42.18). We shall show, by induction on Borel codes in $\mathfrak{M}$, that for every $c \in \mathrm{BC}^{\mathfrak{M}}$,

$$(42.20) \qquad x \in A_c^* \;\leftrightarrow\; [A_c] \in G$$

First we consider Σ_1^0-codes (in $\mathfrak{M}$), and let us start with those $c \in \Sigma_1 \cap \mathfrak{M}$ that code a rational interval, i.e., such that $c(n) = 1$ for exactly one n; then c codes the interval I_n. Let $I_n = (p, q)$. We have

$$\begin{aligned} x \in A_c^* \quad &\text{iff} \quad p < x < q \quad \text{iff} \quad p < \sup\{r : [(r, \infty)] \in G\} < q \\ &\text{iff} \quad [(p, \infty)] \in G \text{ and } [(q, \infty)] \notin G \\ &\text{iff} \quad [(p, q)] \in G \quad \text{iff} \quad [A_c] \in G \end{aligned}$$

Now if $c \in \Sigma_1$, then $A_c = \bigcup_{n=0}^{\infty} I_{k_n}$, where $\{k_n : n = 0, 1, \ldots\}$ is the set $\{k : c(k) = 1\}$, and we have

$$\begin{aligned} x \in A_c^* \quad &\text{iff} \quad x \in \bigcup_{n=0}^{\infty} I_{k_n}^* \quad \text{iff} \quad \exists n(x \in I_{k_n}^*) \quad \text{iff} \quad \exists n([I_{k_n}] \in G) \\ &\text{iff} \quad \sum_{n=0}^{\infty} [I_{k_n}] \in G \quad \text{iff} \quad \Big[\bigcup_{n=0}^{\infty} I_{k_n}\Big] \in G \quad \text{iff} \quad [A_c] \in G \end{aligned}$$

Next let $\alpha < \omega_1^{\mathfrak{M}}$ and let $c \in \Pi_\alpha \wedge \mathfrak{M}$, and let us assume that (42.20) holds for all $c \in \Sigma_\alpha \cap \mathfrak{M}$. We may assume that $c(0) = 0$; then $u(c) \in \Sigma_\alpha \cap \mathfrak{M}$ and $A_{u(c)} = \boldsymbol{R} - A_c$, and we have

$$x \in A_c^* \quad \text{iff} \quad x \notin A_{u(c)}^* \quad \text{iff} \quad [A_{u(c)}] \notin G \quad \text{iff} \quad [A_c] \in G$$

Finally, the induction step for Σ_α is handled in a way similar to the case for $c \in \Sigma_1$. Thus (42.20) holds for every $c \in \mathrm{BC}^{\mathfrak{M}}$, and thus x is the unique real number that satisfies (42.16) (in case of $\mathscr{B}_{\mathrm{m}}$) or (42.17) (in case of $\mathscr{B}_{\mathrm{c}}$). ■

The following lemma provides a characterization of random and generic reals.

Lemma 42.6. *A real number is random over $\mathfrak{M}$ if and only if it does not belong to any null Borel set with code in $\mathfrak{M}$, and is generic over $\mathfrak{M}$ if and only if it does not belong to any meager Borel set with code in $\mathfrak{M}$.*

Hence if $\mathrm{Ra}(\mathfrak{M})$ and $\mathrm{Ge}(\mathfrak{M})$ denote the sets of all random and of all generic reals over $\mathfrak{M}$, we have

$$\mathrm{Ra}(\mathfrak{M}) = \boldsymbol{R}^* - \bigcup \{A_c^* : c \in \mathrm{BC}^{\mathfrak{M}} \text{ and } A_c^* \text{ is null}\} \tag{42.21}$$

$$\mathrm{Ge}(\mathfrak{M}) = \boldsymbol{R}^* - \bigcup \{A_c^* : c \in \mathrm{BC}^{\mathfrak{M}} \text{ and } A_c^* \text{ is meager}\}$$

Note that by Lemma 42.4, A_c is null (in $\mathfrak{M}$) if and only if A_c^* is null (in V).

Proof. On the one hand, if x is random over $\mathfrak{M}$, let G be an $\mathfrak{M}$-generic ultrafilter on $\mathscr{B}_{\mathrm{m}}$ such that $x = x_G$. Then if A_c is null then $[A_c] \notin G$, and by (42.16), $x \notin A_c^*$. Similarly for x that is generic over $\mathfrak{M}$.

On the other hand, let x be such that $x \notin A_c^*$ whenever A_c is null (and $c \in \mathfrak{M}$). First we observe that if $[A_c] = [A_d]$ then $A_c \triangle A_d$ is null, hence $A_c^* \triangle A_d^*$ is null and it follows that x belongs to A_c^* if and only if x belongs to A_d^*. Let

$$G = \{[A_c] : x \in A_c^*\} \tag{42.22}$$

It is easy to see that G is a filter on $\mathscr{B}_{\mathrm{m}}$: If $[A_c] \in G$ and $[A_d] \in G$, then $x \in A_c^* \cap A_d^*$ and hence $[A_c \cap A_d] \in G$; similarly, if $[A_c] \le [A_d]$ and $[A_c] \in G$, then $[A_d] \in G$.

We shall show that G is $\mathfrak{M}$-generic. Since $\mathscr{B}_{\mathrm{m}}$ satisfies the c.c.c., it suffices to show that if $\{A_{c_n} : n \in \omega\} \in \mathfrak{M}$ is such that $\sum_{n=0}^{\infty} [A_{c_n}] \in G$, then some $[A_{c_n}]$ is in G. But this is true because

$$\sum_{n=0}^{\infty} [A_{c_n}] = [\bigcup_{n=0}^{\infty} A_{c_n}] \qquad \text{and} \qquad \left(\bigcup_{n=0}^{\infty} A_{c_n}\right)^* = \bigcup_{n=0}^{\infty} A_{c_n}^*$$

Finally, we claim that $x = x_G$. But this follows from (42.22), by the genericity of G. Thus a real number x is random over $\mathfrak{M}$ if and only if $x \notin A_c^*$ for any null Borel set $A_c \in \mathfrak{M}$.

The proof is entirely similar for generic reals. ■

Solovay Sets of Reals

Let $\mathfrak{M}$ be a transitive model of ZFC. Let S be a set of reals. We say that the set S is *Solovay* over $\mathfrak{M}$ if there is a formula $\varphi(x)$, with parameters in $\mathfrak{M}$, such that for all reals x,

$$x \in S \quad \text{iff} \quad \mathfrak{M}[x] \models \varphi(x) \tag{42.23}$$

The following lemma is the key step in the proof of Theorem 101:

Lemma 42.7. *Let S be a Solovay set of reals over $\mathfrak{M}$. There exist Borel sets A and B such that*

$$S \cap \operatorname{Ra}(\mathfrak{M}) = A \cap \operatorname{Ra}(\mathfrak{M}) \qquad \textit{and} \qquad S \cap \operatorname{Ge}(\mathfrak{M}) = B \cap \operatorname{Ge}(\mathfrak{M})$$

Corollary.

(a) *If the set of all reals that are not random over $\mathfrak{M}$ is null, then S is Lebesgue measurable.*

(b) *If the set of all reals that are not generic over $\mathfrak{M}$ is meager, then S has the property of Baire.*

[*Proof.* Under the assumptions of the corollary, $S \triangle A$ is null and $S \triangle B$ is meager.]

Proof. Let me prove the lemma for random reals. Let us consider the forcing language in $\mathfrak{M}$ associated with $\mathscr{B}_{\mathrm{m}}$. Let $\boldsymbol{G}$ be the canonical name for a generic ultrafilter on $\mathscr{B}_{\mathrm{m}}$, and let $\boldsymbol{a}$ be the canonical name for a random real; i.e., let $\boldsymbol{a}$ be the $\mathscr{B}_{\mathrm{m}}$-valued name defined in $\mathfrak{M}^{\mathscr{B}_{\mathrm{m}}}$ from $\boldsymbol{G}$, by (42.16): $\|\boldsymbol{a} - x_{\boldsymbol{G}}\| = 1$.

Let $\varphi(x)$ be a formula with parameters in $\mathfrak{M}$ such that (42.23) holds for all x. Let $A_c \in \mathscr{B}$ be such that

$$[A_c] = \|\varphi(\boldsymbol{a})\|$$

and let $A = A_c^*$. The set A is a Borel set (in the universe); I claim that for all $x \in \operatorname{Ra}(\mathfrak{M})$, x belongs to S if and only if x belongs to A. But if x is random over $\mathfrak{M}$, let G be $\mathfrak{M}$-generic on $\mathscr{B}_{\mathrm{m}}$ such that $x = x_G$; then $\boldsymbol{a}$ is a name for x and we have

$$\begin{aligned} x \in S \quad &\text{iff} \quad \mathfrak{M}[x] \vDash \varphi(x) \quad \text{iff} \quad \mathfrak{M}[G] \vDash \varphi(x) \\ &\text{iff} \quad \|\varphi(\boldsymbol{a})\| \in G \quad \text{iff} \quad [A_c] \in G \quad \text{iff} \quad x \in A_c^* \quad \blacksquare \end{aligned}$$

The Models $\mathfrak{M}[G]$ and $\mathfrak{N}$

Let $\mathfrak{M}$ be a transitive model of ZFC and let κ be an inaccessible cardinal in $\mathfrak{M}$. Let B be the Lévy algebra for κ, i.e., $B = \text{r.o. } P$ where P is the notion of forcing (20.7) that collapses each $\alpha < \kappa$ onto $\aleph_0$: The conditions are functions p on subsets of $\kappa \times \omega$ such that each $\operatorname{dom}(p)$ is finite, and $p(\alpha, n) < \alpha$ whenever $(\alpha, n) \in \operatorname{dom}(p)$.

Let G be an $\mathfrak{M}$-generic ultrafilter on B. We shall show that in $\mathfrak{M}[G]$ every projective set of reals is Lebesgue measurable, has the property of Baire, and if uncountable, then it contains a perfect subset.

In $\mathfrak{M}[G]$, let S be the class of all infinite sequences of ordinal numbers, $\mathrm{S} = {}^{\omega}Ord$, and let $\mathfrak{N} = \operatorname{HOD}(\mathrm{S})$ be the class of all sets hereditarily ordinal definable over S. The class $\mathfrak{N}$ is a model of ZF (see Exercise 15.21). In fact, $\mathfrak{N}$ is a model of ZF + DC (see Lemma 42.8), and we shall show that in $\mathfrak{N}$ every set of reals is Lebesgue measurable, has the Baire property, and if uncountable, then it contains a perfect subset.

Let s be an infinite sequence of ordinals (in $\mathfrak{M}[G]$), let φ be a formula, and let $X \in G$ be a set such that

$$X = \{x : \mathfrak{M}[G] \models \varphi(x, s)\} \tag{42.24}$$

The method of Section 15 shows that X is (in $\mathfrak{M}[G]$) ordinal definable over $\mathrm{S} = {}^{\omega}Ord$. Conversely, if $X \in \mathrm{OD}(\mathrm{S})$, then for some formula ψ and a finite sequence $\langle s_1, \ldots, s_k \rangle$ of elements of S, $X = \{x : \mathfrak{M}[G] \models \psi(x, \langle s_1, \ldots, s_k \rangle)$. Then clearly there is φ and $s \in \mathrm{S}$ such that (42.24) holds. Hence the class OD(S) consists of all sets X of the form (42.24)—sets *definable* in $\mathfrak{M}[G]$ *from a sequence of ordinals.*

Note that every projective set of reals is definable from a sequence of ordinals: If A is $\Sigma_n^1(a)$ for some $a \in {}^{\omega}\omega$, then A is definable in HC from a, and therefore $A \in \mathrm{OD}(\mathrm{S})$.

Lemma 42.8.

(a) *If* $f \in \mathfrak{M}[G]$ *is a function on* ω *with values in* $\mathfrak{N}$, *then* $f \in \mathfrak{N}$.
(b) *The model* $\mathfrak{N}$ *satisfies the principle of dependent choices.*

Proof. (a) We show that if f is a function from ω into OD(S) then $f \in \mathrm{OD}(\mathrm{S})$. by (42.24), $\mathrm{OD}(\mathrm{S}) = \bigcup \{\mathrm{OD}(s) : s \in \mathrm{S}\}$; by Lemma 15.8 there is a definable function $\mathbf{F}$ on $Ord \times \mathrm{S}$ such that for each $s \in \mathrm{S}$, the function $\mathbf{F}_s(\alpha) = \mathbf{F}(\alpha, s)$ maps Ord onto $\mathrm{OD}(s)$. Let $f \colon \omega \to \mathrm{OD}(\mathrm{S})$. For each n, we choose α_n and s_n such that $f(n) = \mathbf{F}(\alpha_n, s_n)$. Clearly, f is definable from $\langle \alpha_n : n \in \omega \rangle$ and $\langle s_n : n \in \omega \rangle$. It is easy to find a single sequence u of ordinals such that both $\langle \alpha_n : n \in \omega \rangle$ and $\langle s_n : n \in \omega \rangle$ are definable from u. Hence f is definable from u, and so $f \in \mathrm{OD}(\mathrm{S})$.

(b) In $\mathfrak{N}$, let ρ be a relation over a nonempty set A such that for every $x \in A$ there is y such that $y \mathrel{\rho} x$. Since $\mathfrak{M}[G]$ satisfies the axiom of choice, there exists in $\mathfrak{M}[G]$ a sequence $a_0, a_1, \ldots, a_n, \ldots$ such that $a_{n+1} \mathrel{\rho} a_n$ for all n. However, by part (a) of this lemma, the sequence $\langle a_n : n \in \omega \rangle$ is in $\mathfrak{N}$. ■

Measure and Category in $\mathfrak{M}[G]$ and $\mathfrak{N}$

We shall now prove the part of Theorem 101 dealing with Lebesgue measure and the Baire property, using Lemma 42.7.

Lemma 42.9. *Let* $s \in \mathfrak{M}[G]$ *be an infinite sequence of ordinals. The set of all reals (in* $\mathfrak{M}[G]$*) that are random over* $\mathfrak{M}[s]$ *is null; the set of all reals that are not generic over* $\mathfrak{M}[s]$ *is meager.*

Proof. Since the algebra B is κ-saturated, there exists a subalgebra $D \subseteq B$ such that $|D| < \kappa$ and $\mathfrak{M}[s] = \mathfrak{M}[D \cap G]$. It follows that κ is inaccessible in $\mathfrak{M}[s]$; and since $\kappa = \aleph_1^{\mathfrak{M}[G]}$, $\mathfrak{M}[s]$ has only countably many subsets of ω. Thus there are only countably many Borel codes in $\mathfrak{M}[s]$; and by (42.21), the complement of the set $\mathrm{Ra}(\mathfrak{M}[s])$ is the union of countably many null sets and hence null. Similarly, the complement of $\mathrm{Ge}(\mathfrak{M}[s])$ is meager. ■

Lemma 42.10. *Let $X \in \mathfrak{M}[G]$ be a set of reals that is definable in $\mathfrak{M}[G]$ from a sequence s of ordinals. Then X is (in $\mathfrak{M}[G]$) Solovay over $\mathfrak{M}[s]$.*

Proof. The proof uses the properties of the Lévy algebra discussed in Section 25, in particular the factor lemma (Exercise 25.11). We shall first prove the following: Given a formula φ, there is a formula $\tilde{\varphi}$ such that for every sequence of ordinals $x \in \mathfrak{M}[G]$,

$$\mathfrak{M}[G] \vDash \varphi(x) \quad \text{iff} \quad \mathfrak{M}[x] \vDash \tilde{\varphi}(x) \tag{42.25}$$

We recall that P is the notion of forcing (20.7) that corresponds to the Lévy algebra for κ; the forcing conditions are finite and so the definition of P is absolute for all models. If M is a model, we denote M^P the Boolean valued model constructed in M using P and if ψ is a formula and $z \in M^P$, we denote $\|\psi(z)\|^M$ the Boolean value (computed in M using P) of $\psi(z)$. If $a \in M$, then $\check{a} \in M^P$ is the canonical name for a.

Let $\tilde{\varphi}(x)$ be the following formula

$$\|\varphi(\check{x})\|^{\mathfrak{M}[x]} = 1 \tag{42.26}$$

Let x be a countable sequence of ordinals in $\mathfrak{M}[G]$; we shall show that $\mathfrak{M}[G] \vDash \varphi(x)$ if and only if $\mathfrak{M}[x] \vDash \tilde{\varphi}(x)$. By the factor lemma (Exercise 25.11), there exists an $\mathfrak{M}[x]$-generic filter H on P such that $\mathfrak{M}[G] = \mathfrak{M}[x][H]$. Arguing in $\mathfrak{M}[x]$, we invoke the homogeneity of the Lévy algebra (see the corollary of Theorem 63): the Boolean value $b = \|\varphi(\check{x})\|^{\mathfrak{M}[x]}$ is either 0 or 1. Since H is generic on P over $\mathfrak{M}[x]$, $\varphi(x)$ is true in $\mathfrak{M}[x][H]$ if $b = 1$ and false if $b = 0$. Hence $\varphi(x)$ is true in $\mathfrak{M}[G]$ if and only if $\tilde{\varphi}(x)$ is true in $\mathfrak{M}[x]$.

Clearly, the above argument works also for a formula φ with two variables: There is $\tilde{\varphi}$ such that for all $x, y \in \mathfrak{M}[G] \cap {}^{\omega}Ord$,

$$\mathfrak{M}[G] \vDash \varphi(x, y) \quad \text{iff} \quad \mathfrak{M}[x, y] \vDash \tilde{\varphi}(x, y)$$

Now let $X \in \mathfrak{M}[G]$ be a set of reals that is definable in $\mathfrak{M}[G]$ from a sequence of ordinals s. For some formula φ

$$x \in X \quad \leftrightarrow \quad \mathfrak{M}[G] \vDash \varphi(x, s)$$

for all reals $x \in \mathfrak{M}[G]$. Since every real can be considered a countable sequence of ordinals, we have, for all $x \in \boldsymbol{R}^{\mathfrak{M}[G]}$,

$$\begin{aligned} x \in X \quad &\leftrightarrow \quad \mathfrak{M}[s, x] \vDash \tilde{\varphi}(s, x) \\ &\leftrightarrow \quad \mathfrak{M}[s][x] \vDash \tilde{\varphi}(s, x) \end{aligned}$$

which shows that X is Solovay over $\mathfrak{M}[s]$. ■

Corollary 1. *In $\mathfrak{M}[G]$ every set of reals definable from a sequence of ordinals (and in particular, every projective set of reals) is Lebesque measurable and has the property of Baire.*

Proof. This follows from Lemmas 42.7, 42.9, and 42.10. ■

Corollary 2. *In $\mathfrak{N}$ every set of reals is Lebesgue measurable and has the property of Baire.*

Proof. Clearly, the model $\mathfrak{N}$ has the same reals and the same elements of ${}^{\omega}\omega$ as the model $\mathfrak{M}[G]$. In particular, $\mathfrak{N}$ and $\mathfrak{M}[G]$ have the same Borel codes, and since $A_c^{\mathfrak{N}} = A_c^{\mathfrak{M}[G]} \cap \mathfrak{N} = A_c^{\mathfrak{M}[G]}$ for every $c \in BC^{\mathfrak{M}[G]}$, the two models have the same Borel sets.

If $X \in \mathfrak{N}$ is a set of reals, then X is definable in $\mathfrak{M}[G]$ from a sequence of ordinals and hence $\mathfrak{M}[G] \vDash$ (X is Lebesgue measurable and has the Baire property). Thus there are (in $\mathfrak{M}[G]$) Borel sets A, B, N, M such that $X \mathbin{\triangle} A \subseteq N$, $X \mathbin{\triangle} B \subseteq M$, and N is null and M is meager (in $\mathfrak{M}[G]$). By Lemma 42.4, $\mathfrak{N}$ satisfies that N is null and M is meager, and hence $\mathfrak{N}$ satisfies that X is Lebesgue measurable and has the Baire property. ■

A Remark on Σ_2^1 Sets of Reals

The method of the proof of Theorem 101 can be used to prove the following result on Σ_2^1 sets:

Lemma 42.11. *Assume that for every $a \subseteq \omega$, $\aleph_1^{L[a]}$ is countable. Then every Σ_2^1 set of reals is Lebesgue measurable and has the Baire property.*

[Compare this with the Corollary of Theorem 99.]

Proof. We use the tree representation of Σ_2^1 sets. Let A be a Σ_2^1 set. (We may assume that A is a Σ_2^1 subset of the Baire space which is homeomorphic to the irrational members.) Let $a \in {}^{\omega}\omega$ be such that A is $\Sigma_2^1(a)$, and let T be a tree over $\omega \times \omega_1$ such that $T \in L[a]$ and

$$x \in A \quad \text{iff} \quad T(x) \text{ is not well-founded}$$

for all $x \in {}^{\omega}\omega$. Since well foundedness is absolute, we have

$$x \in A \quad \text{iff} \quad L[a][x] \vDash T(x) \text{ is not well-founded}$$

and hence A is Solovay over $L[a]$.

Now each $L[a]$ has only countably many subsets of ω, hence only countably many Borel codes, and by (42.21) the complement of $\mathrm{Ra}(L[a])$ is null and the complement of $\mathrm{Ge}(L[a])$ is meager. Hence by Lemma 42.7, A is Lebesgue measurable and has the Baire property. ■

Exercise 42.3.* Use the above method to show that all Σ_1^1 sets are Lebesgue measurable and have the Baire property.

[Given a Σ^1_1 set A, find a countable transitive model M of $\mathrm{ZF}^- + \mathrm{DC}$ such that A is Solovay over M. Use absoluteness of Σ^1_1 for countable models, Exercise 42.1 and the fact that M has only countably many Borel codes.]

Perfect Subsets in $\mathfrak{M}[G]$ and $\mathfrak{N}$

We shall now finish the proof of Theorem 101 by showing that in $\mathfrak{M}[G]$ every uncountable set of reals definable from a countable sequence of ordinals contains a perfect subset. Then it follows that in $\mathfrak{N}$, every uncountable set A of reals has a perfect subset: If A is uncountable in $\mathfrak{N}$, then A is uncountable in $\mathfrak{M}[G]$ (by Lemma 42.8); and since A is definable from a sequence of ordinals, A has a perfect subset F (in $\mathfrak{M}[G]$); but then $\mathfrak{N} \vDash F$ is a perfect set.

By Lemma 42.10, every set of reals definable in $\mathfrak{M}[G]$ from s is Solovay over $\mathfrak{M}[s]$; thus it suffices to prove that in $\mathfrak{M}[G]$ every uncountable set of reals, Solovay over some $\mathfrak{M}[s]$, contains a perfect subset. Furthermore, it suffices to give the proof only for sets of reals Solovay over $\mathfrak{M}$ since the general case (Solovay over $\mathfrak{M}[s]$) follows from the special case by the factor lemma: $\mathfrak{M}[G] = \mathfrak{M}[s][H]$ is a generic extension of $\mathfrak{M}[s]$ via Lévy forcing. And finally, we can consider subsets of the Cantor space ${}^\omega\{0, 1\}$ instead of sets of reals.

Thus let A be, in $\mathfrak{M}[G]$, an uncountable subset of ${}^\omega\{0, 1\}$, and let φ be a formula (with parameters in $\mathfrak{M}$) such that for all $x \in {}^\omega\{0, 1\}$ in $\mathfrak{M}[G]$,

$$x \in A \qquad \text{iff} \qquad \mathfrak{M}[x] \vDash \varphi(x)$$

Since A is uncountable, there exists $x \in A$ such that $x \notin \mathfrak{M}$. There exists (in $\mathfrak{M}$) a complete subalgebra $C \subseteq B$ such that $|C| < \kappa$ and that $x \in \mathfrak{M}[G \cap C]$. Let us consider the Boolean-valued model $\mathfrak{M}^C$ and the corresponding forcing relation $\Vdash$. There exists a name $\boldsymbol{x} \in \mathfrak{M}^C$ and a condition $p \in C \cap G$ such that

$$p \Vdash \boldsymbol{x} \in {}^\omega\{0, 1\} \text{ and } \boldsymbol{x} \notin \mathfrak{M} \text{ and } (\mathfrak{M}[\boldsymbol{x}] \vDash \varphi(\boldsymbol{x})) \tag{42.27}$$

Since $P^{\mathfrak{M}}(C)$ is countable in $\mathfrak{M}[G]$, let $D_0, D_1, \ldots, D_n, \ldots$ be an enumeration (in $\mathfrak{M}[G]$) of all open dense subsets of C in $\mathfrak{M}$.

We shall construct conditions $p_s \in C$, for all finite 0, 1-sequences s, as follows:

Let $p_\varnothing \leq p$ be such that $p_\varnothing \in D_0$. Given p_s, there exists $n_s \in \omega$ such that p_s does not decide $\boldsymbol{x}(\check{n}_s)$ (because $p \Vdash \boldsymbol{x} \notin \mathfrak{M}$), and we let p_{s0} and p_{s1} be such that $p_{s0} \Vdash \boldsymbol{x}(\check{n}_s) = \check{0}$, and $p_{s1} \Vdash \boldsymbol{x}(\check{n}_s) = \check{1}$; moreover, we choose p_{s0} and p_{s1} so that both are in the open dense set D_k, where k is the length of s.

For every $z \in {}^\omega\{0, 1\}$, let $G_z = \{p \in C : p \geq p_s \text{ for some } s \subset z\}$. Clearly, $G_z \cap D_n \neq \varnothing$ for every n, and hence G_z is an $\mathfrak{M}$-generic ultrafilter on C. Let $f(z) = i_{G_z}(\boldsymbol{x})$ be the interpretation of $\boldsymbol{x}$ by G_z. Since G_z is generic, and by (42.27), we have $f(z) \in A$. Thus f is a function from ${}^\omega\{0, 1\}$ into A.

It follows from the construction of f that f is one-to-one and continuous. Thus $f[{}^\omega\{0, 1\}]$, the one-to-one continuous image of a perfect compact set, is a perfect subset of A. ∎

43. THE AXIOM OF DETERMINACY

With each subset A of ${}^{\omega}\omega$ we associate the following game G_A, played by two players I and II. First I chooses a natural number a_0, then II chooses a natural number b_0, then I chooses a_1, then II chooses b_1, and so on. The game ends after ω steps; if the resulting sequence $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is in A, then I wins, otherwise II wins.

A *strategy* (for I or II) is a rule that tells the player what move to make, depending on the previous moves of both players. A strategy is a *winning strategy* if the player who follows it always wins. The game G_A is *determined* if one of the players has a winning strategy.

The axiom of determinacy (AD) states that for every $A \subseteq {}^{\omega}\omega$, the game G_A is determined.

The axiom of determinacy would be an attractive hypothesis if it were not for the fact that it contradicts the axiom of choice (see Lemma 43.1). In a sense, some consequences of the axiom of determinacy for sets of real numbers are more desirable than the consequences of the axiom of choice; for instance, AD implies that all sets of reals are Lebesgue measurable. However, the consequences of AD for cardinal arithmetic are quite undesirable: For instance, AD implies that all $\aleph_n$, $3 \leq n < \omega$, are singular cardinals.

More interesting is the relation between determinacy and the theory of large cardinals. It has been established that AD implies existence of models with many measurable cardinals, and thus in order to prove the consistency of AD with ZF, one has to make some large cardinal assumption (consistency of AD is an open problem).

A more plausible hypothesis than AD, widely used by descriptive set theorists, is *projective determinacy* (PD) which states that G_A is determined for every projective set $A \subseteq {}^{\omega}\omega$. Although consistency of PD is also an open problem, PD does not seem to contradict the axiom of choice and has pleasing consequences in descriptive set theory.

Some Consequences of the Axiom of Determinacy

First some definitions: Let $A \subseteq {}^{\omega}\omega$ be given and let G_A denote the corresponding game. A *play* is a sequence $\langle a_0\, b_0\, a_1\, b_1 \cdots\rangle \in {}^{\omega}\omega$; for each n, a_n is the nth *move of player* I and b_n is the nth *move of player* II. A *strategy* for I is a function σ with values in ω whose domain consists of finite sequences $s \in Seq$ of even length. The player I plays $\langle a_0 b_0 a_1 b_1 \cdots\rangle$ *by the strategy* σ if $a_0 = \sigma(\emptyset)$, $a_1 = \sigma(\langle a_0\, b_0\rangle)$, $a_2 = \sigma(\langle a_0\, b_0\, a_1\, b_1\rangle)$, and so on; it is clear that if I plays by σ, then the play is determined by σ and the sequence $\langle b_n : n \in \omega\rangle$. We denote the play by $\sigma * b$. A strategy σ is a *winning strategy for I* if

$$\{\sigma * b : b \in \mathcal{N}\} \subseteq A \tag{43.1}$$

in other words, if all plays that I plays by σ are in A. Similarly, a strategy for II is a function τ with values in ω, defined on finite sequences $s \in Seq$ of odd length. If $a \in \mathcal{N}$ and if τ is a strategy for II, then $a * \tau$ denotes the play in which I plays a and II plays by τ. A strategy τ for II is a winning strategy if

$$(43.2) \qquad \{a * \tau : a \in \mathcal{N}\} \subseteq \mathcal{N} - A$$

Exercise 43.1.

(a) The function $f(b) = \sigma * b$ is a one-to-one continuous function.
(b) The set $\{\sigma * b : b \in \mathcal{N}\}$ contains a perfect subset.

[We shall sometimes consider games G_A whose moves are not natural numbers but elements of an arbitrary set S. A play is then a sequence $p \in {}^{\omega}S$, and the result of the game depends on whether $p \in A$ or $p \notin A$ (here A is a subset of ${}^{\omega}S$). The other relevant notions are defined accordingly.]

Since the number of strategies is $2^{\aleph_0}$, an easy diagonal argument shows that the axiom of choice is incompatible with the axiom of determinacy:

Lemma 43.1. *Assuming the axiom of choice, there exists $A \subseteq {}^{\omega}\omega$ such that the game G_A is not determined.*

Proof. Let $\{\sigma_\alpha : \alpha < 2^{\aleph_0}\}$ and $\{\tau_\alpha : \alpha < 2^{\aleph_0}\}$ enumerate all strategies for I and all strategies for II. We construct sets $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ and $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\}$, subsets of $\mathcal{N}$, as follows: Given $\{x_\xi : \xi < \alpha\}$ and $\{y_\xi : \xi < \alpha\}$, let us choose some y_α such that $y_\alpha = \sigma_\alpha * b$ for some b and $y_\alpha \notin \{x_\xi : \xi < \alpha\}$ (such y_α exist because the set $\{\sigma_\alpha * b : b \in \mathcal{N}\}$ has size $2^{\aleph_0}$); similarly, let us choose x_α such that $x_\alpha = a * \tau_\alpha$ for some a and $x_\alpha \notin \{y_\xi : \xi \leq \alpha\}$. It is clear that the sets X and Y are disjoint, that for each α there is b such that $\sigma_\alpha * b \notin X$, and there is a such that $a * \tau_\alpha \in X$. Thus neither I nor II has a winning strategy in the game G_X, and hence G_X is not determined. ■

In contrast with this lemma, the axiom of determinacy implies a weak form of the axiom of choice:

Lemma 43.2. *The axiom of determinacy implies that every countable family of nonempty sets of real numbers has a choice function.*

Proof. We prove that if $\mathcal{X} = \{X_n : n \in \omega\}$ is a family of nonempty subsets of $\mathcal{N}$, then there exists f on $\mathcal{X}$ such that $f(X_n) \in X_n$ for all n. Let us consider the following game: If I plays $\langle a_0 a_1 a_2 \cdots \rangle$ and II plays $\langle b_0 b_1 b_2 \cdots \rangle$, then II wins just in case $b \in X_{a_0}$. It is clear that I does not have a winning strategy: Once I plays a_0, the player II finds some $b \in X_{a_0}$, plays $b = \langle b_0 b_1 b_2 \cdots \rangle$ and wins. Hence II has a winning strategy τ, and we can define f on $\mathcal{X}$ as follows: $f(X_n) = \tau * \langle n 0 0 0 \cdots \rangle$. ■

[Regardless of the above lemma, we assume the principle of dependent choices throughout this section.]

The first theorem of this section shows that the axiom of determinacy eliminates some of the "undesirable" counterexamples that can be obtained if one uses the axiom of choice.

Theorem 102. *Assume the axiom of determinacy. Then:*

(a) *Every set of reals is Lebesgue measurable.*
(b) *Every set of reals has the property of Baire.*
(c) *Every uncountable set of reals contains a perfect subset.*

Let us deal first with the Lebesgue measure. We shall prove the following lemma:

Lemma 43.3. *Assuming* AD, *let S be a set of reals such that every measurable $Z \subseteq S$ is null. Then S is null.*

Recalling Lemma 39.7, it is easy to see that Lemma 43.3 implies that every set X is Lebesgue measurable: Let $A \supseteq X$ be a measurable set with the property that every measurable $Z \subseteq A - X$ is null. Then $A - X$ is null and hence X is measurable.

Thus let S be a set of reals with the property

(43.3) if $Z \subseteq S$ is Lebesgue measurable, then Z is null;

we shall use AD to show that S is null. It is clear that we can restrict ourselves to subsets of the unit interval; thus assume that $S \subseteq [0, 1]$. In order to show that S is null, it suffices to show that the outer measure $\mu^*(S)$ is less than any $\varepsilon > 0$. Thus let ε be a fixed positive real number.

The Covering Game

Given S and ε, let us set up a game as follows: If $\langle a_0 a_1 a_2 \cdots \rangle$ is a sequence of 0's and 1's, let a be the real number

$$a = \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}} \tag{43.4}$$

For each $n \in \omega$, let G_k^n, $k = 0, 1, 2, \ldots$, be an enumeration of the set K_n of all sets G such that

(43.5) (i) G is a union of finitely many intervals with rational endpoints;
(ii) $\mu(G) \leq \varepsilon/2^{2(n+1)}$.

The rules of the game are that player I tries to play a real number $a \in S$, and player II tries to cover the real a by the union $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in K_n$ for all n. More precisely, a play

$$\langle a_0 b_0 a_1 b_1 \cdots \rangle$$

is won by player I if

(i) $a_n = 0$ or 1, for all n;
(ii) $a \in S$; and
(iii) $a \notin \bigcup_{n=0}^{\infty} G^n_{b_n}$.

I claim that player I does not have a winning strategy in the game. To show this, notice that if σ is a winning strategy for I, then the function f that to each $b = \langle b_0 b_1 b_2 \cdots\rangle \in \mathcal{N}$ assigns the real number $a = f(b)$ such that $\langle a_0 b_0 a_1 b_1 \cdots\rangle = \sigma * b$ is continuous and hence the set $Z = f[\mathcal{N}]$ is analytic and hence measurable. Moreover, $Z \subseteq S$, and therefore Z is null. Now a null set can be covered by a countable union $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in K_n$ for all n (see Exercise 39.8), and therefore there is $b \in \mathcal{N}$ such that if II plays $\langle b_0 b_1 b_2 \cdots\rangle$ and I plays by σ, then II wins. Thus σ cannot be a winning strategy for I.

Assuming AD, the covering game is determined, and therefore player II has a winning strategy. Let τ be such strategy. For each finite sequence $s = \langle a_0, \ldots, a_n\rangle$ of 0's and 1's, let $G_s \in K_n$ be the set $G^n_{b_n}$, where $\langle b_0, \ldots, b_n\rangle$ are the moves that II plays by τ in response to $a_0, \ldots, a_n$. Since τ is a winning strategy, every $a \in S$ is in the set $\bigcup \{G_s : s \subset a\}$ and hence

(43.6) $$S \subseteq \bigcup \{G_s : s \in Seq\{0, 1\}\} = \bigcup_{n=1}^{\infty} \bigcup_{s \in {}^n\{0,1\}} G_s$$

Now for every $n \geq 1$, if $s \in {}^n\{0, 1\}$, then $\mu(G_s) \leq \varepsilon/2^{2n}$ and hence

$$\mu\Bigl(\bigcup_{s \in {}^n\{0,1\}} G_s\Bigr) \leq \frac{\varepsilon}{2^{2n}} \cdot 2^n = \frac{\varepsilon}{2^n}$$

It follows that $\mu(\bigcup_{n=1}^{\infty} \bigcup_{s \in {}^n\{0,1\}} G_s) \leq \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$ and thus $\mu^*(S) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, S is null. This completes the proof of Lemma 43.3 and also of part (a) of Theorem 102. Next we consider the property of Baire:

The Game G^{**}_X

Let X be a subset of the Baire space $\mathcal{N}$, and let us consider the following game G^{**}_X: Player I plays a finite sequence $s_0 \in Seq$; then II plays a proper extension $t_0 \supset s_0$; then I plays $s_1 \supset t_0$, etc.:

(43.7) $$s_0 \subset t_0 \subset s_1 \subset t_1 \subset \cdots$$

The sequence (43.7) converges to some $x \in \mathcal{N}$. If $x \in A$, then I wins, and otherwise II wins.

First we verify that this game can be reformulated as a game G_A of the kind introduced at the beginning of the section (i.e., when the moves are natural numbers). Let u_k, $k \in N$, be an enumeration of the set Seq. If $\langle a_0 b_0 a_1 b_1 \cdots\rangle$ is a sequence of numbers, then consider the sequence

(43.8) $$u_{a_0},\ u_{b_0},\ u_{a_1},\ u_{b_1},\ \ldots$$

and let A be the set of all $\langle a_0 \, b_0 \, a_1 \, b_1 \cdots \rangle \in \mathcal{N}$ such that: Either there is n such that

$$u_{a_0} \subset u_{b_0} \subset \cdots \subset u_{a_n} \not\subset u_{b_n}$$

or the sequence (43.8) is increasing and converges to some $x \in X$. It is clear I wins the game G_X^{**} if and only if I wins the game G_A.

Thus if AD holds, the game G_X^{**} is determined, for every $X \subseteq \mathcal{N}$. We will use the $**$-game to show that every $X \subseteq \mathcal{N}$ has the Baire property.

Lemma 43.4. *Player* II *has a winning strategy in* G_X^{**} *if and only if* X *is meager.*

Proof. Let Y be the complement of X in $\mathcal{N}$. For each $s \in Seq$, U_s denotes the basic open set $\{x \in \mathcal{N} : s \subset x\}$.

(a) If X is a meager set, then there exist open dense sets G_n, $n \in N$, such that $Y \supseteq \bigcap_{n=0}^{\infty} G_n$. It is easy to find a winning strategy τ for II: If I plays s_0, let $t_0 = \tau(\langle s_0 \rangle)$ be some $t_0 \supset s_0$ such that $U_{t_0} \subset G_0$; such t_0 exists because G_0 is dense. Then if I plays $s_1 \supset t_0$, let $t_1 = \tau(\langle s_0 \, t_0 \, s_1 \rangle)$ be some $t_1 \supset s_1$ such that $U_{t_1} \subset G_1$, and so on. It is clear that every such play $s_0 \subset t_0 \subset s_1 \subset \cdots$ converges to $x \in \bigcap_{n=0}^{\infty} G_n$, and hence τ is a winning strategy for II.

(b) Conversely, assume that II has a winning strategy τ. A *correct position* is a finite sequence $\langle s_0, t_0, \ldots, s_n, t_n \rangle$ such that $s_0 \subset t_0 \subset \cdots \subset t_n$ and $t_0 = \tau(\langle s_0 \rangle)$, $t_1 = \tau(\langle s_0 \, t_0 \, s_1 \rangle)$, etc. We shall first prove the following claim: Let $x \in \mathcal{N}$ and assume that for every correct position $p = \langle s_0 \cdots t_n \rangle$ with $t_n \subset x$ there exists $s \supset t_n$ such that $\tau(p^\frown s) \subset x$. Then $x \in Y$.

To prove the claim, let x satisfy the condition. To begin, there exists s_0 such that $\tau(s_0) \subset x$; let $t_0 = \tau(\langle s_0 \rangle)$. Then there exists $s_1 \supset t_0$ such that $t_1 = \tau(\langle s_0 \, t_0 \, s_1 \rangle) \subset x$; then there is $s_2 \supset t_1$ such that $\tau(\langle s_0 t_0 s_1 t_1 s_2 \rangle) \subset x$; and so on. The sequence $s_0 \subset t_0 \subset s_1 \subset t_1 \subset \cdots$ converges to x and is a play in which II plays by τ. Hence $x \in Y$.

For every correct position $p = \langle s_0 \cdots t_n \rangle$, let

$$F_p = \{x \in \mathcal{N} : t_n \subset x \text{ and } (\forall s \supset t_n)\, \tau(p^\frown s) \not\subset x\}$$

By the claim, for every $x \in X$ there is a correct position p such that $x \in F_p$; in other words,

$$X \subseteq \bigcup \{F_p : p \text{ is a correct position}\}$$

It is easy to see that for each p, $U_{t_n} - F_p$ is an open dense set in U_{t_n}; hence F_p is a closed rare set. The number of correct positions is countable and hence X is meager. ■

Corollary. *Player* I *has a winning strategy in* G_X^{**} *if and only if for some* $s \in Seq$, $U_s - X$ *is meager.*

Proof. Note that I has a winning strategy in G_X^{**} if and only if there exists $s \in Seq$ (the first move of I in G_X^{**}) such that player II has a winning strategy in the following game $\mathrm{G}^{**s}_{U_s-X}$: I plays $t_0 \supset s$, II plays $s_0 \supset t_0$, I plays $t_1 \supset s_0$, etc.; and I wins if $t_0 \subset s_0 \subset t_1 \subset \cdots$ converges to $x \in U_s - X$. By the lemma, II has a winning strategy in this game just in case $U_s - X$ is meager. ■

Now part (b) of Theorem 102 follows. If $X \subseteq \mathcal{N}$, then since G^{**}_X is determined, either X is meager or for some $s \in Seq$, $U_s - X$ is meager. Thus let $X \subseteq \mathcal{N}$ be arbitrary. If X is meager, then X has the Baire property. If X is nonmeager, then let $G = \bigcup \{U_s : U_s - X \text{ is meager}\}$. Clearly, $G - X$ is meager, and $X - G$ must be meager too because otherwise there would be s such that $U_s - (X - G)$ is meager, which contradicts the definition of G. It follows that X has the Baire property. ■

Now we turn our attention to part (c) of Theorem 102. We will use AD to prove that every uncountable set in the Cantor space $C = {}^{\omega}\{0, 1\}$ has a perfect subset. We consider the following game.

The Game G^*_X

Let X be a subset of ${}^{\omega}\{0, 1\}$. The game G^*_X is defined as follows: Player I plays a sequence $s_0 \in Seq\{0, 1\}$ of 0's and 1's (possibly the empty sequence), then player II plays $n_0 \in \{0, 1\}$, then I plays $s_1 \in Seq\{0, 1\}$, and so on. Let $x = s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \cdots$. Player I wins if $x \in X$, and II wins if $x \notin X$. The game G^*_X can be reformulated as a game G_A, for some $A \subseteq {}^{\omega}\omega$.

Lemma 43.5. *If* II *has a winning strategy in* G^*_X, *then* X *is countable.*

Proof. The proof follows closely the proof of Lemma 43.4b. Let τ be a winning strategy for II. A *correct position* is a finite sequence $\langle s_0, n_0, \ldots, s_k, n_k\rangle$ such that $n_0 = \tau(\langle s_0\rangle)$, $n_1 = \tau(\langle s_0, n_0, s_1\rangle)$, etc. By exactly the same argument as in Lemma 43.4, we get the following claim: Let $x \in {}^{\omega}\{0, 1\}$ and assume that for every correct position $p = \langle s_0 \cdots n_k\rangle$ if $s_0{}^\frown n_0{}^\frown \cdots {}^\frown n_k \subset x$, then there exists $s \in Seq\{0, 1\}$ such that $s_0{}^\frown n_0{}^\frown \cdots {}^\frown n_k{}^\frown s^\frown \tau(p^\frown s) \subset x$. Then $x \notin X$.

It follows that $X \subseteq \bigcup \{F_p : p \text{ is a correct position}\}$, where

$$F_p = \{x \in C : s_0{}^\frown \cdots {}^\frown n_k \subset x \text{ and } \forall s(s_0{}^\frown \cdots {}^\frown n_k{}^\frown s^\frown \tau(p^\frown s) \not\subset x)\}$$

The lemma will follow if we show that each F_p has exactly one element $x \in C$. This element x is uniquely determined as follows (because each $x(m)$ is either 0 or 1): firstly, for some $l \in N$, $\langle x(0), \ldots, x(l-1)\rangle = s_0{}^\frown n_0{}^\frown \cdots {}^\frown n_k$; then $x(l) = 1 - \tau(p^\frown \varnothing)$, $x(l+1) = 1 - \tau(p^\frown \langle x(l)\rangle)$, $x(l+2) = 1 - \tau(p^\frown \langle x(l), x(l+1)\rangle)$, and so on. ■

Now part (c) of Theorem 102 follows. If $X \subseteq C$ is uncountable, then II does not have a winning strategy in G^*_X; and since the game is determined, I has a winning strategy σ. For each $x = \langle n_0, n_1, \ldots\rangle \in C$, let $F(x) \in C$ denote the 0, 1-sequence

$$s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \cdots$$

where $s_0 = \sigma(\emptyset)$, $s_1 = \sigma(\langle s_0{}^\frown n_0\rangle)$, $s_2 = \sigma(\langle s_0{}^\frown n_1{}^\frown s_1{}^\frown n_1\rangle)$, etc. The function f is continuous and one-to-one, and hence $f[C]$ is a perfect set. But $X \supseteq f[C]$ and hence X has a perfect subset. ■

Exercise 43.2. I has a winning strategy in G^*_X if and only if X has a perfect subset. II has a winning strategy if and only if X is countable.

We proved in Section 41 that if $\aleph_1 = \aleph_1^{L[a]}$ for some $a \subseteq \omega$, then there is an uncountable set without a perfect subset. Thus if every uncountable set of reals contains a perfect subset, then every $\aleph_1^{L[a]}$ is countable, and it follows that for every $a \subseteq \omega$, $\aleph_1$ is an inaccessible cardinal in $L[a]$. (Because if $\aleph_1 = \aleph_{\alpha+1}^{L[a]}$, then there exists $x \subseteq \omega$ such that $\aleph_\alpha^{L[a]}$ is countable in $L[a, x]$, and hence for some $b \subseteq \omega$, $\aleph_1 = \aleph_1^{L[b]}$.) Thus we have:

Corollary to Theorem 102. *If* AD *holds, then* $\aleph_1$ *is inaccessible in* $L[a]$, *for every* $a \subseteq \omega$.

AD and Large Cardinals

To illustrate the relationship between the axiom of determinacy and the theory of large cardinals, we shall show that AD implies that $\aleph_1$ and $\aleph_2$ are measurable cardinals. We recall that measurability of successor cardinals (in ZF without choice) is a plausible property (see Theorem 90).

Theorem 103 (Solovay). *The axiom of determinacy implies that*

(a) $\aleph_1$ *is a measurable cardinal, and moreover, the closed unbounded filter over* $\aleph_1$ *is an ultrafilter.*
(b) $\aleph_2$ *is a measurable cardinal.*

We shall first show that AD implies that ω_1 is measurable. We already know that ω_1 is inaccessible in every $L[a]$, $a \subseteq \omega$.

Let us consider the following partial ordering of the Baire space:

$$x \preccurlyeq y \quad \text{iff} \quad x \in L[y] \tag{43.9}$$

and the corresponding equivalence relation

$$x \equiv y \quad \text{iff} \quad x \preccurlyeq y \ \text{ and } \ y \preccurlyeq x \tag{43.10}$$

We say that $A \subseteq \mathcal{N}$ is *$\equiv$-closed* if ($x \in A$ and $y \equiv x$) implies $y \in A$. Note that the collection $\mathcal{B}$ of all $\equiv$-closed sets in $\mathcal{N}$ is a complete Boolean algebra.

If $x_0 \in \mathcal{N}$, then we let

$$\text{cone}(x_0) = \{x \in \mathcal{N} : x_0 \preccurlyeq x\} = \{x : x_0 \in L[x]\} \tag{43.11}$$

and call cone(x_0) a *cone*. Clearly, every cone is $\equiv$-closed. Let

$$\mathcal{F}_c = \{A \in \mathcal{B} : A \text{ contains a cone}\}$$

We claim that $\mathscr{F}_c$ is a σ-complete filter on B. Let $A_0, A_1, \ldots, A_n, \ldots$ be elements of $\mathscr{F}_c$. For each n, we choose $x_n \in \mathcal{N}$ such that $A_n \supseteq \text{cone}(x_n)$. Let $x \in \mathcal{N}$ be defined as follows: $x(\Gamma(n, m)) = x_n(m)$, for all $n, m \in N$ (where Γ is the canonical Γ: $\omega \times \omega \to \omega$). It is clear that for each n, $x_n \in L[x]$ and hence $\text{cone}(x) \subseteq \text{cone}(x_n) \subseteq A_n$. Thus $\bigcap_{n=0}^{\infty} A_n$ is in $\mathscr{F}_c$.

Lemma 43.6. AD *implies that for every* $\equiv$*-closed* $A \subseteq \mathcal{N}$, *either* A *or its complement contains a cone. Hence* $\mathscr{F}_c$ *is a* σ*-complete ultrafilter on* $\mathscr{B}$.

Proof. We show that if I has a winning strategy in the game G_A, then A contains a cone (and similarly, if II has a winning strategy, then $\mathcal{N} - A \in \mathscr{F}_c$). Let σ be a winning strategy for I. It suffices to show that A contains the cone $\{x \in \mathcal{N} : \sigma \in L[x]\}$.

Let $x \in \mathcal{N}$ be such that $\sigma \in L[x]$. Then $a = \sigma * x$ is in A because σ is a winning strategy. Clearly, $x \in L[a]$, and because $\sigma \in L[x]$, we also have $a \in L[x]$ and hence $x \equiv a$. Since A is $\equiv$-closed, we have $x \in A$. ■

Now we can use AD to find a nonprincipal σ-complete ultrafilter U over ω_1. For each $x \in \mathcal{N}$, let $f(x) = \aleph_1^{L[x]}$; $f(x)$ is a countable ordinal. Note that if $x \equiv y$, then $f(x) = f(y)$, and hence for every $X \subseteq \omega_1$, the set $f_{-1}(X) \subseteq \mathcal{N}$ is $\equiv$-closed. Let

$$U = \{X \subseteq \omega_1 : f_{-1}(X) \in \mathscr{F}_c\}$$

Since $\mathscr{F}_c$ is a σ-complete ultrafilter on $\mathscr{B}$, U is a σ-complete ultrafilter over ω_1. It remains to show that U is nonprincipal. But for every $\alpha < \omega_1$, if $x \in \mathcal{N}$ is such that $\aleph_1^{L[x]} = \alpha$, then there is $y \succcurlyeq x$ such that $\aleph_1^{L[y]} > \alpha$; and hence $f_{-1}(\{\alpha\}) \notin \mathscr{F}_c$.

Thus AD implies that ω_1 is a measurable cardinal.

Lemma 43.7. AD *implies that for every* $S \subseteq \omega_1$, *the set* $\{x \in \text{WO} : \|x\| \in S\}$ *is* Π_1^1.

Corollary. AD *implies that for every* $S \subseteq \omega_1$ *there is* $a \subseteq \omega$ *such that* $S \in L[a]$.

[Proof of Corollary: Lemma 41.4.]

Proof of Lemma 43.7. If $x \in \mathcal{N}$, then for each $n \in N$ we let $x_n \in \mathcal{N}$ be such that $x_n(m) = x(\Gamma(n, m))$ for all $m \in N$.

The Solovay Game

Let $S \subseteq \omega_1$. Player I plays $a = \langle a(0), a(1), \ldots\rangle$, and II plays $b = \langle b(0), b(1), \ldots\rangle$. If $a \notin \text{WO}$, then II wins; if $a \in \text{WO}$, then II wins just in case

$$\{\alpha \in S : \alpha \leq \|a\|\} \subseteq \{\|b_n\| : n \in \omega\} \subseteq S$$

We claim that I does not have a winning strategy in the Solovay game. Let σ be a winning strategy for I; for each $b \in \mathcal{N}$, let $f(b)$ be the $a \in \mathcal{N}$ such that

$\langle a(0), b(0), a(1), b(1), \dots\rangle = \sigma * b$. The set $f[\mathcal{N}]$ is a Σ^1_1 subset of WO, and by the boundedness lemma, there is $\alpha < \omega_1$ such that $\|f(b)\| < \alpha$ for all $b \in \mathcal{N}$. Hence let $b \in \mathcal{N}$ be such that $\{\|b_n\| : n \in \omega\} = S \cap \alpha$. Then $\sigma * b$ is a play won by player II, and hence σ cannot be a winning strategy for I.

Now the lemma follows: Let $S \subseteq \omega_1$. By AD, player II has a winning strategy τ in the Solovay game. For each a, let $g(a)$ be the $b \in \mathcal{N}$ such that $\langle a(0), b(0), \dots\rangle = a * \tau$. It follows that for each $a \in \mathrm{WO}$,

$$\|a\| \in S \qquad \text{iff} \qquad \exists n(a = (g(a))_n)$$

and consequently the set $\{x \in \mathrm{WO} : \|x\| \in S\}$ is Π^1_1. ■

We can now complete the proof of Theorem 103a. If $X \subseteq \omega_1$, then $X \in L[a]$ for some $a \subseteq \omega$. Since $\aleph_1$ is a measurable cardinal, $a^\#$ exists, and it follows that either X or $\omega_1 - X$ contains a closed unbounded subset (see Exercise 30.3). Thus the closed unbounded filter over ω_1 is an ultrafilter.

By the countable axiom of choice, the closed unbounded filter is σ-complete, and we therefore conclude that AD implies that the closed unbounded filter over ω_1 is the unique σ-complete normal ultrafilter over ω_1.

Proof of Theorem 103b. We shall now show that, assuming AD, $\aleph_2$ is a measurable cardinal. For each $x \in \mathcal{N}$, let $f(x)$ denote the successor cardinal of the (real) cardinal $\aleph_1$ in $L[x]$:

$$f(x) = ((\aleph_1)^+)^{L[x]}$$

If $x \subseteq \omega$, then because $x^\#$ exists, $f(x)$ is an ordinal less than $\aleph_2$. Moreover, if $x \equiv y$, then $f(x) = f(y)$, and hence $f_{-1}(X)$ is $\equiv$-closed for each $X \subseteq \omega_2$. Thus let us define an ultrafilter U over ω_2 as follows:

$$U = \{X \subseteq \omega_2 : f_{-1}(X) \in \mathcal{F}_c\}$$

We wish to show that U is an $\aleph_2$-complete nonprincipal ultrafilter over ω_2. Since $\mathcal{F}_c$ is σ-complete, U is σ-complete. It is also easy to see that U is nonprincipal: If $\alpha < \omega_2$ and $f(x) = \alpha$, then there exists $S \subseteq \omega_1$ such that α is not a cardinal in $L[S]$; then by the corollary of Lemma 43.7 there is $y \in \mathcal{N}$ such that $x \in L[y]$ and $f(y) > \alpha$. Hence $f_{-1}(\{\alpha\})$ does not contain a cone.

It remains to show that U is $\aleph_2$-complete. Since U is σ-complete, it suffices to show that if

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_\alpha \supseteq \cdots \qquad (\alpha < \omega_1) \tag{43.12}$$

is a descending sequence of subsets of ω_2 such that each $f_{-1}(X_\alpha)$ contains a cone then $f_{-1}(\bigcap_{\alpha<\omega_1} X_\alpha)$ contains a cone.

Let us consider such a sequence (43.12), and let $X = \bigcap_{\alpha<\omega_1} X_\alpha$. We shall use the following game: Player I plays $a = \langle a(0), a(1), \dots\rangle$, and II plays $b = \langle b(0), b(1), \dots\rangle$. If $a \notin \mathrm{WO}$, then I loses; if $a \in \mathrm{WO}$ and $\|a\| = \alpha$, then II wins just in case $\mathrm{cone}(b) \subseteq f_{-1}(X_\alpha)$.

We claim that I does not have a winning strategy in this game: If σ is a winning strategy for I, then the set of all $a \in \mathcal{N}$ that I plays by σ against all possible $b \in \mathcal{N}$, is a Σ^1_1 subset of WO and hence there is α such that $\|a\| < \alpha$ for all these a's. Now II can beat I simply by playing some $b \in \mathcal{N}$ such that $\text{cone}(b) \subseteq f_{-1}(X_\alpha)$.

Thus II has a winning strategy τ, and we intend to show that $f_{-1}(X)$ contains the cone $\{x \in \mathcal{N} : \tau \in L[x]\}$. Let $\alpha < \omega_1$ and let $x \in \mathcal{N}$ be such that $\tau \in L[x]$; we want to show that $f(x) \in X_\alpha$.

Let P_α be the standard notion of forcing that collapses α onto ω: The conditions are finite sequences of ordinals less than α. Since $\aleph_1$ is inaccessible in $L[x]$, $L[x]$ has only countably many subsets of P_α, and therefore there exists an $L[x]$-generic filter G on P_α. Let $a \in \text{WO}$ be such that $\|a\| = \alpha$ and that $L[a] = L[G]$ and let $y \in \mathcal{N}$ be such that $L[y] = L[x][G] = L[x][a]$.

Since G is generic on P_α over $L[x]$, all cardinals in $L[x]$ greater than α are preserved in $L[x][G]$. In particular, $(\aleph_1^+)^{L[x]}$ is preserved and hence $f(y) = f(x)$.

Now if I plays $a = \langle a(0), a(1), \ldots\rangle$ and if II plays against a by his winning strategy τ, II produces $b = \langle b(0), b(1), \ldots\rangle$ such that $\text{cone}(b) \subseteq f_{-1}(A_\alpha)$. But since $b \in L[\tau, a]$ and $\tau \in L[x]$, we have $b \in L[x, a] = L[y]$ and therefore $y \in \text{cone}(b)$. It follows that $f(y) \in X_\alpha$; and because $f(x) = f(y)$, we have $f(x) \in X_\alpha$, as we wanted to prove.

This completes the proof of Theorem 103b. ■

Already Theorem 103 indicates that AD is a very strong large cardinal axiom: The fact that the closed unbounded filter over ω_1 is an ultrafilter has alone strong consequences, namely existence of models with many measurable cardinals. The AD also implies existence of a model of ZFC having a measurable cardinal that is a limit of measurable cardinals. And $\aleph_1$ and $\aleph_2$ are not the only measurable cardinals if AD holds: There are many more measurable cardinals (but not $\aleph_3$; all $\aleph_n$, $3 \leq n < \omega$ are singular and have cofinality $\aleph_2$; the next two measurable cardinals are $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$).

Projective Determinacy

Since the assumption that the game G_A is determined for every $A \subseteq \mathcal{N}$ is inconsistent with the axiom of choice, and perhaps too strong to be plausible, it is natural to ask whether one can assume that G_A is determined whenever A is a simple set; i.e., if A is definable or has a simple structure. One immediate observation is that all open games are determined.

Lemma 43.8. *If $A \subseteq \mathcal{N}$ is an open set, then G_A is determined.*

Proof. Player I plays $\langle a_0 a_1 \cdots\rangle$, player II plays $\langle b_0 b_1 \cdots\rangle$, and I wins just in case $\langle a_0 b_0 a_1 b_1 \cdots\rangle \in A$. Let us assume that player I does not have a winning strategy, and let us show that II has a winning strategy. The strategy for II is as follows: When I plays a_0, then because I does not have a winning strategy, there exists b_0 such that the position $\langle a_0 b_0 \rangle$ is not yet lost for II. That is, I does not have a winning strategy in the game $G_A^{\langle a_0 b_0 \rangle}$ that starts at $\langle a_0 b_0 \rangle$, in which I plays $\langle a_1 a_2 \cdots \rangle$, II plays $\langle b_1 b_2 \cdots \rangle$, and in which I wins just in case $\langle a_0 b_0 a_1 b_1 \cdots \rangle \in A$.

Let II play such b_0. When I plays a_1, then again, because II is not yet lost at $\langle a_0 b_0 \rangle$, there exists b_1 such that II is not yet lost at $\langle a_0 b_0 a_1 b_1 \rangle$. Let II play such b_1. And so on. We claim that this strategy for II is a winning strategy.

Let $x = \langle a_0 b_0 a_1 b_1 \cdots \rangle$ be a play which II plays by the above strategy; we want to show that $x \notin A$. If $x \in A$, then because A is open, there is $s = \langle a_0 b_0 \cdots a_n b_n \rangle \subset x$ such that $U_s \subset A$. But then it is clear that II is lost at s; a contradiction. ■

The same argument (interchanging the players) would show that every closed game is determined. Or, we can show that every closed game is determined as follows: If A is closed, then I has a winning strategy in G_A if and only if there is $a_0 \in \mathcal{N}$ such that II does not have a winning strategy in the open game $G_A^{a_0}$ in which II makes a first move b_0, then I plays a_1, etc., and II wins iff $\langle a_0 b_0 a_1 \cdots \rangle$ is in the open set $\mathcal{N} - A$. Since $G_A^{a_0}$ is determined for all $a_0 \in \mathcal{N}$, G_A is determined.

A similar argument shows

Exercise 43.3. Let $n > 0$. If G_A is determined for every Σ_n^1 set A, then G_A is determined for every Π_n^1 set A, and vice versa.

A useful conjecture for descriptive set theorists is the *axiom of projective determinacy* (PD) which states that G_A is determined for all projective sets $A \subseteq \mathcal{N}$. Unlike AD, the axiom PD does not seem to contradict the axiom of choice.

The reason projective determinacy is so useful in descriptive set theory is that it enables one to prove various properties of projective sets and decides various questions on projective sets which are undecidable in ZF + DC. For instance, the proof of Theorem 102 shows that PD implies that all projective sets are Lebesgue measurable, have the Baire property, and are either countable or contain a perfect subset. Another application of PD is a generalization of the uniformization theorem (Theorem 97): if PD holds then every Π_{2n+1}^1 relation can be uniformized by a Π_{2n+1}^1 function. A similar generalization applies to the reduction principle (cf. Exercise 40.3).

Let us say that all Σ_n^1 (Π_n^1, Δ_n^1) games are determined if G_A is determined for every Σ_n^1 (Π_n^1, Δ_n^1) set $A \subseteq \mathcal{N}$. The following two theorems solve the problem of determinacy of Borel and analytic games.

Theorem 104 (Martin). *All Borel games are determined.*

Theorem 105. *All analytic games are determined if and only if $x^\#$ exists for all $x \subseteq \omega$.*

Corollary. *If there exists a measurable cardinal, then all analytic games are determined.*

[The "if" part of Theorem 105 is due to Martin, the "only if" part is due to Harrington.]

Theorem 105 shows that PD is essentially a large cardinal axiom. In fact, it is known that determinacy of Δ^1_2 games implies existence of a model with many measurable cardinals (a measurable cardinal that is a limit of measurable cardinals). Consistency of PD is an open problem; the best result to date is this theorem of Martin: Δ^1_2 determinacy follows from the existence of an elementary embedding j: $V \to M$ with the property that ${}^{\lambda}M \subseteq M$ for every $\lambda < \sup\{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$, where κ is the least ordinal moved. (Compare this with Exercise 28.9 and the concluding remark in Section 33.)

Theorems 104 and 105 are stated here without proof. I shall prove only the corollary of Theorem 105; the proof can be converted without much effort into a proof of the "if" part of Theorem 105. The proof features an important technique of determinacy, the use of an auxiliary game.

Let κ be a measurable cardinal and let $A \subseteq \mathcal{N}$ be an analytic set. We want to show that the game G_A is determined.

Let us use the tree representation of analytic sets. There is a tree $T \subseteq Seq_2$ such that for all $x \in \mathcal{N}$,

$$x \in A \quad \leftrightarrow \quad T(x) \text{ is not well-founded}$$

(see (40.23)). Let $\preccurlyeq$ be the linear ordering of the set Seq that extends the partial ordering $\supseteq$: If $s, t \in Seq$, then $s \preccurlyeq t$ either if $s \supseteq t$, or if s and t are incompatible, and $s(n) < t(n)$ where n is the least n such that $s(n) \neq t(n)$. Thus (see 40.26)

$$x \in A \quad \leftrightarrow \quad T(x) \text{ is not well-ordered by } \preccurlyeq$$

We also recall that $T(x) = \{t : (s, t) \in T \text{ for some } s \in Seq\}$ and that the first n levels of $T(x)$ depend only on $\bar{x}(n)$. For $s \in Seq$, we let

$$T(s) = \{t : (u, t) \in T \text{ for some } u \subseteq s\}$$

then $T(\bar{x}(n))$ is exactly the first n levels of the tree $T(x)$. We need some further notation. Let $t_0, t_1, \ldots, t_n, \ldots$ be an enumeration of the set Seq. If $s \in Seq$ is a sequence of length $2n$, let K_s be the finite set $\{t_0, \ldots, t_{n-1}\} \cap T(s)$ and let k_s be the size of K_s.

We shall now define an auxiliary game G*: Player I plays natural numbers $a_0, a_1, a_2, \ldots$, and player II plays pairs $(b_0, h_0), (b_1, h_1), (b_2, h_2), \ldots$ where $b_0, b_1, b_2, \ldots$ are natural numbers, and for each n, h_n is an order-preserving mapping from $(K_s, \preccurlyeq)$ into κ where $s = \langle a_0, b_0, \ldots, a_n, b_n\rangle$, such that $h_0 \subseteq h_1 \subseteq h_2 \subseteq \cdots \subseteq h_n \subseteq \cdots$. If player II is able to follow these rules throughout the game, then he (or she) wins. Otherwise, I wins.

It is clear that the game G* is determined: If I does not have a winning strategy, then he cannot prevent II from following the rules and thus II has a winning strategy, namely his each move is to reach a position in which I does not have a winning strategy. (The argument is the same as in the proof of determinacy of open games; in fact, G* is an open game in a suitable topology.)

If II wins a play in the game G*, then he has constructed an order-preserving mapping $h = \bigcup_{n=0}^{\infty} h_n$ of $(T(x), \preccurlyeq)$ into κ, where $x = \langle a_0, b_0, a_1, b_1, \ldots\rangle$; hence $\preccurlyeq$ well-orders $T(x)$ and so $x \notin A$. Thus we can view the game G* as a variant of G_A, but more difficult for player II: II tries to make sure that $x \notin A$, and in addition, he tries to construct an embedding of $(T(x), \preccurlyeq)$ in κ. Hence it is fairly obvious that if II has a winning strategy in the game G*, then II has a winning strategy in G_A: If τ^* is a winning strategy for II in G*, let τ be as follows. When I plays a_0, let $\tau(\langle a_0\rangle) = b_0$ where $(b_0, h_0) = \tau^*(\langle a_0\rangle)$; then when I plays a_1, let $\tau(\langle a_0 b_0 a_1\rangle) = b_1$ where $(b_1, h_1) = \tau^*(\langle a_0, (b_0, h_0), a_1\rangle)$; etc.

Since G* is determined, it suffices to prove the following lemma in order to show that G_A is determined:

Lemma 43.9. *If* I *has a winning strategy in* G*, *then* I *has a winning strategy in* G_A.

Proof. Let σ^* be a winning strategy for I in G*. After $2n + 2$ moves, the players have produced a sequence $s = \langle a_0, b_0, \ldots, a_n, b_n\rangle$, and II has constructed order-preserving functions $h_0 \subseteq \cdots \subseteq h_n$; the strategy σ^* then tells player I what to play next. Let E be the range of h_n; E is a finite subset of κ, and in fact its size is k_s. We observe that there is one and only one way II could have constructed $h_0, \ldots, h_n$ so that $\operatorname{ran}(h_n) = E$; the reason is that h_n is the unique order-preserving one-to-one function between $(K_s, \preccurlyeq)$ and E. Thus σ^* depends (as long as II plays correctly) only on $s \in Seq$ and the finite set $E \subset \kappa$.

For each $s \in Seq$ of even length, let F_s be the following function from $[\kappa]^{k_s}$ into ω:

$$F_s(E) = \sigma^*(s, E) \tag{43.13}$$

Each F_s is a partition of $[\kappa]^{k_s}$ into ω pieces; and because κ is measurable, there exists a set $H \subseteq \kappa$ of size κ homogeneous for each F_s. Let us denote by $\sigma(s)$ the unique value of $F_s(E)$ for $E \in [H]^{k_s}$.

We shall complete the proof by showing that σ is a winning strategy for I in the game G_A. Let $x = \langle a_0 b_0 a_1 b_1 \cdots\rangle$ be a play in which I plays by σ. We shall show that $x \in A$.

Assume that on the contrary, $x \notin A$. Then $(T(x), \preccurlyeq)$ is well-ordered and has order-type $< \omega_1$. Since H is uncountable, there exists an embedding h of $(T(x), \preccurlyeq)$ into H. Let us consider the following play of the game G*: I plays a_0. Then II plays (b_0, h_0) where h_0 is the restriction of h to $K_{\langle a_0 b_0\rangle}$. Then I plays a_1 and II plays (b_1, h_1) where h_1 is restriction of h to $K_{\langle a_0 b_0 a_1 b_1\rangle}$. And so on.

We show that in this play, player I plays by the strategy σ^*. Clearly, $a_0 = \sigma(\emptyset) = \sigma^*(\emptyset, \emptyset)$. Then $a_1 = \sigma(\langle a_0, b_0\rangle)$, and by the definition of σ it is clear that $\sigma(\langle a_0 b_0\rangle) = \sigma^*(\langle a_0 b_0\rangle, h[K_{\langle a_0 b_0\rangle}])$ and therefore a_1 is a move according to σ^*. And so on: All the moves $a_0, a_1, \ldots, a_n, \ldots$ are by σ^*.

This is a contradiction because σ^* is a winning strategy for I in G*, but the play we described is won by player II. It follows that $x \in A$ and hence σ is a winning strategy for I in the game G_A. ■

This completes the proof of Σ_1^1 determinacy assuming a measurable cardinal. This assumption can be weakened to the assumption that $x^\#$ exists for all $x \subseteq \omega$. The above proof is then modified as follows: We play the auxiliary game as before; κ is an uncountable cardinal. The definition of the auxiliary game is absolute for the model $L[T]$, and it follows that either I or II has a winning strategy for G*, which is in $L[T]$. In particular, in Lemma 43.9, we may take $\sigma^* \in L[T]$. Then the collection $\{F_s : s \in Seq\}$, where F_s is defined by (43.13), is in $L[T]$, and an indiscernibility argument shows that there is an uncountable set $H \subseteq \kappa$ of indiscernibles for $L[T]$ such that each F_s has the same value for all $E \in [H]^{k_s}$. The rest of the proof is the same.

44. SOME APPLICATIONS OF FORCING IN DESCRIPTIVE SET THEORY

This last section contains several consistency results in descriptive set theory. Since we use Martin's axiom and generic models, we work again in set theory with the axiom of choice.

Some Consequences of Martin's Axiom

Theorem 106 (Martin–Solovay). *If Martin's Axiom holds, then the union of less than $2^{\aleph_0}$ null sets is null, and the union of less than $2^{\aleph_0}$ meager sets is meager.*

Corollary 1. *If* MA *holds, then both the algebra of Lebesgue measurable sets and the algebra of sets with the Baire property are $2^{\aleph_0}$-complete, and moreover, Lebesgue measure is $2^{\aleph_0}$-additive, i.e., if $\kappa < 2^{\aleph_0}$ and A_α, $\alpha < \kappa$, are pairwise disjoint, then*

$$\mu\Big(\bigcup_{\alpha<\kappa} A_\alpha\Big) = \sum_{\alpha<\kappa} \mu(A_\alpha) \tag{44.1}$$

[See (27.10) for the definition of the sum on the right-hand side of (44.1).]

Corollary 2. *If* MA *holds and if $2^{\aleph_0} > \aleph_1$, then every Σ_2^1 set is Lebesgue measurable and has the property of Baire.*

To deduce Corollary 1 from the theorem, we prove by induction on $\kappa < 2^{\aleph_0}$ that if A_α, $\alpha < \kappa$, are Lebesgue measurable, then $A = \bigcup_{\alpha<\kappa} A_\alpha$ is Lebesgue measurable. Given A_α, $\alpha < \kappa$, let $B_\alpha = A_\alpha - \bigcup_{\beta<\alpha} A_\beta$, for each $\alpha < \kappa$. The sets B_α are Lebesgue measurable (by the induction hypothesis), and being pairwise disjoint, all but countably many are null. It follows from the theorem that $A = \bigcup_{\alpha<\kappa} B_\alpha$ is Lebesgue measurable. The same argument proves (44.1) (see also Lemma 27.4), and the property of Baire is analogous.

As for Corollary 2, we use Lemma 42.7. If A is $\Sigma^1_2(a)$, then A is Solovay over $L[a]$ (see Lemma 42.11). Since $(2^{\aleph_0})^{L[a]} = \aleph_1^{L[a]} \le \aleph_1$, it follows that the set of all reals that are not random over $L[a]$ is the union of at most $\aleph_1$ null sets, hence null (by the theorem). By the corollary of Lemma 42.7, A is Lebesgue measurable. The Baire property is analogous.

Proof of Theorem 106. First we prove that the union of less than $2^{\aleph_0}$ null sets is null. Let $\kappa < 2^{\aleph_0}$ and let A_α, $\alpha < \kappa$, be null sets of reals. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$. In order to prove that A is null, it suffices to find, for each $\varepsilon > 0$, an open set $U \supseteq A$ such that $\mu(U) \le \varepsilon$. Let $\varepsilon > 0$.

We apply Martin's axiom as follows. Let P be the set of all open sets of measure $< \varepsilon$, and let $p \in P$ be stronger than $q \in P$ if $p \supseteq q$. We claim that the notion of forcing $(P, \supseteq)$ satisfies the countable chain condition.

It suffices to show that if W is an uncountable subset of P, then there are $p, q \in W$, $p \ne q$, such that $\mu(p \cup q) < \varepsilon$. Let S be the countable set of all unions of finitely many open intervals with rational endpoints. If $W \subseteq P$ is uncountable, then there is $n \in N$ and an uncountable $Z \subseteq W$ such that $\mu(p) < \varepsilon - 1/n$ for all $p \in Z$. For each $p \in Z$, let $p^* \in S$ be such that $p^* \subseteq p$ and $\mu(p - p^*) < 1/n$. Since S is countable, there exist $p, q \in Z$, $p \ne q$, such that $p^* = q^*$. Then $\mu(p \cup q) < \varepsilon$.

For each $\alpha < \kappa$, let $D_\alpha = \{p \in P : A_\alpha \subseteq p\}$. Each D_α is a dense subset of P: If $p \in P$, then since A_α is null, there exists an open set $q \supseteq A_\alpha$ such that $\mu(p) + \mu(q) < \varepsilon$, and hence $p \cup q \in D_\alpha$ and $p \cup q \supseteq p$.

By Martin's axiom, there exists a filter $G \subseteq P$ such that $G \cap D_\alpha \ne \emptyset$ for all $\alpha < \kappa$. Let $U = \bigcup \{p : p \in G\}$. It is clear that $A \subseteq U$ and it remains to show that $\mu(U) \le \varepsilon$. We use the well-known fact (easy to verify) that if $U = \bigcup \{p : p \in G\}$, then there is a countable $H \subseteq G$ such that $U = \bigcup \{p : p \in H\}$. Thus if $\mu(U) > \varepsilon$, there exist $p_1, \ldots, p_n \in H$ such that $\mu(p_1 \cup \cdots \cup p_n) > \varepsilon$. But this is impossible: Since G is a filter on P, we have $p \cup q \in G$ whenever $p \in G$ and $q \in G$; thus $p_1 \cup \cdots \cup p_n \in G$ and hence $\mu(p_1 \cup \cdots \cup p_n) < \varepsilon$.

This completes the proof that the union of $< 2^{\aleph_0}$ null sets is null if MA holds.

In order to show that the union of less than $2^{\aleph_0}$ meager sets is meager, it suffices to show that the union of less than $2^{\aleph_0}$ closed rare sets is meager. The following lemma will complete the proof:

Lemma 44.1. *Assume Martin's axiom. Let $\kappa < 2^{\aleph_0}$ and let A_α, $\alpha < \kappa$, be closed rare sets of reals. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$. Then there exists a countable family of dense open sets H_i, $i = 0, 1, 2, \ldots$, such that A is disjoint from $\bigcap_{i=0}^\infty H_i$.*

Proof. We apply Martin's axiom as follows. Let P be the set of all finite sequences of pairs

$$p = \langle (U_0, E_0), (U_1, E_1), \ldots, (U_n, E_n) \rangle$$

such that

(44.2) (i) each U_i is the union of finitely many open intervals with rational endpoints;
(ii) each E_i is a finite subset of κ; and
(iii) for each i, U_i is disjoint from $\bigcup_{\alpha \in E_i} A_\alpha$.

A condition $p' = \langle (U'_0, E'_0), \ldots, (U'_m, E'_m)\rangle$ is stronger than a condition $p = \langle (U_0, E_0), \ldots, (U_n, E_n)\rangle$ if

(44.3) (i) $m \geq n$; and
(ii) for each $i \leq n$, $U'_i \supseteq U_i$ and $E'_i \supseteq E_i$.

It is clear that this notion of forcing satisfies the countable chain condition: If two conditions have the same $U_0, \ldots, U_n$, then they are compatible, and there are only countably many sequences $\langle U_0, \ldots, U_n\rangle$.

Let I_k, $k = 0, 1, 2, \ldots$, be an enumeration of all open intervals with rational endpoints. We let, for each $\alpha < \kappa$ and all $i, k = 0, 1, 2, \ldots$

$$D_\alpha = \{p : p = \langle (U_0, E_0), \ldots, (U_n, E_n)\rangle \text{ and } \alpha \in E_i \text{ for some } i \leq n\} \tag{44.4}$$

$$E_{ik} = \{p : p = \langle (U_0, E_0), \ldots, (U_n, E_n)\rangle \text{ and } U_i \cap I_k \neq \emptyset\}$$

Since each A_α, $\alpha < \kappa$, is rare, it is clear that for all i and k, every condition can be extended to a condition $p \in E_{ik}$, and hence each E_{ik} is dense in P. Also, each D_α is dense in P.

By Martin's axiom, there exists a filter $G \subseteq P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$, and $G \cap E_{ik} \neq \emptyset$ for all $i, k \in \omega$. For each $i = 0, 1, 2, \ldots$, we let

$$H_i = \bigcup \{U_i : (\exists p \in G)\, p = \langle \ldots, (U_i, E_i), \ldots\rangle\}$$

Since E_{ik} is a dense set of conditions, for all k, H_i is a dense open set of reals.

Now if $\alpha < \kappa$, then because D_α is dense, there exists $i \in \omega$ such that H_i is disjoint from A_α, and hence A_α is disjoint from $\bigcap_{i=0}^{\infty} H_i$. Therefore A is disjoint from $\bigcap_{i=0}^{\infty} H_i$. ■

We showed in Section 27 (Corollary 2 of Lemma 27.9) that if Martin's axiom holds, then $2^{\aleph_0}$ is not real-valued measurable. Using Theorem 106, one can get a better result:

Corollary 3. *If* MA *holds, then there is no real-valued measurable cardinal* $\leq 2^{\aleph_0}$.

Proof. If there is a real-valued measurable cardinal $\leq 2^{\aleph_0}$, then there exists a σ-additive measure ν on $\boldsymbol{R}$ that extends the Lebesgue measure μ (see the remark following Lemma 27.5). Theorem 106 implies that every set $B \subseteq \boldsymbol{R}$ such that $\nu(B) > 0$ has size $2^{\aleph_0}$. Using this fact and σ-additivity of ν, the proof of Lemma 27.9 shows that there is no $2^{\aleph_0}$-scale; but MA implies that a $2^{\aleph_0}$-scale exists. ■

A set of reals A has *strong measure* 0 if for every sequence $a_0 \geq a_1 \geq \cdots \geq a_n \geq \cdots$ of positive reals, there exists a sequence of open intervals I_n, $n = 0, 1, \ldots$, such that length $(I_n) \leq a_n$ and that $A \subseteq \bigcup_{n=0}^{\infty} I_n$. It is clear that every set of strong measure 0 is null, but not every null set has necessarily strong measure 0 (compare this with Exercise 39.8, where intervals are replaced by finite unions of intervals). For instance the Cantor set C (cf. Fig. 4.3) is null but does not have strong measure 0; if $a_0 = \frac{1}{3}$, $a_1 = \frac{1}{9}$, $a_2 = \frac{1}{27}$, etc., then C cannot be covered with open intervals of lengths $a_0, a_1, a_2, \ldots$.

Exercise 44.1. If $A \subseteq \boldsymbol{R}$ contains a perfect subset, then it does not have strong measure 0.

[Either prove directly, or use the fact that A contains a subset homeomorphic to the Cantor set and that a uniformly continuous image of a set of strong measure 0 has strong measure 0.]

Clearly, every countable set has strong measure 0. Martin's axiom generalizes this.

Exercise 44.2. Martin's axiom implies that every set $A \subseteq \boldsymbol{R}$ of size $< 2^{\aleph_0}$ has strong measure 0.

[May assume $2^{\aleph_0} > \aleph_1$. Consider the forcing notion $(P, <) = (Seq, \supset)$ which adjoins a Cohen generic $x \in {}^{\omega}\omega$. Since $|A| < 2^{\aleph_0}$, MA implies that there is $x \in {}^{\omega}\omega$ P-generic over every $L[a]$, $a \in A$. Let $r_0, r_1, \ldots, r_n, \ldots$ be an enumeration of all rational numbers. Given positive reals $a_0 \geq a_1 \geq \cdots \geq a_n \geq \cdots$, let for each n, I_n be the interval with center $r_{x(n)}$ and diameter a_n. Use the genericity of x to show that $a \in \bigcup_{n=0}^{\infty} I_n$, for each $a \in A$.]

While every countable set has strong measure 0, the problem of whether only countable sets have strong measure 0 (*Borel's conjecture*) is independent of ZFC. The following two exercises show that the continuum hypothesis implies that there is an uncountable set of strong measure 0; and there is a generic model in which no uncountable set of reals has strong measure 0.

Exercise 44.3. If $2^{\aleph_0} = \aleph_1$, then there exists an uncountable set $E \subseteq \boldsymbol{R}$ such that for every rare set F, $E \cap F$ is at most countable. (E is called a *Luzin set*.) More generally, MA implies that there is a set E of size $2^{\aleph_0}$ whose intersection with every rare set has size $< 2^{\aleph_0}$.

[Let $F_0, F_1, \ldots, F_\alpha, \ldots, \alpha < 2^{\aleph_0}$, be an enumeration of all closed rare sets. Let $E = \{e_\alpha : \alpha < 2^{\aleph_0}\}$, where for each α, $e_\alpha \notin \bigcup_{\beta < \alpha} F_\beta$. Each e_α exists; the MA case uses Theorem 106.]

Exercise 44.4. Martin's axiom (and in particular the continuum hypothesis) implies that there is an uncountable set of reals of strong measure 0.

[Let E be the set from Exercise 44.3. Let $a_0 \geq a_1 \geq \cdots \geq a_n \geq \cdots$ be given positive reals. For each n, let I_{2n} be the interval of length a_{2n} around the nth rational. The set $U = \bigcup_{n=1}^{\infty} I_{2n}$ is open dense and hence $E - U$ is countable, $E - U = \{x_0, x_1, \ldots, x_n, \ldots\}$. For each n, let I_{2n+1} be the interval of length a_{2n+1} around x_n.]

I shall mention one more consequence of Martin's axiom. Let $\mathscr{B}_{\mathrm{m}}$ be the complete Boolean algebra of Borel subsets of the unit interval modulo null sets. The algebra $\mathscr{B}_{\mathrm{m}}$ does not have a countable dense subset. (To see this, let $[A_n]$, $n = 0, 1, 2, \ldots$, be nonzero elements of $\mathscr{B}_{\mathrm{m}}$. For each n, let $B_n \subseteq A_n$ be such that $0 < \mu(B_n) \leq 1/2^{n+2}$, and let $B = \bigcup_{n=0}^{\infty} B_n$. The complement C of B has positive measure, and clearly $[A_n] \nleq [C]$, for all n.)

Exercise 44.5. Martin's axiom implies that every dense subset of $\mathscr{B}_{\mathrm{m}}$ has size $2^{\aleph_0}$.

[Let $\kappa = 2^{\aleph_0}$. Let $x_\alpha, \alpha < \kappa$, be an enumeration of all reals. MA implies that for every α, $\{x_\beta : \beta \geq \alpha\}$ has positive measure; let K_α be a compact subset of $\{x_\beta : \beta \geq \alpha\}$ such that $\mu(K_\alpha) > 0$. If $\mathscr{B}_{\mathrm{m}}$ has a dense subset of size $< \kappa$, then since κ is regular, there exists $W \subseteq \kappa$ of size κ and a set X of positive measure such that $X - K_\alpha$ is null for all $\alpha \in W$. Hence every finite subset of $\{K_\alpha : \alpha \in W\}$ has nonempty intersection and so $\bigcap_{\alpha \in W} K_\alpha$ is nonempty; a contradiction.]

In contrast to Exercise 44.5, there is a model of set theory in which $\mathscr{B}_{\mathrm{m}}$ has a dense subset of size $< 2^{\aleph_0}$.

Recall that a complete Boolean algebra is weakly (ω, ω)-distributive if it satisfies

$$\prod_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{nm} = \sum_{g:\,\omega\to\omega} \prod_{n=0}^{\infty} \sum_{m<g(n)} u_{nm}$$

(see (25.8)). The algebra $\mathscr{B}_{\mathrm{c}}$ of Borel sets modulo meager sets is not weakly (ω, ω)-distributive (cf. Exercise 25.4).

*Exercise 44.6.** The Boolean algebra $\mathscr{B}_{\mathrm{m}}$ is weakly (ω, ω)-distributive.

[Let u_{nm} $(n, m = 0, 1, 2, \ldots)$ be given. Clearly, the right-hand side is $\leq$ the left-hand side. Conversely, let $0 \neq u \leq$ LHS, and let $r = \mu(u)$. Let g: $\omega \to \omega$ be defined as follows: $g(n)$ is such that

$$\mu\Bigl(u - \sum_{m<g(n)} u_{nm}\Bigr) < \frac{r}{2^{n+2}}$$

Then $u \cdot (\prod_{n=0}^{\infty} \sum_{m<g(n)} u_{nm}) \neq 0$. Hence LHS $\leq$ RHS.]

(Compare with Lemma 27.9.)

Two Examples on Measure and Category

If Martin's axiom holds and $2^{\aleph_0} > \aleph_1$, then every set of reals of size $\aleph_1$ is null and meager; in particular, the set $\boldsymbol{R} \cap L$ of all constructible reals is null and meager. We shall now give an example of a model where $2^{\aleph_0}$ is large and the set $\boldsymbol{R} \cap L$ is not Lebesgue measurable, and another example where $2^{\aleph_0}$ is large and $\boldsymbol{R} \cap L$ does not have the Baire property.

Lemma 44.2. *If there exists a nonconstructible real, then:*

(a) *$\boldsymbol{R} \cap L$ is either null or not Lebesgue measurable.*
(b) *$\boldsymbol{R} \cap L$ is either meager or does not have the Baire property.*

Proof. We use the fact that both measure and category are preserved under translation.

(a) Let S be the set of all constructible reals in the unit interval $[0, 1]$. Let a be a nonconstructible real. For each $n > 0$, let S_n be the translation of S by a/n, $S_n = \{x + (a/n) : x \in S\}$. The sets S_n are pairwise disjoint, $\mu(S_n) = \mu(S)$ for all n, and $\bigcup_{n=0}^{\infty} S_n$ is a bounded set. It is therefore clear that if S is measurable, then $\mu(S) > 0$ is impossible.

(b) Let S be the set of all constructible reals. First we prove that $\boldsymbol{R} - S$ is not meager. Let a be a nonconstructible real and let $S_a = \{x + a : x \in S\}$; clearly, $S \cap S_a = \emptyset$. Thus $\boldsymbol{R} = (\boldsymbol{R} - S) \cup (\boldsymbol{R} - S_a) \cup (S \cap S_a) = (\boldsymbol{R} - S) \cup (\boldsymbol{R} - S_a)$, and if $\boldsymbol{R} - S$ were meager, then $\boldsymbol{R} - S_a$ would also be meager, a contradiction.

It follows that for any nonempty interval I, $I - S$ is not meager: For each rational r, let $A_r = \{x + r : x \in I - S\}$; if $I - S$ is meager, then each A_r is meager, and $\boldsymbol{R} - S = \bigcup \{A_r : r \text{ rational}\}$.

If S has the Baire property, then because $U - S$ is not meager for any nonempty open set U, S is meager. ■

Example I. A model where $2^{\aleph_0} > \aleph_1$ and the set of all constructible reals is not Lebesgue measurable.

Let λ be a regular uncountable cardinal and let B be the measure algebra defined in (34.3); i.e., let $(S, \mathscr{F}, m)$ be the product measure space, where S is the product of $\lambda \times \omega$ copies of $\{0, 1\}$, $\mathscr{F}$ is the least σ-complete field of subsets of S containing all $\{t \in S : t(\alpha, n) = 0\}$, and m is the product measure, and let B be the measure algebra, $B = \mathscr{F}/\text{sets of measure } 0$.

Let us consider the generic extension of the constructible universe by the measure algebra B. As we proved in Section 34, the generic extension $L[G]$ satisfies $2^{\aleph_0} = \lambda$. We shall show that in $L[G]$ the set of all constructible reals is not Lebesgue measurable.

In view of Lemma 44.2, it suffices to show that the set of all constructible reals is not null. Thus assume that it is null and let I_k, $k = 0, 1, 2, \ldots$, be an enumeration (in L) of all intervals with rational endpoints. Let μ denote Lebesgue measure.

Assuming that $L[G] \vDash \mu(\boldsymbol{R} \cap L) = 0$, there is a B-valued name X for a subset of ω, and a rational $\varepsilon > 0$ such that the Boolean value

(44.5)

$$\| \textstyle\bigcup \{I_k : k \in X\} \text{ contains all constructible reals, and has Lebesgue measure } \leq \varepsilon \|$$

is in G. We may assume, without loss of generality, that the Boolean value (44.5) is 1.

For each $k \in N$, let $A_k \in \mathscr{F}$ be such that

$$\|k \in X\| = [A_k]$$

Let us consider (in L) the product measure space $(\boldsymbol{R}, \mu) \times (S, \mathscr{F}, m)$ with the product measure $\nu = \mu \times m$. Let $E \subseteq \boldsymbol{R} \times S$ be the set

$$E = \bigcup_{k=0}^{\infty} (I_k \times A_k)$$

We claim that $\nu(E) \leq \varepsilon$. It suffices to show that for every k_0, $\nu(\bigcup_{k<k_0} (I_k \times A_k)) \leq \varepsilon$. Let $k_0 \in N$. By (44.5), for every condition $a = [A]$ there exists a stronger condition $c = [C]$ and a set $Y \subset k_0$ such that

$$c \Vdash X \cap \check{k}_0 = \check{Y}$$

and that $\mu(\bigcup_{k \in Y} I_k) \leq \varepsilon$. Clearly, $[C] \leq [A_k]$ if $k \in Y$, and $[C] \cdot [A_k] = 0$ if $k \in k_0 - Y$ and hence

$$\nu\left(\bigcup_{k<k_0} I_k \times (A_k \cap C)\right) = \mu\left(\bigcup_{k \in Y} I_k\right) \cdot m(C) \leq \varepsilon \cdot m(C) \tag{44.6}$$

Thus the set of all $[C]$ for which (44.6) holds is dense in the algebra B and hence

$$\nu\left(\bigcup_{k<k_0} I_k \times A_k\right) \leq \varepsilon$$

Since $\nu(E) \leq \varepsilon$, the complement of E has positive measure and hence there exists, by Fubini's theorem, a number $x \in \boldsymbol{R}$ such that

$$m(\{t \in S : (x, t) \notin E\}) > 0$$

It follows that there exists $A \in \mathscr{F}$ of positive measure such that

$$(x, t) \notin E, \qquad \text{for all} \quad t \in A \tag{44.7}$$

We shall show that

$$[A] \Vdash \check{x} \notin \bigcup \{I_k : k \in X\} \tag{44.8}$$

thus completing the proof.

If (44.8) were not true, there would exist some $k \in N$ and some $C \subseteq A$ of positive measure such that $x \in I_k$ and $C \Vdash \check{k} \in X$. But then $[C] \leq [A_k]$ and hence there is $t \in C$ such that $(x, t) \in E$, contrary to (44.7). ■

Example II. A model where $2^{\aleph_0} > \aleph_1$ and the set of all constructible reals does not have the property of Baire.

Let λ be a regular uncountable ordinal and let P be the notion of forcing that adjoins λ Cohen reals: A condition is a finite 0,1-function whose domain is a subset of λ.

Let us consider the generic extension of the constructible universe by P. In $L[G]$, $2^{\aleph_0} = \lambda$. We shall show that in $L[G]$ the set of all constructible reals does not have the Baire property.

In view of Lemma 44.2, it suffices to show that the set of all constructible reals is not meager. For every $S \subseteq \lambda$ (in L), let $P_S = \{p \in P : \operatorname{dom}(p) \subseteq S\}$, and let $G_S = G \cap P_S$.

Lemma 44.3. *If $L[G] \vDash \boldsymbol{R} \cap L$ is meager, then there exists a countable $S \subseteq \lambda$ in L) such that $L[G_S] \vDash \boldsymbol{R} \cap L$ is meager.*

Proof. Let I_k, $k \in N$, be an enumeration of all open intervals with rational endpoints. If $L[G] \vDash \boldsymbol{R} \cap L$ is meager, then there exists a sequence $\langle U_n : n \in N \rangle \in L[G]$ such that for every $n \in N$, $L[G] \vDash U_n$ is dense open, and that $\boldsymbol{R} \cap L \subseteq \bigcup_{n=0}^{\infty} (\boldsymbol{R} - U_n)$. Let $A = \{(n, k) : I_k \subseteq U_n\}$, and let $\dot{A}$ be a name for A. Since P satisfies the countable chain condition, there is a countable $S \subset \lambda$ such that $\dot{A}$ is r.o.(P_S)-valued. It is easy to verify that if $U'_n = U_n \cap L[G_S]$, then for each $n \in N$, $L[G_S] \vDash U'_n$ is dense open, and that $\boldsymbol{R} \cap L \subseteq \bigcup_{n=0}^{\infty} (\boldsymbol{R} - U'_n)$. Thus $L[G_S] \vDash \boldsymbol{R} \cap L$ is meager. ■

Since P_S is countable, it suffices to prove the following lemma:

Lemma 44.4. *If P is a countable notion of forcing in L and if G is an L-generic filter on P, then $L[G] \vDash \boldsymbol{R} \cap L$ is not meager.*

Proof. It suffices to show that if $\mathcal{U} = \langle U_n : n \in N \rangle$ is (in $L[G]$) a sequence of dense open sets of reals, then there is a constructible real a such that $a \in \bigcap_{n=0}^{\infty} U_n$. Let $\dot{\mathcal{U}}$ be a name for $\mathcal{U}$ and let us assume, without loss of generality, that every condition forces that each $\dot{\mathcal{U}}_n$ is dense open. It is enough to find (in L) a real number x such that for each n and each $p \in P$, there is $q \leq p$ such that $q \Vdash \check{x} \in \dot{\mathcal{U}}_n$.

Let t_k, $k = 0, 1, \ldots$, be an enumeration of all pairs $t = (n, p)$ where $n \in N$ and $p \in P$. Let us construct a sequence $I_0 \supseteq I_1 \supseteq \cdots \supseteq I_k \supseteq \cdots$ of closed bounded intervals as follows: Let $t_k = (n, p)$. Since $p \Vdash \dot{\mathcal{U}}_n$ is open dense, there is an open interval $J \subseteq I_{k-1}$ with rational endpoints such that some $q \leq p$ forces $J \subseteq \dot{\mathcal{U}}_n$. Let $I_k \subseteq J$. The intersection $\bigcap_{k=0}^{\infty} I_k$ is nonempty; and if x is in it, then for each n and each $p \in P$ there is $q \leq p$ such that $q \Vdash \check{x} \in \dot{\mathcal{U}}_n$. ■

A Nonconstructible Δ_3^1-Set of Integers

By Shoenfield's absoluteness theorem, every Π_2^1 or Σ_2^1 subset of N is constructible. The model presented below (due to Jensen) shows that it is consistent with ZFC that there exists a nonconstructible subset of N that is Δ_3^1. (Recall that another example of a nonconstructible Δ_3^1 subset of N is $0^\#$.)

Example III. A generic extension of L which has a nonconstructible Δ_3^1 subset of N.

Let us assume the axiom of constructibility, i.e., the ground model is the constructible universe L. We shall construct a notion of forcing P such that if G is an L-generic filter on P, then $L[G] \vDash$ there exists a nonconstructible Δ_3^1-set $A \subseteq N$.

The construction is a combination of perfect set forcing (Section 26) and arguments using the $\diamondsuit$-principle (Section 22). Let us consider *perfect binary trees*: $Seq\{0, 1\}$ denotes the set of all finite 0,1-sequences; a nonempty set $p \subseteq Seq\{0, 1\}$ is a perfect tree if: (a) if s is an initial segment of t and $t \in p$, then $s \in p$; and (b) for every $s \in p$, there exists $t \supseteq s$ such that both $t^\frown 0$ and $t^\frown 1$ are in p.

We denote by $[p]$ the set of all branches in p:

$$(44.9) \qquad a \in [p] \quad \leftrightarrow \quad a \in {}^\omega\{0, 1\} \text{ and } \forall n(\bar{a}(n) \in p)$$

The set $[p]$ is a perfect subset of the Cantor space.

The *stem* of a perfect tree is the maximal $s \in Seq\{0, 1\}$ such that for every $t \in p$, either $t \subseteq s$ or $s \subseteq t$. If p is a perfect tree and if $s \in p$, we denote by $p^{(s)}$ the perfect tree

$$(44.10) \qquad p^{(s)} = \{t \in p : t \subseteq s \text{ or } t \supseteq s\}$$

Assume that P is a set of perfect trees, partially ordered by $\subseteq$, such that if $p \in P$ and $s \in p$, then $p^{(s)} \in P$, and let G be an L-generic filter on P. Then there is a unique $a \in {}^\omega\{0, 1\}$ which is a branch in every $p \in G$; and conversely, $G = \{p \in P : a \text{ is a branch in } p\}$. Therefore $L[G] = L[a]$, and we call a P-generic over L.

A set $X \subseteq P$ is a *cover* (see Exercise 16.4) if for every $p \in P$, there is a stronger $q \in P$ such that $q \leq x$ for some $x \in X$. (Clearly, every dense set is a cover.) Note that $a \in {}^\omega\{0, 1\}$ is P-generic over L just in case for every constructible cover $X \subseteq P$, a is a branch in some $x \in X$.

Let $P \times P$ denote the product of P with itself; i.e., $(p', q') \leq (p, q)$ means $p' \subseteq p$ and $q' \subseteq q$. Similarly as for P, a generic filter G on $P \times P$ corresponds to a unique pair (a, b) such that for each $(p, q) \in G$, a is a branch in p and b is a branch in q. A pair (a, b) is $(P \times P)$-generic over L just in case for every constructible cover $X \subseteq P \times P$, there is $(x, y) \in X$ such that a is a branch in x and b is a branch in y.

When we used perfect sets as forcing conditions in Section 26, we used Lemma 26.1 (the fusion lemma) which enabled us to apply the operation of fusion to perfect sets. Let $T = \{T(s) : s \in Seq\{0, 1\}\}$ be a collection of perfect trees such that for every s,

(44.11) (a) $T(s)$ is a perfect tree whose stem has length $\geq$ length(s).
(b) $T(s^\frown 0) \subseteq T(s)$ and $T(s^\frown 1) \subseteq T(s)$.
(c) $T(s^\frown 0)$ and $T(s^\frown 1)$ have incompatible stems.

If T satisfies (44.11), we say that T is *fusionable* and we let

$$(44.12) \qquad \mathscr{F}(T) = \bigcap_{n=0}^{\infty} \bigcup_{s \in {}^n\{0, 1\}} T(s)$$

For each fusionable T, $p = \mathscr{F}(T)$ (the *fusion* of T) is a perfect tree; and for each s, if t is the stem of $p_s = T(s)$, then $p^{(t)}$ is stronger than both p and p_s.

We shall not use the set of all perfect trees as the notion of forcing; rather we shall construct a set P of perfect trees with the property that if $p \in P$ and $s \in p$, then $p^{(s)} \in P$. We shall construct P such that if a is P-generic over L, then a is the only P-generic set in $L[a]$, and such that $\{n \in N : a(n) = 1\}$ is (in $L[a]$) a Δ^1_3 subset of N.

We shall construct P as the union of countable sets

$$P_0 \subseteq P_1 \subseteq \cdots \subseteq P_\alpha \subseteq \cdots \qquad (\alpha < \omega_1)$$

of perfect trees. The construction uses the $\diamondsuit$-principle (cf. 22.15):

($\diamondsuit$) There exists an ω_1-sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ such that for every $X \subseteq \omega_1$, the set $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary.

By Exercise 22.17, there is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ that satisfies $\diamondsuit$ such that the function $\alpha \mapsto S_\alpha$ is Δ_1 over L_{ω_1}; let us fix such a sequence. Also, let us fix a Δ_1 over L_{ω_1} function τ that is a one-to-one mapping of L_{ω_1} onto ω_1.

We shall now construct the sequence $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_\alpha \subseteq \cdots$:

(44.13) $P_0 =$ the set of all $p_0^{(s)}$ where p_0 is the full binary tree, $p_0 = Seq\{0, 1\}$, and $s \in p_0$;

$P_\alpha = \bigcup_{\beta < \alpha} P_\beta$ if α is a limit ordinal.

$P_{\alpha+1} = P_\alpha \cup Q_{\alpha+1}$ where $Q_{\alpha+1}$ is a set of perfect trees defined as follows:

Let $P_\alpha = \{p_n^\alpha : n \in \omega\}$; and let us consider the $<_L$-least such enumeration. Let $\mathscr{X}_\alpha$ and $\mathscr{Y}_\alpha$ be the following countable collections of subsets of P_α and $P_\alpha \times P_\alpha$ respectively:

(44.14) $\mathscr{X}_\alpha$ contains:

(a) all Q_β, $\beta \leq \alpha$,

(b) all $X \subseteq P_\alpha$ such that $\tau[X] = S_\beta$ for some $\beta \leq \alpha$.

$\mathscr{Y}_\alpha$ contains:

(a) $Q_\beta \times Q_\beta$ for all $\beta \leq \alpha$;

(b) all $Y \subseteq P_\alpha \times P_\alpha$ such that $\tau[Y] = S_\beta$ for some $\beta \leq \alpha$.

There exists a family $\{T_n; n \in \omega\}$ of fusionable collections of elements of P_α such that:

(44.15) (a) $T_n(\emptyset) = p_n^\alpha$ for all n;

(b) for every $X \in \mathscr{X}_\alpha$, if X is a cover in P_α, then for every $n \in N$ and every $h \in N$, there is $k \geq h$ such that for each $s \in {}^k\{0, 1\}$, there exists $x \in X$ such that $T_n(s) \leq x$;

(c) for every $Y \in \mathscr{Y}_\alpha$, if Y is a cover in $P_\alpha \times P_\alpha$, then for every n, every m, and every h there is $k \geq h$ such that for each $s \in {}^k\{0, 1\}$ and each $t \in {}^k\{0, 1\}$; if either $n \neq m$ or $s \neq t$, then there exists $(x, y) \in Y$ such that $(T_n(s), T_m(t)) \leq (x, y)$.

A family $\{T_n : n \in \omega\}$ with properties (44.15a–c) is easily constructed because $\mathcal{X}_\alpha$ and $\mathcal{Y}_\alpha$ are countable. We denote $\{T_n^\alpha : n \in \omega\}$ the $<_L$-least such family, and let

$$Q_{\alpha+1} = \{p^{(s)} : p = \mathcal{F}(T_n^\alpha) \text{ for some } n, \text{ and } s \in p\} \tag{44.16}$$

We let $P = \bigcup_{\alpha<\omega_1} P_\alpha$. The following sequence of lemmas will show that if $a \in {}^\omega\{0, 1\}$ is P-generic over L, then in $L[a]$ the set $\{n : a(n) = 1\}$ is Δ_3^1.

Lemma 44.5. *For each α, $Q_{\alpha+1}$ is dense in $P_{\alpha+1}$, and $Q_{\alpha+1} \times Q_{\alpha+1}$ is dense in $P_{\alpha+1} \times P_{\alpha+1}$.*

Proof. It suffices to show that below each $p \in P_\alpha$ there is some $q \in Q_{\alpha+1}$. If $p \in P_\alpha$, then $p = p_n^\alpha$ for some n, and $\mathcal{F}(T_n) \subseteq T_n(\emptyset) = p$. ■

Lemma 44.6. *For each α, if $X \in \mathcal{X}_\alpha$ is a cover in P_α, then X is a cover in $P_{\alpha+1}$; if $Y \in \mathcal{Y}_\alpha$ is a cover in $P_\alpha \times P_\alpha$, then Y is a cover in $P_{\alpha+1} \times P_{\alpha+1}$. Consequently, if $X \in \mathcal{X}_\alpha$ is a cover in P_α (if $Y \in \mathcal{Y}_\alpha$ is a cover in $P_\alpha \times P_\alpha$), then X is a cover in P (then Y is a cover in $P \times P$).*

Proof. Let $X \in \mathcal{X}_\alpha$ be a cover in P_α; we have to show that for each $p^{(u)} \in Q_{\alpha+1}$ there is a stronger $q \in Q_{\alpha+1}$ such that $q \leq x$ for some $x \in X$. Let $p = \mathcal{F}(T_n)$ and let $u \in p$. Let $h = \text{length}(u)$. There is $k \geq h$ such that n and k satisfy (44.15b). There is $s \in {}^k\{0, 1\}$ such that $u \in T_n(s)$; let v be the stem of $T_n(s)$. Then $p^{(v)} \leq T_n(s)$ and $T_n(s) \leq x$ for some $x \in X$.

A similar argument, using (44.15c), shows that if $Y \in \mathcal{Y}_\alpha$ is a cover in $P_\alpha \times P_\alpha$, then Y is a cover in $P_{\alpha+1} \times P_{\alpha+1}$.

Since the sequences $\mathcal{X}_\alpha$, $\alpha < \omega_1$, and $\mathcal{Y}_\alpha$, $\alpha < \omega_1$, are increasing, it follows by induction that X is a cover in every P_β, $\beta < \omega_1$, and hence in P. Similarly for Y. ■

Lemma 44.7. *$P \times P$ satisfies the countable chain condition (and hence P also satisfies the countable chain condition).*

Proof. Here we use $\diamondsuit$. Let us assume that $Y \subset P \times P$ is a maximal incompatible set of conditions in $P \times P$ and that Y is uncountable. Since each P_α is countable, it is easy to see that the set of all $\alpha < \omega_1$ such that $\tau[Y \cap (P_\alpha \times P_\alpha)] = \tau[Y] \cap \omega_1$, is closed unbounded ($\tau$ is the one-to-one function of L_{ω_1} onto ω_1). Then it is not much more difficult to see that the set of all $\alpha < \omega_1$ such that $Y \cap (P_\alpha \times P_\alpha)$ is maximal incompatible in $P_\alpha \times P_\alpha$, is closed unbounded (compare this argument with the $\diamondsuit$-construction of a Suslin tree in L).

By $\diamondsuit$, there exists α such that $Y' = Y \cap (P_\alpha \times P_\alpha)$ is a cover in $P_\alpha \times P_\alpha$ and that $\tau[Y'] = S_\alpha$. Therefore $Y' \in \mathcal{Y}_\alpha$ and by Lemma 44.6, Y' is a cover in $P \times P$. It follows that $Y' = Y$. Thus Y is countable. ■

Lemma 44.8.

(a) *If $a \in {}^{\omega}\{0, 1\}$, then a is P-generic over L if and only if for every $\alpha < \omega_1$ there is $n \in N$ such that a is a branch in $\mathscr{F}(T_n^\alpha)$.*

(b) *If $a \neq b \in {}^{\omega}\{0, 1\}$, then (a, b) is $(P \times P)$-generic over L if and only if for every $\alpha < \omega_1$ there exist $n, m \in N$ such that a is a branch in $\mathscr{F}(T_n^\alpha)$ and b is a branch in $\mathscr{F}(T_m^\alpha)$.*

Proof. (a) Let a be P-generic and let $\alpha < \omega_1$. Since $Q_{\alpha+1}$ is dense (hence a cover) in $P_{\alpha+1}$ and because $Q_{\alpha+1} \in \mathscr{X}_{\alpha+1}$, $Q_{\alpha+1}$ is a cover in P. By the genericity of a, there exists $q \in Q_{\alpha+1}$ such that a is a branch in q. But $q = p^{(s)}$ where $p = \mathscr{F}(T_n^\alpha)$ for some n and $s \in p$, and clearly a is a branch in p.

Conversely, let us assume that the condition is satisfied. Let $X \subseteq P$ be maximal incompatible; we wish to show that a is a branch in some $x \in X$. By Lemma 44.7, X is countable, and (by the proof of Lemma 44.7) there is α such that $X \in \mathscr{X}_\alpha$. Let $n \in N$ be such that a is a branch in $\mathscr{F}(T_n^\alpha)$.

By (44.15b), there is $k \in N$ such that each $T_n(s)$, $s \in {}^k\{0, 1\}$, is stronger than some $x \in X$. Since a is a branch in $\mathscr{F}(T_n)$, it is clear that there is a unique $s \in {}^k\{0, 1\}$ such that a is a branch in $T_n(s)$. But if $x \in X$ is such that $T_n(s) \subseteq x$, then a is also a branch in x.

(b) The proof that the condition is necessary is analogous to (a). Thus let us assume that the condition is satisfied and let $Y \subseteq P \times P$ be maximal incompatible; we want to find $(x, y) \in Y$ such that a is a branch in x and b is a branch in y. Again, there is α such that $Y \in \mathscr{Y}_\alpha$. Let n and $m \in N$ be such that a is a branch in $\mathscr{F}(T_n^\alpha)$ and that b is a branch in $\mathscr{F}(T_m^\alpha)$.

Let $h \in N$ be such $\bar{a}(h) \neq \bar{b}(h)$; by (44.15c), there is $k \in N$ such that for each $s, t \in {}^k\{0, 1\}$, if either $n \neq m$ or $s \neq t$, then $(T_n(s), T_m(t)) \leq (x, y)$ for some $(x, y) \in Y$. There is a unique pair s, t such that a is a branch in $T_n(s)$ and b is a branch in $T_m(t)$: It is clear that either $n \neq m$ or $s \neq t$. Then a is a branch in x and b is a branch in y where (x, y) is some element of Y such that $(T_n(s), T_m(t)) \leq (x, y)$. ∎

Corollary 1. *If a and b are P-generic over L and $a \neq b$, then (a, b) is $(P \times P)$-generic over L.* ∎

Corollary 2. *If a is P-generic over L, then $L[a] \vDash$ a is the only P-generic x over L.*

Proof. If $a \neq b$ and if both a and b are P-generic over L, then by the product lemma 20.1, b is a P-generic over $L[a]$ and hence $b \notin L[a]$. ∎

Lemma 44.9. *The set $H = \{a : a$ is P-generic over $L\}$ is Π_1 over* HC.

Proof. It follows from the construction of P that the function $\alpha \mapsto \langle T_n^\alpha : n \in \omega\rangle$ is Δ_1 over $L_{\omega_1^L}$. Since $L_{\omega_1^L}$ is a Σ_1 set over HC, the function is Δ_1 over HC. By Lemma 44.8,

$$a \in H \quad \leftrightarrow \quad (\forall \alpha < \omega_1^L)\,(\exists n \in \omega)\,[a \text{ is a branch in } \mathscr{F}(T_n^\alpha)]$$

and hence H is Π_1 over HC. ∎

Corollary. *If a is P-generic over L and if $A = \{n \in \mathcal{N} : a(n) = 1\}$, then $L[a] \models A$ is a Δ^1_3 subset of $\mathcal{N}$.*

Proof. We have in $L[a]$:

$$(44.17) \qquad \begin{aligned} n \in A &\leftrightarrow (\exists a \in \mathcal{N})\,[a \in H \wedge a(n) = 1] \\ &\leftrightarrow (\forall a \in \mathcal{N})\,[a \in H \rightarrow a(n) = 1] \end{aligned}$$

By Lemma 41.1, since H is Π_1 over HC, it is a Π^1_2 subset of $\mathcal{N}$. Hence (44.17) shows that A is Δ^1_3. ∎

A Long Π^1_2 Well-Ordering

By Theorem 100, every Σ^1_2 well-ordering of a set of reals has order-type less than ω_2. The model given below (due to Harrington) shows that it is consistent with ZFC that there exists a Π^1_2 well-ordering of a set of reals whose order-type is as large as prescribed.

Example IV. A generic extension of L having a Π^1_2 well-ordering of a set of reals of order type $\geq \omega_2$.

A long Π^1_2 well-ordering is obtained as follows: Let θ be an arbitrary ordinal. First we extend the constructible model generically by adjoining at least θ Cohen reals: in $L[G]$, $2^{\aleph_0} \geq \theta$. In $\mathfrak{M} = L[G]$ (which has the same cardinals as L), we consider a well-ordering $A \subseteq \boldsymbol{R} \times \boldsymbol{R}$ whose order-type is θ. The following lemma then shows that there is a generic extension $\mathfrak{M}[H]$ of $\mathfrak{M}$, with the same cardinals, in which A is a Π^1_2 relation. Thus in $\mathfrak{M}[H]$, there is a Π^1_2 well-ordering of a set of reals of order type θ.

Lemma 44.10. *Let $\mathfrak{M}$ be a model of* ZFC *such that $\omega_1^{\mathfrak{M}} = \omega_1^L$. Let A* be (in $\mathfrak{M}$) a subset of ${}^\omega\omega$. There is a generic extension of $\mathfrak{M}$ that preserves all cardinals and in which A is a Π^1_2 set.

The generic extension will be obtained by a two-step iteration of forcing, and both steps will use a variant of the almost disjoint forcing introduced in Sections 23 (Lemma 23.10) and 26 ((26.1)-(26.3)).

Let $\{s_n : n \in \omega\}$ be an arithmetical enumeration of the set *Seq*. If $x \in \mathcal{N}$, let

$$(44.18) \qquad S(x) = \{n \in \omega : s_n \subset x\}$$

Each $S(x)$ is an infinite set; and if $x \neq y$, then $S(x) \cap S(y)$ is finite. Thus $\{S(x) : x \in \mathcal{N}\}$ is an almost disjoint family of subsets of ω.

With each $X \subseteq {}^\omega\omega$ we associate the following notion of forcing P_X: A condition is a pair $p = (u, E)$ where u is a finite subset of ω, and E is a finite subset of X. A condition (u_2, E_2) is stronger than (u_1, E_1) if

$$(44.19) \qquad E_1 \subseteq E_2,\ u_1 \subseteq u_2, \text{ and for every } x \in E_1,\ S(x) \cap u_2 \subseteq u_1.$$

Note that if (u, E) and (v, F) are compatible, then $(u \cup v, E \cup F)$ is a condition. Also, if (u, E) and (v, F) are incompatible, then $u \neq v$; and since there are only countably many u's, P_X satisfies the c.c.c.

Let G be a generic filter on P_X, and let

$$a = \{n \in \omega : n \in u \text{ for some } (u, E) \in G\} \tag{44.20}$$

It is easy to see that $G = \{(u, E) \in P_X\colon$ for every $x \in E$, $S(x) \cap a \subseteq u\}$ and hence $\mathfrak{M}[G] = \mathfrak{M}[a]$.

If $x \in X$, then the set of all (u, E) such that $x \in E$ is dense in P_X and hence there is $(u, E) \in G$ such that $x \in E$. Since $S(x) \cap v \subseteq u$ for every stronger condition $(v, F) \leq (u, E)$, it follows that $S(x) \cap a \subseteq u$ and hence $S(x) \cap a$ is finite.

If $x \in {}^{\omega}\omega - X$ (in the ground model), then for every condition (u, E) and every $k \in \omega$ there is $n \geq k$ such that $n \in S(x)$ and that $(u \cup \{n\}, E)$ is stronger than (u, E); namely, since $S(x) \cap S(y)$ is finite for every $y \in E$, we take $n \geq k$ such that $n \in S(x)$, and $n \notin S(y)$ for all $y \in E$. It follows that $S(x) \cap a$ is infinite.

Thus we have shown that for every $x \in {}^{\omega}\omega$ in the ground model,

$$x \in X \qquad \text{iff} \qquad S(x) \cap a \text{ is finite} \tag{44.21}$$

Let $\mathfrak{M}$ be a model of ZFC such that $\omega_1^{\mathfrak{M}} = \omega_1^L$, and let $A \in \mathfrak{M}$ be a subset of ${}^{\omega}\omega$. We consider first the notion of forcing P that is the product of $\aleph_1$ copies of P_A: A forcing condition is a sequence $P = \langle p_\alpha : \alpha < \omega_1\rangle$ such that:

$$\text{for each } \alpha,\ p_\alpha \in P_A, \text{ and } \{\alpha : p_\alpha \neq (\emptyset, \emptyset)\} \text{ is finite.}$$

A condition $\langle p_\alpha : \alpha < \omega_1\rangle$ is stronger than $\langle q_\alpha : \alpha < \omega_1\rangle$ if for each α, p_α is stronger than q_α.

It is easy to see that the product of a finite number of copies of P_A satisfies the countable chain condition; hence by Lemma 23.11, P satisfies the c.c.c. Let G be an $\mathfrak{M}$-generic filter on P and let for each $\alpha < \omega_1$, $G_\alpha = \{p_\alpha : p \in G\}$, and

$$a_\alpha = \{n \in \omega : n \in u \text{ for some } (u, E) \in G_\alpha\} \tag{44.22}$$

Clearly,

$$\mathfrak{M}[G] = \mathfrak{M}[\langle a_\alpha : \alpha < \omega_1\rangle]$$

Lemma 44.11. *$\mathfrak{M}[G]$ satisfies that for every $x \in {}^{\omega}\omega$,*

$$x \in A \qquad \text{iff} \qquad (\forall\alpha < \omega_1)\ S(x) \cap a_\alpha \text{ is finite} \tag{44.23}$$

Proof. If $x \in A$, then for every α, G_α is an $\mathfrak{M}$-generic filter on P_A, and hence by (44.21), $S(x) \cap a_\alpha$ is finite.

If $x \in {}^{\omega}\omega - A$, then, because P satisfies the c.c.c., there exists $\alpha < \omega_1$ such that $x \in \mathfrak{M}[\langle a_\beta : \beta < \alpha\rangle]$. Then (by the product lemma 20.1) G_α is an $\mathfrak{M}[\langle a_\beta : \beta < \alpha\rangle]$-generic filter on P_A, and since $x \notin A$, (44.21) implies that $S(x) \cap a_\alpha$ is infinite. ■

Let $\langle d_{\alpha n} : \alpha < \omega_1 \text{ and } n \in \omega\rangle$ be a constructible one-to-one sequence of constructible reals. Moreover, let us choose the sequence so that the function

$(\alpha, n) \mapsto d_{\alpha n}$ is Δ_1 over L_{ω_1}. In $\mathfrak{M}[G]$ let

$$B = \{d_{\alpha n} : n \in a_\alpha\} \tag{44.24}$$

and let us consider, in $\mathfrak{M}[G]$, the notion of forcing P_B.

Let H be an $\mathfrak{M}[G]$-generic filter on P_B and let

$$a = \{n \in \omega : n \in u \text{ for some } (u, E) \in H\} \tag{44.25}$$

For every $x \in {}^\omega\omega$ in $\mathfrak{M}[G]$, we have

$$x \in B \quad \text{iff} \quad S(x) \cap a \text{ is finite}$$

and therefore, for every $\alpha < \omega_1$ and every $n \in \omega$, we have

$$n \in a_\alpha \quad \text{iff} \quad S(d_{\alpha n}) \cap a \text{ is finite} \tag{44.26}$$

It follows from (44.26) that $\langle a_\alpha : \alpha < \omega_1 \rangle \in \mathfrak{M}[a]$ and therefore $\mathfrak{M}[G][H] = \mathfrak{M}[a]$.

Lemma 44.12. *$\mathfrak{M}[a]$ satisfies that for every $x \in {}^\omega\omega$,*

$$x \in A \quad \text{iff} \quad (\forall \alpha < \omega_1)\ S(x) \cap a_\alpha \text{ is finite} \tag{44.27}$$

Proof. If $x \in A$, then $S(x) \cap a_\alpha$ is finite for every α, by (44.23). Thus it suffices to show that if $x \in \mathfrak{M}[a]$ is such that $x \in {}^\omega\omega - A$, then $S(x) \cap a_\alpha$ is infinite for some α.

The model $\mathfrak{M}[a] = \mathfrak{M}[G][H]$ is obtained from $\mathfrak{M}$ by a two-step forcing, first using P and then P_B. We note that the model $\mathfrak{M}[a]$ can be obtained as a generic extension via the following notion of forcing:

For each $\alpha < \omega_1$, let $\boldsymbol{a}_\alpha$ be the canonical r.o. P-valued name for a_α. Let Q be the set of all pairs (p, q) such that:

(44.28) (i) $p \in P$;
(ii) $q = (u, E)$ where u is a finite subset of ω and E is a finite subset of $\omega_1 \times \omega$ such that for every $(\alpha, n) \in E$, $p \Vdash \check{n} \in \boldsymbol{a}_\alpha$.

A condition $(p', (u', E'))$ is stronger than $(p, (u, E))$ if $p' \leq p$ and for every $(\alpha, n) \in E$, $u' \cap S(d_{\alpha n}) \subseteq u$.

Since $B = \{d_{\alpha n} : n \in a_\alpha\}$, it is clear that the generic extension by Q corresponds to the two-step extension by P and P_B.

The following is a trivial but important observation: If (p, q) and $(\bar{p}, \bar{q})$ are conditions in Q and if $p = \langle (u_\alpha, E_\alpha) : \alpha < \omega_1 \rangle$, $\bar{p} = \langle (\bar{u}_\alpha, \bar{E}_\alpha) : \alpha < \omega_1 \rangle$, $q = (u, E)$ and $\bar{q} = (\bar{u}, \bar{E})$; and if (p, q) and $(\bar{p}, \bar{q})$ are compatible, then $(\langle (u_\alpha \cup \bar{u}_\alpha, E_\alpha \cup \bar{E}_\alpha) : \alpha < \omega_1 \rangle, (u \cup \bar{u}, E \cup \bar{E}))$ is a condition in Q; let us denote this condition $(p \vee \bar{p}, q \vee \bar{q})$.

Let $\boldsymbol{x}$ be an r.o. Q-valued name for an element of ${}^\omega\omega$. For each $n \in \omega$, let W_n be a maximal incompatible set of conditions in Q that force $\check{n} \in S(\boldsymbol{x})$. Since Q satisfies the c.c.c., each W_n is countable and therefore there exists $\alpha < \omega_1$ such that for every $n \in \omega$ and every $(p, q) \in W_n$, $p_\alpha = (\emptyset, \emptyset)$.

We shall show that for every $(p, q) \in Q$ that forces $\boldsymbol{x} \notin A$ and for every $k \in \omega$, there exists a stronger condition (p', q') and $n \geq k$ such that $(p', q') \Vdash \check{n} \in S(\boldsymbol{x}) \cap \boldsymbol{a}_\alpha$. Let $(p, q) \in Q$ be such that $(p, q) \Vdash \boldsymbol{x} \notin A$, and let $k \in \omega$.

Let $p_\alpha = (u_\alpha, E_\alpha)$ be the αth coordinate of p. If $y \in E_\alpha$, then $(p, q) \Vdash \boldsymbol{x} \neq \check{y}$, and hence $(p, q) \Vdash S(\boldsymbol{x}) \cap S(\check{y})$ is finite. Hence there is $n \geq k$ such that $n \notin S(y)$ for all $y \in E_\alpha$ and such that (p, q) does not force $\check{n} \notin S(\boldsymbol{x})$. Thus there exists $(\bar{p}, \bar{q}) \in W_n$ that is compatible with (p, q). By the important remark that we made earlier in the proof, $(p \vee \bar{p}, q \vee \bar{q})$ is a condition, and forces $\check{n} \in S(\boldsymbol{x})$. Since $(\bar{p}, \bar{q}) \in W_n$, the αth coordinate of p is $(\emptyset, \emptyset)$, and hence the αth coordinate of $(p \vee \bar{p}, q \vee \bar{q})$ is $p_\alpha = (u_\alpha, E_\alpha)$.

Let (p', q') be the condition obtained from $(p \vee \bar{p}, q \vee \bar{q})$ by changing the αth coordinate from $p_\alpha = (u_\alpha, E_\alpha)$ to $p'_\alpha = (u_\alpha \cup \{n\}, E_\alpha)$. Clearly, p'_α is a stronger condition than p_α (in P_A) and hence $(p', q') \leq (p \vee \bar{p}, q \vee \bar{q})$. Moreover, $(p', q') \Vdash \check{n} \in \boldsymbol{a}_\alpha$, and therefore $(p', q') \Vdash \check{n} \in S(\boldsymbol{x}) \cap \boldsymbol{a}_\alpha$.

Thus we have shown that if $x \in {}^\omega\omega - A$, then $S(x) \cap a_\alpha$ is infinite. ■

Lemma 44.13. *In* $\mathfrak{M}[a]$ *the set* A *is* $\Pi^1_2(a)$.

Proof. First we recall that the function $(\alpha, n) \mapsto d_{\alpha n}$ is Δ_1 over HC. Then we use (44.26) to show that the function $\alpha \mapsto a_\alpha$ is Δ_1 in the parameter a over HC:

$$y = a_\alpha \quad \leftrightarrow \quad (\forall n \in \omega)\,[n \in y \quad \leftrightarrow \quad S(d_{\alpha n}) \cap a \text{ is finite}]$$

Hence by (44.27), A is Π_1 in a over HC:

$$x \in A \quad \leftrightarrow \quad \forall \alpha (S(x) \cap a_\alpha \text{ is finite})$$

By Lemma 41.1, A is a $\Pi^1_2(a)$ subset of ${}^\omega\omega$. ■

This completes the proof of Lemma 44.10.

HISTORICAL NOTES AND GUIDE TO THE BIBLIOGRAPHY

CHAPTER 1

Set theory was invented by Georg Cantor. The first attempt to consider infinite sets is attributed to Bolzano (who also introduced the term *Menge*). It was however Cantor who realized the significance of one-to-one functions between sets and introduced the notion of cardinality of a set. Cantor originated the theory of cardinal and ordinal numbers as well as the investigations of the topology of the real line. Much of the development in the first four sections follows Cantor's work. The main reference to Cantor's work is his collected works, Cantor [1932]. Another source of references to the early research in set theory is Hausdorff's book [1914].

Cantor started his investigations in [1874], where he proved that the set of all real numbers is uncountable, while the set of all algebraic reals is countable. In [1878] he gave the first formulation of the celebrated continuum hypothesis.

Section 1

The axioms for set theory (except Replacement and Regularity) are due to Zermelo [1908]. Prior to Zermelo, Frege attempted to axiomatize set theory, using the false comprehension schema; Zermelo formulated his axioms following the discovery of several paradoxes by Burali-Forti, Cantor, Russell, and Zermelo. The Replacement schema is due to Fraenkel [1922a] and Skolem (see [1970, pp. 137-152]).

Exercises 1.12 and 1.13: Tarski [1925a].

Section 2

The theory of well-ordered sets was developed by Cantor, who also introduced transfinite induction. The idea of identifying an ordinal number with the set of smaller ordinals is due to Zermelo and von Neumann.

Section 3

Cardinal numbers and alephs were introduced by Cantor. The proof of the Cantor–Bernstein theorem is Bernstein's; see Borel [1898, p. 103]. The first proof of $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ appeared in Hessenberg [1906, p. 593]. Regularity of cardinals was investigated by Hausdorff, who also raised the question of existence of regular limit cardinals. D-finiteness was formulated by Dedekind.

Section 4

Theorems 10 and 11 are due to Cantor. The construction of real numbers by completion of the rationals is due to Dedekind [1872].

Theorem 13: Baire [1899].

Theorem 14: Cantor, Bendixson [1883].

Exercises 4.3, 4.4, 4.5a: Cantor.

Suslin's problem: Suslin [1920].

Section 5

The axiom of choice was formulated by Zermelo, who used it to prove the well-ordering theorem in [1904]. The present form of the Kuratowski-Zorn lemma is as in Zorn [1935]; for a related principle, see Kuratowski [1922]. (Hausdorff in [1914, pp. 140–141] proved that every partially ordered set has a maximal linearly ordered subset.) The principle of dependent choices was formulated by Bernays in [1937–1954, III].

Exercise 5.1: Felix Bernstein.

Section 6

König's theorem appeared in J. König [1905]. Theorem 18 was found independently by Bukovský [1965] and Hechler. The discovery that cardinal exponentiation is determined by the gimel function (Theorem 20) was made by Bukovský; cf. [1965]. The inductive computation of $\aleph_\alpha^{\aleph_\beta}$ is as in Jech [1973b]; that paper also investigates properties of the gimel function.

Exercise 6.6: L. Patai.

Hausdorff's formula (6.18): Hausdorff [1904].

Exercise 6.8: Tarski [1925b].

Exercises 6.15 and 6.19: Jech [1973b].

Inaccessible cardinals were introduced in the paper by Sierpiński and Tarski [1930]; see Tarski [1938] for more details.

Section 7

The notion of filter is, according to Kuratowski's book [1966], due to H. Cartan. The ultrafilter theorem was first proved by Tarski in [1930]. Fodor's theorem appeared in Fodor [1956].

Reduced products were first investigated by Łoś in [1955]; see Section 28. Lemma 7.6 is a special case of Theorem 85 of Solovay proved in Section 35.

Exercise 7.12: Friedman [1974]. (For related problems, see Prikry and Solovay [1975]).

Mahlo cardinals are named after P. Mahlo, who in 1911–1913 investigated what we call weakly Mahlo cardinals.

Section 8

Theorem 23 is due to Silver. Silver's proof [1974] combines forcing with the ultrapower method; see Section 35. An elementary proof was subsequently found by Baumgartner and Prikry [1976, 1977] and Jensen. The combinatorial argument of Baumgartner and Prikry was extended by Galvin and Hajnal in [1975] to obtain Theorem 24. The proof of Lemma 8.7 is a modification of Galvin-Hajnal's method.

Lemma 8.5: Erdös, Hajnal, and Milner [1968].

Models where the singular cardinals hypothesis fails have been constructed by Magidor, cf. [1977b] and [1977c].

Section 9

The axiom of regularity was introduced by von Neumann in [1925], although a similar principle had been considered previously by Skolem (see [1970, pp. 137–152]). The transitive collapse is defined in Mostowski [1949]; see Theorem 28 in Section 11. Induction on well-founded relations (Theorem 25′) was formulated by Montague in [1955].

Appendix

The axiomatic system BG was introduced by Bernays in [1937–1954, I] and used by Gödel in his consistency proof [1940]. Shoenfield's result was published in [1954].

For more references on the history of axioms of set theory consult Fraenkel *et al.* [1973].

CHAPTER 2

The main results, namely consistency of the axiom of choice and of the generalized continuum hypothesis, are due to Kurt Gödel, as is the concept of constructible sets. The results were announced in [1938], and an outline of proof appeared in [1939]. Gödel's monograph [1940] contains a detailed construction of L and the proof that L satisfies AC and GCH.

Section 10

For concepts of model theory, the history of the subject and for model-theoretical terminology, I refer the reader to Chang and Keisler's book [1973]. The impossibility of a consistency proof of existence of inaccessible cardinals follows from Gödel's theorem [1931]. An argument that more or less establishes the consistency of the axiom of regularity appeared in Skolem's work in 1923 (see [1970, pp. 137–152]).

Exercise 10.6: Montague and Vaught [1959].

Section 11

The investigation of transitive models of set theory was of course motivated by Gödel's construction of the model L. The first systematic study of transitive models was done by Shepherdson in [1951–1953]. Bernays in [1937–1954, I], employed a finite number of operations on classes to give a finite axiomatization of BG (see Exercise 11.9). Theorem 31 is explicitly stated by Hajnal in [1956].

The collapsing theorem 28 appeared in Mostowski [1949]. The reflection principle was introduced by Montague; see [1961] and Lévy [1960a].

Exercise 11.2: Galvin.

Section 12

In [1939] Gödel defined constructible sets using $L_{\alpha+1} = \{\text{all subsets of } L_\alpha \text{ definable over } L_\alpha\}$; in [1940] he used finitely many operations (and worked in the system BG).

Exercise 12.5: The J_α hierarchy was used by Jensen in [1972].

Section 13

Both Theorems 33 and 34 are due to Gödel.

Section 14

The Σ_n hierarchy was introduced by Lévy in [1965b]. Lévy proved in [1965b] Theorem 36 (the absoluteness lemma), using the principle of dependent choices and Shoenfield's absoluteness theorem (Theorem 98 in Section 41)—see Exercise 41.5; the present proof is due to Barwise and Fisher [1970].

Another result of Lévy [1965] is that the truth predicate $\models_{n+1}$ is Σ_{n+1} (Exercise 14.20); for Tarski's theorem, see, e.g., Tarski [1939].

Karp's paper [1967] investigates Σ_1 relations and gives a detailed computation verifying that constructibility is Σ_1. Primitive recursive functions appear in Jensen and Karp [1971]. The characterization of the sets L_α as transitive models of a single sentence σ is a result of Boolos [1970].

Exercises 14.23 and 14.24: Jensen [1972].

Section 15

Relative constructibility was investigated by Hajnal [1956], Shoenfield [1959] and most generally by Lévy [1957] and [1960b].

Lemma 15.1: Vopěnka and Balcar [1967].

Lemma 15.2 and Exercise 15.10: Lévy [1957] and [1960b].

Exercise 15.16: see Theorem 98.

The concept of ordinal definability was suggested by Gödel in his talk [1946]; the theory was developed independently by Myhill and Scott in [1971] and by Vopěnka, Balcar, and Hájek in [1968]. The former authors use the definition from Exercise 15.18, the latter use (15.16).

CHAPTER 3

The method of forcing was invented by Paul Cohen who used it to prove the independence of the continuum hypothesis and of the axiom of choice (see [1963–1964] and the book [1966]). The present formulation of Cohen's method using Boolean-valued models has been developed by Scott, Solovay, and Vopěnka.

Section 16

The two fundamental theorems of the method of forcing, the forcing theorem and the generic model theorem, are due to Cohen. (To be accurate, Cohen's original method was formulated for particular examples of a notion of forcing, and under the assumption that $\mathfrak{M}$ is a countable transitive model of ZFC. The definition (16.3) of an $\mathfrak{M}$-generic filter on a partially ordered set is due to Solovay.) The forcing conditions defined in (16.4) are the ones used by Cohen in his proof of independence of the continuum hypothesis; the "typical example," a collapsing mapping of ω onto ω_1 is also due to Cohen.

Section 17

Most of the facts about Boolean algebras can be found in Sikorski's book [1964], which also contains an extensive bibliography. The representation theorem for Boolean algebras (Theorem 41) is due to Stone [1936]. The important lemma 17.6 on the chain condition was proved by Erdös and Tarski in [1943].

For a more detailed discussion of the role of the prime ideal theorem, see my book on the axiom of choice [1973c]; for the model-theoretical concepts involved, consult Chang and Keisler [1973]. The independence of the prime ideal theorem from AC is due to Feferman [1965] and is proved in Section 21 (Example V); the independence of the axiom of choice from the prime ideal theorem is a result of Halpern and Lévy [1971].

Lemma 17.5: Rasiowa and Sikorski [1950].

Exercise 17.14: Makinson [1969].

Exercise 17.21: Pierce [1958].

Section 18

The Boolean-valued version of Cohen's method has been formulated by Scott, Solovay, and Vopěnka. Following an observation of Solovay that the forcing relation can be viewed as assigning Boolean values to formulas, Scott formulated his version of Boolean-valued models in [1967]. Vopěnka developed a theory of Cohen's method of forcing, using open sets in a topological space as forcing conditions (in [1964], [1965–1966], [1967a] and Vopěnka-Hájek [1967]), eventually arriving at the Boolean-valued version of forcing more or less identical to Scott–Solovay's version (Vopěnka [1967b]). Vopěnka's paper [1964] contains the observation that Cohen's construction can be modified so that one can construct a (nontransitive two-valued) model directly, without assuming existence of a countable model or a generic filter (see Lemma 18.1). Shoenfield's article [1971] uses a slightly different approach to forcing, proving the main theorems without using Boolean-valued models.

Section 19

The two independence results, independence of the continuum hypothesis (Theorem 44) and of the axiom of choice (Theorem 45) are due to Cohen; cf. [1963–1964] and [1966]. Our method of proof of Theorem 45 (using $\mathfrak{N} = \mathrm{HOD}(A)$) is basically that of Cohen (who used a model $\mathfrak{M}(A)$—adjoining an infinite set of mutually generic reals to a countable transitive model of ZFC).

Shortly after Cohen's discoveries, Solovay (in [1963]) noticed that Cohen's construction of a model for $2^{\aleph_0} = \aleph_2$ can be generalized so that for a regular κ one obtains a model with $2^\kappa = \lambda$ (Lemma 19.7); see also Vopěnka [1965–1966]. (For the general result, see Easton's theorem in Section 20.) The notion of forcing that collapses a given cardinal (lemma 19.9) is due to Cohen (see [1966]).

The relation between the chain condition and preservation of cardinals (Lemma 19.2) is basically due to Cohen; the observation that a λ-closed notion of forcing does not produce new subsets of λ (Lemmas 19.5 and 19.6) is due to Solovay.

The combinatorial lemmas used in computing the chain condition (Lemmas 19.3 and 19.8) are borrowed from point set topology: Marczewski [1941] and Engelking and Karłowicz [1965].

Exercises 19.3–19.7 contain facts noticed by many people including this author.

Section 20

Easton's theorem (Theorem 46) was published in [1970]. The generalization of Cohen's method allowing a class of forcing conditions is due to Easton. The model in Lemma 20.4 (the Lévy collapse) was constructed by Lévy; cf [1970].

The product lemma (Lemma 20.1) is due to Solovay [1970]. Lemma 20.3 follows from the aforementioned combinatorial result of Engelking and Karłowicz.

The chain condition of a product of two c.c.c. partial orderings: Using a construction of Galvin, Fleissner showed that it is consistent that $2^{\aleph_0}$ is large and that there are c.c.c. P and Q such that $\mathrm{sat}(P \times Q) = (2^{\aleph_0})^+$.

Exercise 20.8: Jensen [1965].

The concluding remark on the axiom of choice in BG uses an observation made by various people, e.g., Felgner [1971].

Section 21

The idea of using symmetry arguments to construct models in which the axiom of choice fails goes back to Fraenkel [1922b]; the two examples of models of ZFA (an infinite set of atoms without a countable subset, and a countable set of pairs that has no choice function) are basically due to him. Further examples of permutation models were given by Mostowski who (in [1939]) developed a theory of such models. The present definition using normal filters (21.5) was given by Specker [1957].

Cohen incorporated the symmetry arguments into his method and constructed the models that are presented in Examples I and II. The formulation of Cohen's method in terms of symmetric submodels of Boolean-valued models is due to Scott (unpublished) and Jech [1971a]; the latter's version was a reformulation of a topological version of Vopěnka and Hájek [1965].

The embedding theorem (Theorem 47) is due to Jech and Sochor [1966a, b]. Numerous applications of the theorem are given in the second paper [1966b]. The method has been generalized by Pincus in [1971] and in [1972], extending further the analogy between permutation models of ZFA and symmetric models of ZF.

Lévy showed that in Cohen's model in Example I every set can be linearly ordered; consequently, Halpern and Lévy [1971] proved that the prime ideal theorem holds in the model. Example III (a prescribed partially ordered set of cardinals) is due to Jech [1966]; see also Takahashi [1968]. Example IV (singularity of $\aleph_1$) is due to Feferman and Lévy [1963]. Example V (independence of the prime ideal theorem) is due to Feferman [1965]; the same model establishes the independence of the principle of dependent choices from the axiom of choice; see Feferman [1964]. Jech's paper [1971a] uses a similar construction to obtain two models of ZF that have the same sets of ordinal numbers. A. Blass constructed in [1977] a model, similar to Feferman's model, in which every ultrafilter is principal.

CHAPTER 4

Section 22

Suslin's problem was formulated by Suslin in [1920]. Tennenbaum [1968] and Jech [1967] discovered models of set theory in which a Suslin line exists; Solovay and Tennenbaum [1971] proved that existence of a Suslin line is not provable in ZFC. Subsequently, Jensen proved that a Suslin line exists in the constructible universe (cf. [1968, 1972]).

The present proof of Theorem 48 is as in Jech [1967] (countable forcing conditions); Tennenbaum's proof (finite conditions) is presented in Exercises 22.9 and 22.10.

Suslin's problem is independent of the continuum hypothesis: Tennenbaum's construction yields a Suslin line, with or without the continuum hypothesis (see the remark following Exercise 22.10); the continuum hypothesis is false in Solovay-Tennenbaum's model (cf. Section 23), but an alternative consistency proof of Jensen gives a model where there are no Suslin lines but the continuum hypothesis holds.

For more on Suslin's problem and trees in general, consult Jech [1971b] and Devlin and Johnsbråten [1974]. The first systematic study of trees appears in Kurepa [1935]; Kurepa's paper states that Suslin's problem is equivalent to the problem of existence of a Suslin tree. This equivalence (Lemma 22.1) was later rediscovered by Miller [1943] and Sierpinski [1948]. The first construction of an Aronszajn tree was given by Aronszajn and appears in Kurepa [1935]; see also Specker [1951].

The Boolean algebraic reformulation of Suslin's problem (Lemma 22.2 and Exercise 22.3) is due to Maharam [1948]. The construction of rigid and homogeneous Suslin trees in *L* (Exercises 22.15 and 22.16) is due to Jensen [1969]. Automorphisms of Suslin trees in general are investigated in Jech [1972a].

Exercise 22.8: Fukson [1971].

The Δ-lemma was formulated and proved by Šanin in [1946].

Exercise 22.1: D. König [1927].

Exercise 22.4: Kurepa [1950].

Section 23

Following Solovay and Tennenbaum's construction (by iterated forcing) of a model in which there are no Suslin trees (Theorem 50), Martin formulated an axiom ($MA_{\aleph_1}$) which implies that there are no Suslin trees (Lemma 23.1) and whose consistency was obtained by an easy generalization of Solovay-Tennenbaum's construction. The consistency proof of $MA + 2^{\aleph_0} > \aleph_1$ (Theorem 51, and also Exercise 23.9) appears in Solovay and Tennenbaum [1971].

Martin's axiom is investigated in detail in the paper [1970] of Martin and Solovay. The paper contains various equivalent formulations of Martin's axiom and numerous applications (including

Theorem 52); for applications of MA in descriptive set theory, see Section 44. In recent years, MA has been very popular with point set topologists; cf. Juhász's book [1971].

Iterated forcing, using direct limits of complete Boolean algebras, is developed in Solovay and Tennenbaum's paper [1971]; the paper contains, e.g., Lemmas 23.6 and 23.7 on the c.c.c.

The proof of Lemma 23.10 (using the notion of forcing (23.32)) is a variation of Solovay's method of "almost disjoint forcing." This method was introduced in Martin and Solovay [1970] and Jensen and Solovay [1970]; see also Sections 26 and 44.

Marczewski proved in [1947] that the strong c.c.c. is preserved under (topological) products (of any number of copies). Lemma 23.13 ($MA_{\aleph_1}$ implies that every c.c.c. space satisfies the strong c.c.c.) is due to Kunen, Rowbottom, and Solovay; Theorem 53 is a consequence of this and of Marczewski's result.

Exercise 23.7: Krivine [1969].

Section 24

Baumgartner, Malitz, and Reinhardt [1970] proved that $MA_{\aleph_1}$ implies that every Aronszajn tree can be embedded in the rationals (Theorem 54). Special Aronszajn trees have applications in model theory (this fact is due to Rowbottom and Silver) and are investigated in Mitchell's paper [1972]. The equivalence in Lemma 24.1 is due to Galvin.

Exercises 24.2 and 24.3: Baumgartner [1970].

Exercise 24.4: Jech [1972b].

Construction of a rigid Aronszajn tree is an unpublished result of Baumgartner.

Existence of a Kurepa tree was a problem formulated (in terms of a family of sets, see Lemma 24.6) by Kurepa in [1935]. A model in which there is a Kurepa tree (using the notion of forcing (24.5)) was constructed by D. H. Stewart (unpublished). The model (24.7) in which there is no Kurepa tree is due to Silver [1971b] (for a modification of Silver's model such that $2^{\aleph_0} > \aleph_1$, see Devlin [1975]). The construction of a Kurepa tree in L is due to Solovay; as Jensen's construction of a Suslin tree in L, it is an example of application of the fine structure of L, a method extensively developed by Jensen in [1972]. The observation that Kurepa trees exist unless $\aleph_2$ is inaccessible in L is due to Silver. For more on Kurepa trees, consult Jech [1971b] and Devlin's book [1973].

Lemma 24.4: Erdös and Rado [1960] and Michael [1962].

The notion of almost disjoint sets is due to Sierpiński [1928a] (Lemma 24.7).

Exercise 24.6: Tarski [1928].

Exercise 24.7: Erdös and Hajnal [1971].

The model where $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and no tree of size $\aleph_1$ has $\aleph_2$ branches of length ω_1 is due to Mitchell [1972]; so is the observation that in the model obtained by adjoining $\aleph_{\omega_1}$ Cohen reals there is no dense linearly ordered set of size $2^{\aleph_1}$ with a dense subset of size $\aleph_1$ (the equivalence of the dense linear ordering problem with the problem of existence of a tree with a large number of branches is due to Baumgartner).

Jensen's paper [1972] introduces the fine structure of L; the combinatorial principles $\diamondsuit$, $\square$ and (24.17) are proved (in L) in that paper. Silver's principle (W) replaces (in several proofs) another, quite complicated, combinatorial principle of Jensen (existence of a morass—see Devlin [1973]) true in L.

Exercises 24.10 and 24.11: Baumgartner, Hajnal, and Máté [1973]. The implication $\square \Rightarrow (24.17)$ is due to Magidor. By Jensen [1972], $\square$ holds unless $\aleph_2$ is a Mahlo cardinal in L; Baumgartner in [1976a] uses a weakly compact cardinal to construct a model in which (24.17) (and hence $\square$ too) fails.

Theorem 56 is due to Pospíšil [1937]; the present proof uses independent sets (Lemma 24.8); cf. Fichtenholz and Kantorovich [1934] ($\kappa = \omega$) and Hausdorff [1936].

W. Rudin [1956] proved that p-points exist if $2^{\aleph_0} = \aleph_1$ (Theorem 57a); a recent result of Shelah shows that existence of p-points is unprovable in ZFC. Galvin showed that $2^{\aleph_0} = \aleph_1$ implies existence of Ramsey ultrafilters; construction of p-points and Ramsey ultrafilters (and

p-points that are not Ramsey) under the assumption of Martin's axiom is due to Booth [1970]. Our proof of Theorem 57 follows Ketonen [1976a].

Exercise 24.15: Booth [1970].

Scales were investigated extensively by Hechler [1974]. Hechler introduced the notion of forcing used in the proof of Lemma 24.17 (the lemma itself was explicitly formulated by Kunen). Hechler, among others, showed that if cf $\kappa > \omega$, then there is a generic extension in which $2^{\aleph_0} > \kappa$ and a κ-scale exists.

Section 25

The relationship between automorphism properties of complete Boolean algebras and definability in generic extensions had been studied by several authors. Lévy in [1965a] showed consistency of $L = \text{HOD} \neq V$ virtually as in Exercises 25.8 and 25.9. McAloon in [1971a] constructed a model in which $L \neq \text{HOD} = V$ and remarked that according to Solovay the algebra used in the construction must be rigid. (A direct construction of a rigid complete Boolean algebra is more recent and is due to Shelah; cf [1974].) Theorem 59 which gives the exact description of definable sets in generic extensions is due to Vopěnka and appears in the book [1972] by Vopěnka and Hájek.

The relationship between distributivity laws in B and the size of "new" sets in $\mathfrak{M}[G]$ (Theorem 58b) had been known already to Scott, Solovay, and Vopěnka; for instance, cf. Vopěnka [1965–1966]. Exercises 25.1 and 25.2 (reduced to a triviality by the forcing argument) had been known before forcing; see Sikorski [1964].

McAloon [1971a] constructed models where either $L \neq \text{HOD} = V$ or $L \neq \text{HOD} \neq V$ and noted that it is also consistent that $\text{HOD}^{\text{HOD}} \neq \text{HOD}$. McAloon's paper [1974] has an example of a strictly descending sequence of models $\mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \cdots \supset \mathfrak{M}_n \supset \cdots$, $n < \omega$, such that $\mathfrak{M}_{n+1} = \text{HOD}^{\mathfrak{M}_n}$; Jech's paper [1974b] has an example of a transfinite sequence $\mathfrak{M}_0 \supset \cdots \supset \mathfrak{M}_\alpha \supset \cdots$, of arbitrary length such that $\mathfrak{M}_{\alpha+1} = \text{HOD}^{\mathfrak{M}_\alpha}$, with intersections at limit steps. The sequence $\text{HOD}^0 = V$, $\text{HOD}^1 = \text{HOD}$, $\text{HOD}^2 = (\text{HOD})^{\text{HOD}}, \ldots, \text{HOD}^n, \ldots$ is not necessarily expressible in ZF (by an unpublished result of Harrington); but if $V = L[X]$, then the sequence HOD^α is expressible (Grigorieff [1975]). S. Roguski announced recently that for any transitive model $\mathfrak{M}$ of ZFC there exists a generic extension $\mathfrak{N}$ (via a proper class of conditions) such that $\mathfrak{M} = \text{HOD}^{\mathfrak{N}}$.

The construction of a rigid complete Boolean algebra in L is a modification of Jensen's construction of a rigid Suslin tree in L [1969]. Simple complete Boolean algebras were investigated by McAloon [1971b] and Jech [1974a]. Lemmas 25.8 and 25.9 are due to McAloon [1971b]. Existence of simple complete Boolean algebras (in ZFC) is an open problem. Jensen's Example III in Section 44 (a minimal Δ^1_3 real) (cf. [1970]) gives, by McAloon, a simple complete Boolean algebra in L. The construction of a simple complete Boolean algebra using $\diamondsuit$ mentioned in the text appears in Jech [1974a]; it has subsequently been used in Harrington and Jech [1976] to construct a generic extension of L that admits a Σ_1 well-ordering.

Theorem 60 is an unpublished result of Solovay; prior to Solovay's proof, this author had an alternative proof using the continuum hypothesis. Our paper [1973a] shows that it is consistent that a Suslin algebra exists of arbitrarily prescribed size κ. Subsequently, Laver showed (using $\diamondsuit$ and W) that in L, there is a Suslin algebra of size $\aleph_2$.

Theorem 61 is due to Gaifman [1964a] and Hales [1964]; the present proof using Lemma 25.10 is Solovay's; see [1966]. Theorem 62 is due to Kripke [1967]. In [1970] Solovay proved the factor lemma for Lévy collapse (Exercise 25.11) as the main technical lemma for his consistency proof on Lebesgue measure (see Section 42). Jensen used the factor lemma to prove Theorem 63. Subsequently, Krivine proved the more general factor lemma (Exercise 25.10) in [1969]. Lemmas 25.13 and 25.14: Jech [1972c] and [1971c].

Section 26

The method of almost disjoint forcing appears in Martin and Solovay [1970] and Jensen and Solovay [1970]. Theorem 64 is due to Sacks [1971]. Example II (changing the cofinality of ω_2 to ω)

is due to Namba [1970]; see also Buk'ovský [1976]. Theorem 65 is due to Vopěnka and appears in the book [1972] of Vopěnka and Hájek.

A recent result of Z. Adamowicz provides various examples of well-founded initial segments of degrees of constructibility; Jech's paper [1974b] contains an example of an initial segment of L-degrees of order type ω^*. Hájek in [1974] constructs two Cohen generic reals x, y without greatest lower bound: $L[x] \cap L[y]$ is not a model of ZFC.

Lemma 26.6: Jech [1974b].

Exercise 26.5: Hájek [1974].

Exercise 26.6: Grigorieff [1975].

CHAPTER 5

The study of measurable cardinals originated around 1930 with the work of Banach, Kuratowski, Tarski, and Ulam. Ulam showed in [1930] that measurable cardinals are large, that the least measurable cardinal is at least as large as the least inaccessible cardinal. Around 1960 Keisler and Tarski introduced the method of ultraproducts in the study of measurable cardinals, and it was established that the least measurable cardinal is greater than the least inaccessible cardinal. Scott [1961] used the method of ultrapowers to prove that existence of measurable cardinals contradicts the axiom of constructibility. In the 1960s two methods have been successfully applied to measurable cardinals: Rowbottom and Silver initiated applications of infinitary combinatorics (developed by Erdös and his collaborators), and Gaifman and Kunen introduced the method of iterated ultrapowers.

Section 27

The main result on measurable and real-valued measurable cardinals (Theorem 66) is due to Ulam [1930]. The fact that a measurable cardinal is inaccessible (Lemma 27.2) was discovered by Ulam and Tarski (cf. Ulam [1930]). Prior to Ulam, Banach and Kuratowski proved in [1929] that if $2^{\aleph_0} = \aleph_1$, then there is no measure on the continuum; their proof is as in Lemma 27.9. Real-valued measurable cardinals were introduced by Banach in [1930].

The notion of a κ-saturated ideal was introduced and studied by Tarski in [1939–1945, II].

The example of a Lebesgue nonmeasurable set in Exercise 27.1 was given by Vitali in [1905]. For further information on measure theory, we refer the reader to Halmos' book [1950].

Section 28

The fundamental theorem on ultraproducts (Lemma 28.1) is due to Łoś [1955]. Keisler and Tarski [1964] introduced ultrapowers in the study of large cardinals. Theorem 67 is due to Scott [1961], and Theorem 68 is due to Kunen [1971a].

In [1964a] Hanf studied compactness of infinitary languages (see Sections 32 and 33); his work led to the systematic study of Keisler and Tarski. Hanf proved that the least inaccessible cardinal is not measurable (in fact not weakly compact); Erdös and Hajnal then pointed out (cf. [1962]) that the same result can be proved using infinitary combinatorics. Keisler and Tarski introduced the Mahlo operation (28.23) and showed that the least measurable cardinal is much greater than, e.g., the least Mahlo cardinal, etc. (Lemma 28.12).

In [1962] Vopěnka gave another proof of Scott's theorem: Observing that $|(2^{\kappa})^{Ult}| = |j(\kappa)|$, it follows that the cardinals in the model Ult are not absolute and hence $V \neq L$.

Lemma 28.8: Erdös and Hajnal [1966].

Exercise 28.12: For a related result see Ketonen [1973].

Exercise 28.14: Solovay.

Lemma 28.13 follows from Lemma 28.15, which appears in the abstract [1961] of Hanf and Scott. The result in Exercise 28.18 was announced by Vopěnka in [1970]; more applications of a measurable cardinal to the singular cardinal problem (including Exercise 28.20) are in Jech [1973b].

Section 29

Lemma 29.1 (Ramsey's theorem) is due to Ramsey [1930]. The theory of partition relations has been developed by Erdös, who has written a number of papers on the subject, some coauthored by Rado, Hajnal, and others. The arrow notation is introduced in Erdös and Rado [1956]. Other major comprehensive articles on partition relations are Erdös, Hajnal, and Rado [1965] and Erdös and Hajnal [1971].

Theorem 69 appears in Erdös and Rado [1956]. The present proof incorporates ideas from a model-theoretical proof of the Erdös–Rado theorem due to Simpson [1970], as published in Chang and Keisler's book [1973].

Weakly compact cardinals are investigated in the paper [1961] of Erdös and Tarski. The investigations started in that paper were subsequently completed by others; see the notes for Section 32. For applications of partition relations in model theory, we refer the reader to Chang and Keisler's book [1973].

Lemma 29.3: Sierpiński [1933].

Exercise 29.2: Kurepa [1962].

Exercises 29.3 and 29.4: Erdös and Rado [1952].

The partition relation $\aleph_1 \to (\alpha)^2$ for $\alpha < \omega_1$ was proved by Baumgartner and Hajnal [1973]; the consistency $\aleph_2 \not\to (\omega_1 + \omega)^2$ by forcing is due to Hajnal, and in L due to Rebholz [1974]. The partition relation $\omega \to (\omega)^\omega$ is consistent with ZF (without AC), by Mathias [1968] and [1977] (using an inaccessible cardinal). For further results on the partition relation $\omega \to (\omega)^\omega$ for Borel and analytic partitions, see Galvin and Prikry [1973], Silver [1970b], and Ellentuck [1974].

That every measurable cardinal is a Ramsey cardinal had been proved by Erdös and Hajnal (see their paper [1962]). The stronger version of that result (Theorem 70) is due to Rowbottom [1971], as are Theorem 71 and Lemma 29.10.

No $\aleph_n$ is a Jónsson cardinal: Erdös and Hajnal [1966]. The least Jónsson cardinal is weakly inaccessible or has cofinality ω: Rowbottom. If $V = L$, then there is no Jónsson cardinal: Keisler and Rowbottom [1965]. A model where every Jónsson cardinal is a Ramsey cardinal: Kunen [1970]. A model in which there is a Rowbottom cardinal of cofinality ω: Prikry [1970a]. Consistency of Chang's conjecture: Silver.

Exercise 29.14: Silver; cf. Benda and Ketonen [1974].

The generalization of Aronszajn's construction for successors of regular cardinals (assuming the GCH) is due to Specker [1951]. A model in which there exists no special Aronszajn tree on ω_2 was constructed by Mitchell [1972], who also showed that Mahlo cardinals are necessary for the construction. An analogous result for Aronszajn trees and weakly compact cardinals is due to Silver, see Mitchell [1972].

The generalization of forcing construction producing Suslin trees of larger heights is due to Hrbáček [1967]. Jensen's construction of a Suslin tree in L generates to larger cardinals, using $\diamondsuit$ and $\square$; see Jensen [1972]. The construction of a Suslin ω_2-tree using $\square$ and the GCH is due to Gregory [1976]. The model without Suslin ω_2-trees is due to Laver.

Section 30

Theorem 72 was discovered by Gaifman. Gaifman's results were announced in [1964b] and the proof was published in [1974]. Gaifman's proof used iterated ultrapowers (see also Gaifman [1967]). Silver in his thesis (1966, published in [1971a]) developed the present method of proof, using infinitary combinatorics, and proved the theorem under the weaker assumption of existence of κ with the property $\kappa \to (\aleph_1)^{<\omega}$. Gaifman proved that if there is a measurable cardinal, then there exists $A \subseteq \omega$ such that the conclusion of Theorem 72 holds in $L[A]$. Solovay formulated $0^\#$ and proved that it is a Δ^1_3 set of integers; Silver deduced the existence of $0^\#$ under weaker assumptions.

Construction of models with indiscernibles was introduced by Ehrenfeucht and Mostowski in [1956]; for further details and applications in model theory; see Chang and Keisler [1973].

The equivalence of existence of $0^{\#}$ with existence of a nontrivial elementary embedding of L (Theorem 73) is due to Kunen; the present proof is due to Silver.

Theorem 74 (and its corollaries) is due to Jensen. A proof of the theorem appeared in Devlin and Jensen [1975].

Exercise 30.5: Solovay.

Exercise 30.7: Magidor.

For further work on Silver indiscernibles, see Paris [1974].

Section 31

Most of the results in this section are due to Kunen, who in [1970] developed the method of iterated ultraproducts invented by Gaifman (cf. [1964b] and [1974]). Kunen found the representation of iterated ultraproducts (Lemma 31.11) and generalized the construction for M-ultrafilters. Kunen applied the method to obtain the main results on the model $L[D]$ (Theorem 76).

Theorem 75 (the proof of the GCH in $L[D]$) is due to Silver [1971c]; using a similar technique, Silver proved in [1971d] that in $L[D]$ there is a Δ^1_3 well-ordering of the reals.

The description of κ-complete ultrafilters over κ in $L[D]$ (Lemma 31.18) is due to both Kunen [1970] and Paris [1969]. Lemma 31.3 was first proved by Solovay. Lemma 31.5 is due to Gaifman; cf. [1974]. Lemmas 31.17 and 31.20 are results of Kunen [1970].

Kunen generalized the basic results on $L[D]$ to the model $L\langle D_\alpha : \alpha < \theta\rangle$ constructed from a sequence of measures (with $\theta <$ the least measurable cardinal in the sequence). Mitchell in [1974] generalized the construction further; one of the results in Mitchell's paper is that the model from Exercise 31.13 has a measurable cardinal with exactly two normal measures.

Jensen extended some of his results (e.g., $\diamondsuit$) from [1972] from L to $L[D]$. $0^{\dagger}$ was formulated by Solovay, in analogy with $0^{\#}$. The model $L(j)$ was considered by Jech in [1971d] and also by Hrbáček.

The result that in $L[D]$ the unique normal measure is the only measure that extends the closed unbounded filter (Exercise 31.12) appeared in Jech [1973a].

For further work on iterated ultrapowers of $L[D]$, see Bukovský [1973] and Dehornoy [1975].

Section 32

The equivalence of various formulations of weak compactness is a result of several papers. In [1964a] Hanf initiated investigations of compactness of infinary languages. Erdös and Tarski listed in [1961] several properties that were subsequently shown mutually equivalent (for inaccessible cardinals) and proved several implications. These properties included the partition property $\kappa \to (\kappa)^2_2$, the tree property from Section 29, and the properties from Exercises 32.3, 32.4, and 32.5. Hanf and Scott [1961] introduced Π^n_m indescribability and announced Theorem 77. Further contributions were made in the papers Hanf [1964b], Hajnal [1964], Keisler [1962], Monk and Scott [1964], Tarski [1962], and Keisler and Tarski [1964]. A complete list of equivalent formulations with the proofs appeared in Silver [1971a].

Without inaccessibility, some of the formulations of weak compactness are no longer equivalent; this is discussed by Boos in [1976]. Silver showed that every weakly compact cardinal is weakly compact in L, that every indiscernible is weakly compact in L (both in [1971a]), and that if $\aleph_2$ has the tree property, then $\aleph_2$ is weakly compact in L (cf. Mitchell [1972]). Mitchell pointed out that a weakly compact cardinal is not necessarily weakly compact in a submodel. Nonexistence of simple complete Boolean algebras of weakly compact size is proved in Jech [1974a].

Lemma 32.2: Hanf and Scott [1961].

Exercise 32.6: Kunen.

Ineffable cardinals were defined by Jensen (using the property from Lemma 32.7); Kunen proved the equivalence in Lemma 32.7. Kunen also showed that ineffable cardinals are Π^1_2 indescribable, and Jensen and Kunen proved that every ineffable cardinal is ineffable in L.

The main results on partition cardinals are due to Rowbottom, Reinhardt, and Silver. Rowbottom proved that if η_{ω_1} exists, then there are only countably many constructible reals (see [1971]); Silver then used the same assumption to prove Theorem 72 (i.e., that $0^\#$ exists; see [1971a]). Theorem 78e was proved by Silver in [1970a]; Theorem 78c and Exercise 32.13 were proved by Reinhardt and Silver (see [1965]).

Lemma 32.8 is due to Rowbottom [1971], and Lemma 32.9 is due to Silver. Lemma 32.10 is in effect proved in Jech and Powell [1971].

Kunen showed that both the existence of Jónsson cardinals and Chang's conjecture imply that $0^\#$ exists. Kunen also showed that in $L[D]$ every Jónsson cardinal is a Ramsey cardinal; see [1970]. Kleinberg in [1972] and [1973] showed that existence of Jónsson and of Rowbottom cardinals are equiconsistent. Silver used a Ramsey cardinal to construct a model in which Chang's conjecture holds and showed that if $\aleph_\omega$ is a Jónsson cardinal and a strong limit, then there exists a transitive model with a measurable cardinal.

CHAPTER 6

Section 33

Compact cardinals were introduced by Keisler and Tarski in [1964]; supercompact cardinals were defined by Reinhardt and Solovay. Theorem 79 is due to Vopěnka and Hrbáček [1966]; Theorem 80 is due to Kunen [1971b].

Solovay discovered that the singular cardinal hypothesis holds above a compact cardinal (Theorem 81); see [1974].

Menas and Magidor obtained several results on the relative strength of compact and supercompact cardinals. Menas in [1975] showed that it is consistent (relative to existence of compact cardinals) that there is a compact cardinal that is not supercompact. Magidor in [1976] improved Menas' result by showing that it is possible that the least measurable cardinal is compact (while by Lemma 33.10 it is not supercompact) and also showed that it is consistent (relative to supercompact cardinals) that there exists just one compact cardinal and is supercompact.

Kunen's proof of Theorem 80 uses a lemma of Ketonen (Lemma 33.3); Lemmas 33.5 and 33.6, which Solovay used in his proof of Theorem 81, are also due to Ketonen; see [1972].

Most results on supercompact cardinals (e.g., Lemmas 33.10 and 33.11) are due to Solovay; see Solovay, Reinhardt, and Kanamori [1978]; Magidor's paper [1971a] gives a generalization of the corollary of Lemma 33.11: the number of normal measures on $P_\kappa(\lambda)$. The example of a compact nonsupercompact cardinal (Lemma 33.12) is due to Menas [1975].

Exercises 33.7 and 33.8: Solovay [1974]; for the existence of an E-H function, see Erdös and Hajnal [1966].

The closed unbounded filter over $P_\kappa(\lambda)$ (Exercise 33.4) was introduced in Jech [1973a]. So was the partition property for $P_\kappa(\lambda)$. The equivalence of (iii) and (iv) in Exercise 33.9 as well as the results in Exercises 33.10 and 33.11 are due to Solovay. Menas in [1976a] showed that if κ is supercompact, then a strongly normal measure exists on every $P_\kappa(\lambda)$. Further results on the partition property and on stationary sets in $P_\kappa(\lambda)$ are in Magidor [1974].

Extendible cardinals were introduced by Reinhardt; he proved that extendible cardinals are supercompact; see [1974]. The present proof of Lemma 33.14, as well as Lemmas 33.13 and 33.15, are due to Magidor [1971b].

Lemma 33.16: Powell [1972].

Exercise 33.12: Solovay, Reinhardt, and Kanamori [1978].

Exercise 33.13: Morgenstern [1977].

Section 34

Theorem 82 is due to Solovay [1971]. Theorem 83 is due to Prikry [1975], and so is the model in which κ carries a σ-saturated ideal; see [1970a].

Exercise 34.4: Silver.
Exercises 34.5, 34.6, and 34.7: Prikry [1970a].

Section 35

Saturation of ideals and the problem of almost disjoint sets have been investigated by various authors, e.g., Solovay [1971], Kunen [1978], Baumgartner [1976b], and Jech and Prikry [∞]. The Jech–Prikry paper gives applications of generic ultrapowers, as well as of almost disjoint functions.

Theorem 84a on almost disjoint sets is due to Baumgartner; cf. [1976b]. The present proof follows Jech and Prikry [∞]. For Theorem 84b, see Jech and Prikry [1976] and [∞]. Theorem 85 is due to Solovay; cf. [1971]. The present proof of Theorem 85 uses ideas of Máté and Simpson; Solovay's proof uses Lemmas 35.3 and 35.11. (The present proof of Lemma 35.11 is as in Baumgartner, Hajnal, and Máté [1973].)

Examples I and II are due to Baumgartner (Taylor showed that the thin ideal in Example II is in fact $\aleph_3$-saturated.) The result that if κ carries a κ-saturated ideal, then κ is a weakly Mahlo cardinal is due to Solovay [1971]. If $\aleph_1$ carries an $\aleph_2$-saturated ideal then there are models with many measurable cardinals: Kunen [1970] and [1971b]. There are several consistency results on saturation of ideals using measurable cardinals: A model where $\kappa = 2^{\aleph_0}$ and $\operatorname{sat}(I) = \kappa$ (Kunen and Paris [1971]), a model where κ is inaccessible and $\operatorname{sat}(I) = \kappa$ (Kunen [1978]), models where κ is inaccessible and $\operatorname{sat}(I) = \kappa^+$ (Kunen and Paris [1971] and Boos [1974]). In [1978] Kunen constructs a model (using a huge cardinal) in which $\aleph_1$ carries an $\aleph_2$-saturated ideal.

Lemma 35.5 is in Baumgartner, Taylor, and Wagon [1977]; Corollary 2 of the lemma appeared in Namba [1974].

The problem of existence of ω-closed uncountable subsets of sets $A \subseteq \omega_2$ is discussed in Prikry and Solovay [1975].

Exercise 35.1: Smith and Tarski [1957].
Exercise 35.2: Solovay [1971].
Exercise 35.4: Kanamori [1976].
Exercise 35.5: Lévy, Silver.
Exercise 35.6: Prikry [1973].
Exercise 35.7: Baumgartner, Hajnal, and Máté [1973].
Exercise 35.8: Baumgartner, Taylor, and Wagon [1977].

Generic ultrapowers are studied in detail in Jech and Prikry [∞]. Solovay [1971] used the Boolean-valued ultrapower in the case of κ^+-saturated ideal. Precipitous ideals were introduced in Jech and Prikry [1976]. Theorem 86a is due to Jech and Prikry; the proof generalizes an earlier proof of Kunen, who in [1970] obtained the result for κ^+-saturated ideals. Theorem 86b was proved by Mitchell. Magidor subsequently constructed a model where the thin ideal is precipitous. Cf. Jech *et al.* [∞].

The proof of Silver's theorem (application 1) is a simplified version of Silver's original proof in [1974]. Applications 2, 3, and 4 are from Jech and Prikry [1976] and [∞]; for a related result on almost disjoint stationary sets, see Ketonen [1974].

The result (35.15) on κ-saturated ideals is due to Solovay [1971]. The method of changing cofinalities of cardinals is due to this author, and Namba [1974].

Lemmas 35.8, 35.9, 35.10: Jech and Prikry [1976].
Exercises 35.10, 35.11: Solovay [1971].
Exercises 35.12, 35.13: Jech and Prikry [∞].

The observation that the conclusion of Exercise 35.13 contradicts Chang's conjecture is due to Benda; previously, Magidor had shown by a different argument that Chang's conjecture implies $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$ if $\aleph_{\omega_1}$ is a strong limit, cf. [1977a].

Section 36

Theorem 87 is due to Kunen [1971b]. Theorem 88 is an unpublished result of Silver.

Lemma 36.2 (with the proof as in Exercise 36.1) is due to Lévy and Solovay [1967]. Consistency of measurable cardinals with the GCH by forcing (using the construction from Exercise 20.8) is due to Jensen; cf. [1974]. Lemma 36.3: Kunen and Paris [1971].

The method of inverse limits in interated forcing was developed by Silver and worked out in detail and applied by Menas in [1976b].

Section 37

Theorem 89 is due to Prikry [1970a]. Theorem 90 is due to Jech [1968] (see also Takeuti [1970]).

Magidor in [∞] generalized Prikry's construction, to change a supercompact cardinal into a singular cardinal of cofinality ω_1. Another construction of Magidor (using a supercompact cardinal) gives a model (of ZF) in which cf ω_1 = cf $\omega_2 = \omega$.

The proof of Theorem 86b is due to Mitchell.

The construction of a model with many normal measures (the example) is due to Kunen and Paris; cf. [1971]. The core model K has been constructed by Jensen; see Dodd and Jensen [∞]. Mitchell in [∞] shows that every Ramsey cardinal is Ramsey in K. In [1976b] Menas used Silver's technique of inverse limits to construct models with a supercompact cardinal κ, in which either the GCH holds, or $2^\kappa > \kappa^+$. Silver's method also proves the consistency of a weakly compact (or Ramsey) cardinal κ with $2^\kappa > \kappa^+$. In [1977b] and [1977c], Magidor uses a supercompact (respectively, huge) cardinal to construct a model where $2^{\aleph_n} < \aleph_\omega$ (respectively, $2^{\aleph_n} = \aleph_{n+1}$) for all n, and $2^{\aleph_\omega} > \aleph_{\omega+1}$. Martin and Mitchell prove in [∞] that if the closed unbounded filter is an ultrafilter, then there exist models with many measurable cardinals.

The measurable case of Lemma 37.1 is due to Lévy and Solovay [1967]; Lemma 37.2a: Kakuda [1972], 37.2b: Kunen [1978].

Exercise 37.1: Jensen.

Exercise 37.3: Prikry [1970a].

Exercises 37.5, 37.6: Ketonen [1973].

Section 38

Ramsey ultrafilters are investigated in Booth [1970]. Theorem 91 is due to Kunen [1976].

Exercises 38.1 and 38.2: Frolík [1968]; M. E. Rudin [1971].

The notion of a regular ultrafilter was introduced by Keisler who initiated its applications in model theory.

Lemma 38.6: Frayne, Morel, and Scott [1962]; Keisler [1964].

Exercise 38.6: Frayne, Morel, and Scott [1962].

Exercise 38.7: Keisler [1964].

Exercise 38.8: Keisler [1963].

Exercise 38.9: Jorgensen [1975].

In [1970b] Prikry showed that a nonregular ultrafilter over ω_1 contradicts the axiom of constructibility. Benda and Ketonen [1974] improved on his result, proving Lemma 38.9. Subsequent work of Kanamori (Lemma 38.8), cf. [1974] and [1976], and Ketonen (Lemma 38.11) led to Ketonen's Theorem 92; cf. [1976b].

In [∞∞], Magidor constructed a model that has a nonregular ultrafilter over ω_2.

Corollary (a) of Lemma 38.9: Benda and Ketonen [1974].

Corollary (b) of Lemma 38.9 and Corollary of Lemma 38.10: Jech and Prikry [∞].

CHAPTER 7

Descriptive set theory was developed by Luzin and others following the work of Borel and Lebesgue on real-valued functions. After Kleene's investigations of hierarchies of number theoretic predicates it became clear that these hierarchies are closely related to Luzin's hierarchy of projective sets. The introduction of recursion theoretical methods brought about new developments in descriptive set theory.

"Classical" descriptive set theory is treated in the books of Luzin [1930] and Kuratowski [1966]; both books have an extensive bibliography. For "modern" descriptive set theory, we refer the reader to Martin's expository article [1977].

Section 39

Borel sets were introduced by Borel in [1905]. Lebesgue in [1905] proved in effect Lemma 39.1. Suslin's discovery of an error in a proof in Lebesgue article led to a construction of an analytic non-Borel set and introduction of the operation $\mathscr{A}$. The basic results on analytic sets as well as Theorem 93 appeared in Suslin's article [1917].

Projective sets were introduced by Luzin [1925] and [1927b], and Sierpiński [1925] and [1927]. The present notation (Σ and Π) appeared first in the paper [1959b] of Addison who noticed the analogy between Luzin's hierarchy of projective sets and Kleene's hierarchy of analytical predicates [1955].

Lemma 39.4: Luzin [1930].

Lemma 39.5: Luzin [1927a].

Exercise 39.7: Suslin, Lebesgue.

For detailed treatment of Lebesgue measure, we refer the reader to Halmos' book [1950]; Lebesgue introduced his measure and integral in his thesis [1902]. Sets of first and second category were introduced by Baire in [1899].

Lemmas 39.7 and 39.9: Marczewski [1930a].

Lemma 39.8: Kuratowski and Ulam [1932].

Exercise 39.1: Marczewski [1934].

Theorem 94a (measurability of analytic sets) is due to Luzin [1917]. Theorem 94b (Baire property) is due to Luzin and Sierpiński [1923] and Theorem 94c (perfect subset) is due to Suslin; cf. Luzin [1930]. The present proof of (a) and (b) follows Marczewski [1930a]. Prior to Suslin (and following the Cantor–Bendixson theorem for closed sets) Young proved in [1906] the perfect subset result for G_δ and F_σ sets; and Hausdorff [1916] and Aleksandrov [1916] proved the same for Borel sets.

Section 40

The subject of this section is basically classical descriptive set theory, as in the books of Luzin [1930] and Kuratowski [1966]; the terminology is that of modern descriptive set theory based on the analogy with Kleene's hierarchies (see [1955]).

Theorem 95 appeared in Sierpiński [1925]. Theorem 96 (as well as Exercise 40.2) is due to Kondô [1938].

The basic facts on Σ^1_1 and Π^1_1 sets (Lemmas 40.2–40.5) are all in Luzin's book [1930] and some are of earlier origin: Lemma 40.3 was in effect proved by Lebesgue in [1905], and Corollary 1 of Lemma 40.4 and Lemma 40.5 were proved by Luzin and Sierpiński in [1923].

Reduction and separation principles were introduced by Kuratowski; they are discussed in detail in Kuratowski's book [1966] and in Addison [1959b]. Exercise 40.3: Kuratowski [1936].

The tree representation of Σ^1_2 sets is implicit in Shoenfield's proof in [1961].

Section 41

Theorem 97 and corollaries: In his announcement [1938] Gödel stated that the axiom of constructibility implies that there exists a nonmeasurable Δ^1_2 set and an uncountable Π^1_1 set without a perfect subset. Gödel did not publish the proof but gave an outline in the second printing (in 1951) of his monograph [1940]. Novikov in [1951] gave a proof of the corollaries (Kuratowski's paper [1948] contains somewhat weaker results) and Addison in [1959a] worked out the details of Gödel's outline of the proof of the theorem.

Theorem 98 is due to Shoenfield [1961]. Previously, Mostowski had established absoluteness of Σ^1_1 and Π^1_1 predicates (Exercise 41.6). In [1965b] Lévy used Shoenffield's theorem to prove Theorem 36.

Lemma 41.3: Solovay [1967]. Lemma 41.4 is due to Kechris and Moschovakis [1972] (earlier, Solovay proved the lemma under the assumption of the axiom of determinacy); the present proof is due to Blass [1975].

Theorem 99 and its corollary are due to Mansfield [1970] and Solovay [1969]. Lemma 41.5 was formulated and first proved by Mansfield; the present proof is Solovay's.

Part (b) of Theorem 100 is due to Martin; both (a) and (b) appear in Martin's paper [∞], where (a) is attributed to the classical descriptive set theorists.

Theorem 100c is due to Mansfield [1975]; the present proof is as in Kechris [∞].

Section 42

The results in this Section are due to Solovay; cf. [1970].

Section 43

Infinite games were first considered in the 1930s when Banach formulated the game G** and pointed out its relation to the Baire property. (Mazur then proved part (b) of Theorem 102; the first published proof of this result appeared in Oxtoby [1957].)

In [1953] Gale and Stewart investigated infinite games in general and proved that the axiom of choice implies that there exist undetermined games (Lemma 43.1) and that open games are determined (Lemma 43.8).

In [1962] Mycielski and Steinhaus proposed an axiom and called it axiom of determinateness (AD). In [1964–1966] Mycielski gave a comprehensive account of consequences of AD and related open problems.

Following Solovay's discovery that AD implies that $\aleph_1$ is a measurable cardinal, attention has been turned to the relation between determinacy and large cardinals. There have been numerous results in this direction, and I shall return to subject below.

Concerning determinateness of Borel games: Wolfe in [1955] proved that Σ^0_2 games are determined. Then Davis in [1964] proved determinacy for Σ^0_3, and in [1972], Paris proved Σ^0_4-determinacy. Finally, in [1975] Martin showed that all Borel games are determined (Theorem 104).

Theorem 102a is due to Mycielski and Świerczkowski [1964]; the present proof (and the covering game) is due to Harrington. Theorem 102c and its corollary (and Exercise 43.2) are due to Davis [1964].

Theorem 103 is due to Solovay; the present proof of measurability of $\aleph_1$ (Lemma 43.6) is due to Martin [1968].

Theorem 105: In [1970] Martin proved that analytic games are determined if $x^\#$ exists for all $x \subseteq \omega$; the converse was proved by Harrington [∞].

Various consequences of AD and PD are investigated in the forthcoming paper [∞] of Martin. Martin proved, among others, that AD implies that each $\aleph_n$, $3 \leq n < \omega$, is singular and that $\aleph_{\omega+1}$ is measurable. Other consequences of AD: There exist exactly two normal measures on $\aleph_2$ (Martin and Paris); $\aleph_{\omega+2}$ is measurable (Kunen); various other cardinals are measurable (Martin, Kunen); the partition property $\omega_1 \to (\omega_1)^{\omega_1}$ (Martin; see Kleinberg [1970] for further details on the partition property); there exist transitive models of ZFC with many measurable cardinals (Solovay); there exists a transitive model in which $\aleph_2$ is a measurable limit of measurable cardinals (Kunen). Further results relating determinacy with large cardinals are in Moschovakis [1970] and Friedman [1971]. Martin and Mitchell [∞] showed that if the filter over ω_1 is an ultrafilter, then there exist models with many measurable cardinals.

Among applications of projective determinacy in descriptive set theory are the results of Addison and Moschovakis [1968] and Martin [1968] (reduction principles), and Moschovakis [1971] (uniformization).

Section 44

Theorem 106 is due to Martin and Solovay [1970]; Corollary 3 is due to Kunen.

Example I: Solovay.

Example II: Vopěnka and Hrbáček [1967], and Solovay.

Example III: Jensen [1970]; for another construction of a Δ_3^1 nonconstructible real see Jensen and Solovay [1970], Jensen and Johnsbråten [1974].

Example IV: Harrington [1977].

Strong measure zero and Borel conjecture: Borel [1919].

Exercise 44.1: Marczewski [1930b].

Exercise 44.2: Kunen.

The construction of a Luzin set using the continuium hypothesis (Exercise 44.3) is due to Luzin [1914]; its application to get an uncountable strong measure zero set if the CH is true (Exercise 44.4) is due to Sierpiński [1928b]. For more on the Borel conjecture, see Sierpiński [1937].

Laver in [1976] constructed a model in which the Borel conjecture is true.

The argument in Exercise 44.5 is due to Erdös. The model in which $\mathscr{B}_{\mathrm{m}}$ has a dense subset of size $< 2^{\aleph_0}$ is due to Kunen and Solovay.

Exercise 44.6: Banach and Kuratowski [1929].

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NOTATION

Symbol	*Page*	*Symbol*	*Page*
V	3	α^β	20
$\{a, b\}$	4	Γ	20
$\{a\}$	4	$\rho_E(x)$	21
(a, b)	4	$\text{length}(E)$	21
$\varnothing$	6	$\|X\|$	22
$\bigcup X$	6	$\aleph_\alpha$	25
$P(X)$	7	ω_α	25
$X \times Y$	7	$\text{cf}\,\alpha$	26
$\text{dom}(R)$	8	$\boldsymbol{Q}$	29
$\text{ran}(R)$	8	$\boldsymbol{R}$	31
$\text{field}(R)$	8	$\boldsymbol{C}$	31
$f\colon X \to Y$	8	CH	31
${}^X Y$	8	$\mathcal{N}$	35
$f \mid X$	9	U_s	35
$f[X]$	9	Seq	36
$f_{-1}(X)$	9	$[T]$	37
f^{-1}	9	AC	38
$[x]$	9	DC	41
$X/\equiv$	9	$[A]^\kappa$	43
$\sup(X)$	12	$[A]^{<\kappa}$	43
$\inf(X)$	12	$\sum_{i \in I} \kappa_i$	43
o.t. W	14	$\prod_{i \in I} \kappa_i$	44
Ord	14	GCH	46
ω	16	$2^{<\kappa}$	47
$\boldsymbol{N}$	16	$\gimel(\kappa)$	48
$\langle a_\xi : \xi < \alpha \rangle$	16	$\triangle \{X_\alpha : \alpha < \kappa\}$	57
$s^\frown x$	16	$\\|\varphi\\|$	67
$\lim_{\xi \to \alpha} \gamma_\xi$	18	$\text{TC}(S)$	71
$\alpha + \beta$	18	V_α	71
$\alpha \cdot \beta$	19	$\text{rank}(x)$	72

Symbol	*Page*	*Symbol*	*Page*
$\beth_\alpha$	72	MA	230
$\mathscr{H}(\mathbf{A})$	74	$B * C$	233
$\mathrm{ext}_{\mathbf{E}}(x)$	74	$P * Q$	235
$\mathfrak{M} \models \varphi$	78	$D : B$	237
$\mathfrak{B} \prec \mathfrak{A}$	81	$\lim \mathrm{dir}_{i \to \alpha} B_i$	237
HF	86	$h(d)$	238
HC	86	W	254
$\mathfrak{F}_1, \ldots, \mathfrak{F}_{10}$	93	$\Box$	255
$\mathrm{cl}(M)$	96	$X \subseteq A$ mod.f.	257
$\mathrm{def}(U)$	100	$f \prec g$	259
L_α	100	B^*	269
$<_L$	109	$Col(\kappa, \lambda)$	276
Θ	112	$\sum_{i \in I} r_i$	299
Σ_n	114	$I \mid Z$	300
Π_n	114	$=^*$	305
Δ_n	114	$\in^*$	305
OD	133	$Ult_U(\mathfrak{A})$	307
HOD	134	c_a	308
$p \Vdash \sigma$	142	$[f]$	308
$u \triangle v$	148	Ult	308
B/I	148	$\mathrm{ext}(f)$	309
$\sum\{u : u \in X\}$	150	d	311
$\prod\{u : u \in X\}$	150	$\mathrm{M}(X)$	314
$\mathrm{r.o.}(P)$	153	$f_*(U)$	317
$\mathrm{sat}(B)$	157	$D \leq U$	318
$\mathrm{sat}(P)$	158	$D \equiv U$	318
$\lVert\varphi\rVert$	159	$[A]^n$	321
V^B	161	$\kappa \to (\lambda)^n_m$	322
$p \Vdash \varphi$	165	$\exp_n(\kappa)$	324
$\mathfrak{M}^B$	166	$\kappa \to (\alpha)^n_m$	325
i_G	167	$\kappa \to (\lambda)^\alpha$	327
$\mathfrak{M}[G]$	167	$\kappa \to (\alpha)^{<\omega}_m$	327
$\check{x}$	169	$[\kappa]^{<\omega}$	327
G	170	$\mathbf{S}$	337
$P_1 \times P_2$	187	$0^\#$	339
$B_1 \oplus B_2$	187	$\mathscr{H}^{\mathfrak{A}}(X)$	342
$\mathrm{s}(P)$	189	$\Sigma(\mathfrak{A}, I)$	342
$P^\alpha(S)$	198	$x^\#$	356
$\mathrm{sym}_{\mathscr{G}}(x)$	199	$Ult_U^{(\alpha)}(M)$	366
$\mathrm{fix}_{\mathscr{G}}(E)$	200	$Ult^{(\alpha)}$	366
HS	202	$i_{\alpha\beta}$	366
$\mathrm{o}(x)$	217	$\forall^*\alpha$	369
$\mathrm{height}(T)$	217	$X_{(\alpha)}$	369
$\Diamond$	226	U_n	369
MA_κ	230	U_α	371

Symbol	*Page*	*Symbol*	*Page*
$0^\dagger$	380	$\Delta^1_n(a)$	510
$\mathscr{L}_{\kappa,\omega}$	381	WF	515
$\mathscr{L}_{\kappa,\kappa}$	381	WO	515
η_α	392	LO	516
$\hat{P}$	398	$\|x\|$	516
$\triangle_{a \in A} C_a$	408	WF_α	516
κ_P	409	WO_α	516
$\hat{\lambda}_P$	409	$p[T]$	525
$f_*(I)$	417	$A^{\mathfrak{M}}$	530
$f_*(\mu)$	417	Σ_α	538
$\mathrm{sat}(I)$	428	Π_α	538
$Ult_G(\mathfrak{M})$	438	BC	538
$W_1 \leq W_2$	438	A_c	538
$F < G$	439	$\mathscr{B}_{\mathrm{m}}$	542
$\lim \mathrm{inv}^{(\mathscr{F})}_{i \to \alpha} B_i$	456	$\mathscr{B}_{\mathrm{c}}$	542
$\lim \mathrm{inv}_{i \to \alpha} P_i$	457	$\mathrm{Ra}(\mathfrak{M})$	544
$\mathbf{\Sigma}^0_\alpha$	495	$\mathrm{Ge}(\mathfrak{M})$	544
$\mathbf{\Pi}^0_\alpha$	495	AD	550
$\bar{a}(n)$	499	G_A	550
$\mathscr{A}$	499	$\sigma * b$	550
$\mathbf{\Sigma}^1_n$	500	$a * \tau$	551
$\mathbf{\Pi}^1_n$	500	G^{**}_X	553
$\mathbf{\Delta}^1_n$	500	G^*_X	555
μ	503	$\mathrm{cone}(x)$	556
$A \triangle B$	504	$\mathscr{F}_c$	556
$\bar{A}$	505	PD	560
$[T]$	509	$\mathscr{F}(T)$	571
$\mathbf{\Sigma}^1_n(a)$	510	$S(x)$	575
$\mathbf{\Pi}^1_n(a)$	510		

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LIST OF CORRECTIONS

Page	*Should be*
3, line -3	$\varphi(x, p_1, \ldots, p_n) \leftrightarrow \psi(x, q_1, \ldots, q_m)$
28, line 17	$\kappa = \lim_{\xi \to \mathrm{cf}\, \kappa} \alpha_\xi$
34, line -16	For each α such that $\alpha + 1 < \theta$
68, line -4	$\Vert\varphi\Vert_S = \Vert\varphi\Vert$
71, line 2	$S_{n+1} = \bigcup S_n$
72, line 8	$\mathrm{rank}(x)$
76, line 2	(Note that if $\mathbf{F} = \mathbf{G}(\mathbf{F}\vert\mathrm{ext}(x))$ for some $\mathbf{G}$, then
81, line 5	$\langle M, E\rangle \models \ulcorner\varphi\urcorner[a_1, \ldots, a_n]$
83, line -6	$\mathbf{P} \models [\cdots]$
100, line -17	$X = \{u \in U : U \models \varphi(u, x_1, \ldots, x_n)\}.$
102, line -18	$(\forall z \in \mathrm{dom} X) z \in Z$
102, lines -11 to -10	$(\forall z \in Z)(\exists x \in X)(\exists u \in \mathrm{ran}\, X)(\exists v \in \mathrm{dom}\, X)$ $(x = (v, u) \wedge z = (u, v)) \wedge (\forall x \in X)(\forall u \in \mathrm{ran}\, X)$ $(\forall v \in \mathrm{dom}\, X)(\exists z \in Z)(x = (v, u) \to z = (u, v))$
103, line 11	$\neg\exists f(\exists\beta \in \alpha)\psi(\alpha, \beta, f)$
104, line -5	to the formula $z \in \mathfrak{G}(y)$.
105, line 9	α is an ordinal $\wedge \exists f \varphi(f, \alpha) \wedge y = \mathbf{G}(f)$
105, line 14	if $(\alpha, y) \in \mathbf{F}^{\mathfrak{M}}$,
109, line 6	$W^\alpha_{n+1} = \{\mathfrak{F}_i(X, Y) : X, Y \in W^\alpha_n, i = 1, \ldots, 10\}$
111, lines -17 to -15	f is a function on $\alpha{+}1$ and $(\forall\xi < \alpha)[f(\xi{+}1) = \mathrm{def}(f(\xi))]$ and $(\forall\xi < \alpha + 1)[\xi$ is a limit $\to f(\xi) = \bigcup \mathrm{ran}(f\vert\xi)]$ and $f(\alpha) = y$
116, line -8	$\exists u\, [u \in x \wedge \varphi(u, x, \ldots)]$

I am indebted to all those who pointed out to me the numerous misprints and errors in the text, and above all to Randall Dougherty, who supplied the longest list by far.

124, lines 13–14	If M is a transitive model of ZF^-, then $\mathrm{TC}(S)$
124, lines 15–16	If M is a transitive model of ZF^-, then $\mathrm{rank}(x)$
130, line 6	(ii) $L_{\omega_\alpha}[A] \subseteq S$, $A \in S$, $X \in S$, $\beta \in S$;
131, lines 15–16	$L(T)$ is a model of ZF and $T \subset L(T)$.
131, line 18	is not necessarily well-ordered. Also, T need not be an element of $L(T)$.
131, lines 19–20	then (15.13) defines a model $L(\mathbf{T})$ of ZF such that $\mathbf{T} \subseteq L(\mathbf{T})$.
132, line -11	Exercise 14.18.]
155, line 11	$X_D \subseteq G$ and hence
155, line 12	then $e[D]$ is a dense set
163, line 11	By induction on unordered triples $\{\rho(x), \rho(y), \rho(z)\}$,
164, line 1	By the definition of $\|x = z\|$ we have
164, lines 9–10	over all $t \in \mathrm{dom}(x)$, we get $\sum_{t \in \mathrm{dom}\, x}$
164, line -9	equal $\|X \subseteq Y\|$
171, line -10	$\mathfrak{M}[G] \models \exists v \varphi(u, v)$,
172, line -11	$\|(\exists y \in x)\varphi(y)\|$
176, line 5	we can show that, letting $\mathbf{G}$ be the canonical name for a generic filter on P,
176, line 16	($\forall$ model $\mathfrak{M}$ of ZFC)($\forall G \subset P^{\mathfrak{M}}$)
180, lines 18–19	Note that if P is dense in B, then
187, line -11	(In fact $B_1 \oplus B_2$ is the completion of the direct sum
195, line -5	$p \supset q$
196, line 11	$\mathfrak{M}[\mathbf{G}] \models \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad (\exists p \in \mathbf{G})\, p \Vdash \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$
196, lines 17–18	as when we forced with a set. (Separation also needs some extra work which we leave to the reader.)
197, line -16	$\mathrm{sat}(P^{<\alpha}) \leq \beth_\alpha{}^+$
204, line 16	that maps ω one-to-one into A.
204, line 18	f maps $\check{\omega}$ one-to-one into A
210, line 9	let $\bar{\sigma}$ be the set of all permutations
210, line 13	$\bar{H} = \bigcup\{\bar{\sigma} : \sigma \in H\}$
211, line 2	$x \in \mathfrak{V}$
213, line -12	$(\pi p)(\pi(n, i)) = p(n, i)$
219, lines 2–13	Hence T_1 satisfies condition (v). Next we satisfy property (vi): For every chain C in T_1 with the property that $C = \{z : z < x\} = \{z : z < y\}$ and $x \neq y$, we add an extra node a_C and stipulate that $z < a_C$ for all $z \in C$, and $a_C < x$ for every x such that $x > z$ for all $z \in C$. The resulting tree T_2 satisfies (iii), (v) and (vi). For each $x \in T_2$ there are uncountably many branching points $z > x$, i.e. points that have at least two immediate successors (because there

	is no uncountable chain and T_2 satisfies (v)). The tree $T_3 =$ {the branching points of T_2} satisfies (iii), (v) and (vi) and each $x \in T_3$ is a branching point. To get property (iv), let T_4 consist of all $x \in T_3$
224, lines -11 to -10	The forcing (22.6)-(22.7) is equivalent to (i.e. has the same r.o. as) the forcing in Exercise 19.7.
227, lines -3 to -2	$(P, <)$ is equivalent to the forcing in Exercise 19.7.
237, line 3	Assume that P and $\mathbf{Q}$ are separative.
242, line -3	For the rest of this Section, $\prod_{i\in I} P_i$ denotes the product with finite support (20.4).
244, lines -20 to -19	trees of height ω_1 with no uncountable branches such that each
244, line -7	[First modify f so that in addition to (24.1), $f(x) \neq f(y)$ whenever x and y have the same immediate predecessor. Then, if $x \in T$ is at level α,
245, line 8	such that π is order-preserving and so that the range of π is finite so far, we can easily
246, line 14	By Lemma 24.2
246, line -14	for any Aronszajn tree T satisfying (22.3)(vi):
254, line -9	[We may assume that $\mathfrak{M} \models$ CH. A forcing condition
264, line 16	Let κ and λ be infinite cardinals.
266, line 2	by Corollary 2 of Lemma 25.2.
266, line 14	$h(x) = \prod\{u \in B : u \geq x\}$
273, line -11	$\mathfrak{M} \subset \mathfrak{N} \subset \mathfrak{M}[G]$
275, line -14	is a normal Suslin tree
276, line 2	a normal Suslin tree
277, line -13	$\lvert Col(\aleph_0, \lambda)\rvert = \lvert P_\lambda\rvert^\lambda = 2^\lambda$,
278, lines -4 to -3	by Theorem 62 (we are still in V^C),
279, line 1	$C * (B : C) = B$
289, line -7	$\forall T \in G\, \exists s \in T$
298, line -5	(cf. (17.25))
303, line 7	$\omega_1 - \bigcup_{n=0}^{\infty} A_{\alpha n}) \subseteq \alpha + 1$
314, line -17	If $\alpha < 2^\lambda$,
317, line 8	$\{X \subseteq \kappa : f_{-1}(X) \in U\}$
318, line 12	Since $P(\kappa) \subset M$, we have $Ult_U(\kappa, <) \in M$.
319, line 7	almost everywhere mod U. (Otherwise let $f(\xi)$ be the least α
322, lines 4–7	By the induction hypothesis, there exists for each $a \in \omega$ and each infinite $S \subseteq \omega$ an infinite set $H_a^S \subseteq S - \{a\}$ such that F_a is constant on $[H_a^S]^n$. We construct an infinite sequence

	$\{a_i: i = 0, 1, 2, \dots\}$: We let $S_0 = \omega$ and $a_0 = 0$, and $S_{i+1} = H^{S_i}_{a_i}$, $a_{i+1} =$ the least element of S_{i+1} greater than a_i.
328, line 1	$\kappa \to (\omega)^{<\omega}$
333, line 6	then $\mathfrak{A}$ has an elementary submodel
334, line -3	if T has properties (22.3) (ii) and (vi), and if each $x \in T$ has at most two immediate successors
336, line 6	a limit ordinal of cofinality $< \lambda$,
347, line -8	The next two lemmas prove the uniqueness of Silver indiscernibles, and of the corresponding E.M. set.
348, lines 4–5	it follows that $I \cap \mathbf{S}$ is infinite, and $\Sigma(L_\kappa, I) = \Sigma(L_\kappa, I \cap \mathbf{S}) = \Sigma(L_\kappa, \mathbf{S} \cap \kappa)$. Hence
348, line -10	if and only if L_λ satisfies
348, line -7	L_λ satisfies
348, line -5	$L_\lambda \models$
349, line 1	indiscernibles for L_λ
349, line 7	$t(x_1, \dots, x_m, y_1, \dots, y_n) < y_1$
349, line 10	Let $x_1, \dots, x_m$ be the first m elements of I.
350, line -4	is a Δ_1 property
359, line -20	delete the sentence
370, line 5	such that $h = [F]_{U_n}$
372, line 5	$F(t) = f_{(t)}$
372, line 6	$\tilde{f}$
372, line -10	$\{\alpha < \gamma: X_{(\alpha)} \in U_n\} \in U$
372, line 19	definition (31.29)
372, line -4	If ${}^\omega M \subset M$ then each U_α is an iterable M-ultrafilter
374, line -11	**Lemma 31.13.** *Assume that* $L[D] \models D$ *is a normal measure on* κ.
374, line -7	*Then* M *contains all subsets of* κ *in* $L[D]$.
374, line -2	every $X \subseteq \kappa$ in $L[D]$ is
374, line -1	for all $X \subseteq \kappa$ in $L[D]$.
376, line -17	Let $\lambda = \lvert\gamma\rvert^{++}$,
382, line -2	$\Sigma^* \cap \{\sigma_\alpha : \alpha \in \mathrm{dom}(t)\}$
384, line 5	and on the other hand that, for each $\alpha < \kappa$, if $\forall\beta < \alpha\ c_\kappa > c_\beta$, then $c_\kappa > c_\alpha$;
384, line -14	the least nontrivial κ-complete algebra
388, line 5	$\langle V_\kappa, \in, U' \cap V_\kappa \rangle$
388, line 7	$\langle V_\alpha, \in, U' \cap V_\alpha \rangle$
388, line 9	$\langle V_\alpha, \in, U \cap V_\alpha \rangle$
388, line 11	$\langle V_\alpha, \in, U \cap V_\alpha \rangle$

389, line 2	*for every* $\alpha < (\kappa^+)^L$,
389, line 7	Let $\alpha < (\kappa^+)^L$.
389, line 8	an enumeration, in L, of
395, line -16	*Proof of Theorem 78d.*
397, line -5	and $\lvert\eta\rvert = \lvert\gamma\rvert < \lvert\alpha\rvert$,
401, line -13	**Lemma 33.3.**
406, line -4	Let D be the ultrafilter over λ constructed in Lemma 33.5. It follows from the construction of D that $[d]_D = \lim_{\gamma\to\lambda} j_D(\gamma)$. For almost all α
407, line 1	and since $[d] = \lim_{\gamma\to\lambda} j_D(\gamma)$, it follows
412, line -17	restriction of $j_{-1}(F)$ to P.
412, line -15	$Q = F_Q[{}^{\omega}P] = F_P[{}^{\omega}P] = P$
412, line -11	F_P is an E–H function on P and λ_P is regular and P has the property that
412, line -2	if for each nonempty $P \in X$,
413, line -11	hence $M \models \exists P, Q \in j(X)$ such that
413, line -3	extendibility
414, line -1	then κ is extendible; and for $\kappa < \gamma < \alpha$, if there is an elementary embedding $j : V_\gamma \to V_\delta$ with critical point κ, then $V_\alpha \models$ (there is an elementary embedding).
415, line 5	cofinality
415, line 7	$j\lvert V_\gamma$ reflects to a witness to extendibility.
415, line -5	ultrafilter U over $\{X \subseteq \lambda : \text{o.t.}(X) = \kappa\}$.
419, lines 15–16	The proof of Lemma 34.3 uses Skolem functions and arguments similar to those in Theorem 71 and Lemma 29.10. The key ingredient is the following lemma:
420, line 7	goes through in the present context (use $P = \{Y \subseteq \lambda : Y \le_{L[D]} X\}$).
420, lines 13–18	Let I be the ideal of sets of measure zero. I is a normal σ-saturated κ-complete ideal over κ. Let $J = I \cap L[I]$. We have $L[J] = L[I]$ and in $L[I]$, J is a normal $\aleph_1$-saturated κ-complete ideal over κ. (If we could assume that $\aleph_1^{L[I]} = \aleph_1$, it would now follow that κ is a measurable cardinal in $L[I]$.) Since I am not able to show directly
426, line -2	regular uncountable cardinal
427, line 4	Let $F : [\mathcal{G}]^2 \to \lambda$
436, line -10	*ideal over* κ
437, line 9	Lemma 35.5
444, lines -5 to -4	we have $\mathfrak{M}[G] \models \lvert P^{\mathfrak{M}}(\kappa)\rvert \le (2^{\aleph_0})^{\mathfrak{N}}$ and $\mathfrak{N} \models 2^{\aleph_0} = \aleph_{j\kappa}$. Now $\aleph_{j\kappa}^{\mathfrak{N}} \le \aleph_{j\kappa}^{\mathfrak{M}[G]}$;
445, line 1	holds in $\mathfrak{M}[G]$;

445, line 6	we have $\mathfrak{M}[G] \models$
445, line -6	if $X \subseteq \kappa$ has measure one in a normal κ-saturated ideal, then
445, line -4	$\mathrm{cf}\,\alpha > \omega$
447, line -3	such that $\mathrm{cf}\,\nu \geq \mathrm{sat}(I)$
448, lines 16–17	Now if $X \subseteq \kappa$ and $X \in L[A]$, then because π is the identity on κ, we have $X = \pi(Y) = Y \cap \kappa$ for some $Y \in H$, and $Y = \{\xi\colon L[A] \models \varphi(\xi, E, A)\}$.
449, line -13	λ-complete
456, line -3	(iii) for every $i < \alpha$, if $b \in B_i - \{0\}$ and $f \in \mathcal{F}$ are such that $f(i) \cdot b \neq 0$, then there is a $g \in \mathcal{F}$ such that $g(j) = f(j) \cdot b$ for all $j \geq i$.
456, line -2	Let $B = \mathrm{r.o.}(\mathcal{F}, \leq)$;
457, line -17	(iii) if $p \in P_i$, $q \in P_j$ and $p \leq h_{ij}(q)$ then $p \cdot q \in P_j$.
457, line -10	Section 23
462, line -14	let $\mathbf{Q}_\alpha = \{\mathit{1}\}$
463, lines -11 to -10	I claim that in $\mathfrak{N}$ we can apply the factor lemma to jP at $\alpha = \kappa + 1$. First we note that $(jP)_\alpha = P_\alpha$ for all $\alpha < \kappa$, and since $(jP)_\kappa$ is the direct limit, we have $(jP)_\kappa = P_\kappa$. Since $\mathbf{Q}_\kappa$ is the same in $\mathfrak{M}$ and $\mathfrak{N}$, it follows that $(jP)_{\kappa+1} = P_{\kappa+1}$. The first nontrivial step above α
468, line 7	somewhat easier. We may assume that λ is regular in $\mathfrak{M}$ (by Lemma 17.6). Replace B by B_a, where $a = \Vert\kappa$ carries a λ-saturated κ-complete ideal$\Vert$. Let $\mathbf{J} \in \mathfrak{M}^B$
468, line -14	(Again we may assume that λ is regular in $\mathfrak{M}$.)
472, line 20	Let $\lambda > \kappa^+$
473, line 8	$p \cup q_\alpha \Vdash \check{\alpha} \in \mathbf{X}$
475, line 6	$j(x) = i^{\mathfrak{N}}_{G \times H}(j\mathbf{x})$
477, line -17	is a one-to-one function of a subset of ω
479, line 4	$\in D$, $a_0 \in X_0$ and
479, line -13	such that $a_0 \in X_0$ and
480, line 4	Neither X_0 nor X_1
480, line 7	$f^k(x) = f^m(y)$
481, lines 5–7	For every $\alpha \leq \kappa$, let B_α be the measure algebra of ${}^\alpha\{0,1\}$. B_α is a subalgebra of B and if $\mathrm{cf}\,\alpha \geq \omega_1$ then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$.
481, lines -18 to -17	where G^α is $\mathfrak{M}[G_\alpha]$-generic on $B : B_\alpha$. We state without proof that $B : B_\alpha$ is in $\mathfrak{M}[G_\alpha]$ a measure algebra.
482, line 16	$\sum_{n \in Y} m(\Vert\check{n} \in \mathbf{X}\Vert) < \varepsilon$
482, line 17	$\mu_a(\mathbf{X}) = 0$ for all $a \in B$.
482, line 18	$\lim_U m(\Vert\check{n} \in \mathbf{X}\Vert) = 0$
491, line 11	$g^\alpha_n \in L_{f_\delta(\alpha)}$

496, line -3	$a_n \in \mathbf{\Pi}^0_{\alpha_n}$
498, line 13	and if $s \subset t$ then center$(C_t) \in C_s$.
499, line 17 (left)	$\bigcap_{n=0}^{\infty}$
499, line 17 (right)	$\bigcap_{n=0}^{\infty}$
503, line 4	**Exercise 39.7.***
503, lines 6–10	[First show that every Borel set is a one-to-one continuous image of a closed set. Then, using (39.17) and Exersises 39.5 and 39.6, show that a one-to-one continuous image of a closed set is Borel.]
507, line 2	since $\mathcal{G}$ is countable
508, line -11	Since *Seq* is
508, line -10	$\bigcup_{k=0}^{\infty}$
516, line 8	is a $\mathbf{\Sigma}^1_1$ set;
517, line -11	$A = \{x \in \mathcal{N} : \exists z \in \mathcal{N}\, (x, z) \in T\}$.
522, line 8	(b) $\Vert(T(y))_{t_n}\Vert \leq \Vert(T(z))_{t_n}\Vert$ for all $z \in P_n$
522, lines 9–10	(c) $y(n) \leq z(n)$ for all $z \in P_n$ such that $\Vert(T(y))_{t_n}\Vert = \Vert(T(z))_{t_n}\Vert$
522, line -2	$(y, z, k) \in A \wedge (z, y, k) \in A$
522, line -1	$\wedge[(z, y, n) \in B \vee ((z, y, n) \in A \wedge z(n) < y(n))]]$
524, line 19	$(x, h(u, v)) \in B$ iff $(x, v) \in U$,
526, line 2	$\mathrm{dom}(h^*) \cap U(x, y)$,
526, line -4	decomposition into $\aleph_1$ Borel sets
526, line -2	$x \in B^{\gamma}_{\alpha} \quad \leftrightarrow$
527, line 15	$x <_L y$ iff $y \not\leq_L x$
527, line -7	($\exists$ transitive set M)
527, line -6	By the principle of dependent choices $\mathrm{TC}(\{u, x\})$ is
529, line 6	There is a sentence Θ (provable in ZF) such that
529, lines -7 to -6	the projection dom(P) is also uncountable
530, line 8	Exercise 14.17
533, lines 4–5	if and only if it holds in $\mathfrak{N}$. [delete the rest of the sentence]
535, line 8	$x \in \mathrm{LO} \wedge (\exists f : \omega \to \mathcal{N})$
536, line 15	is not well-founded
536, line 16	is not well-founded
539, line -15	$((a, c) \in Q \to (a, d) \in P)$
539, line -5	there is a unique smallest countable set
545, line 14	$\Vert\mathbf{a} = x_{\mathbf{G}}\Vert = 1$.
546, line 4	let $X \in \mathfrak{M}[G]$
546, line -6	*that are not random over*
558, line 8	$\exists n(\Vert a\Vert = \Vert(g(a))_n\Vert)$

559, line 17	such that cone$(b) \subseteq f_{-1}(X_\alpha)$.
561, line 10	embedding $j : V \to M$ with the property that $P(\lambda) \subseteq M$ for every
566, lines 19–21	adjoins a Cohen generic $x \in {}^\omega\omega$. Let $a_0 \geq a_1 \geq \ldots \geq a_n \geq \ldots$ be positive reals. Since $\|A\| < 2^{\aleph_0}$, MA implies that there is $x \in {}^\omega\omega$ P-generic over every $L[a, \langle a_n : n \in \omega\rangle]$, $a \in A$. Let $r_0, r_1, \ldots, r_n, \ldots$ be an enumeration of all rational numbers; let for each n, I_n be the interval
566, lines -2 to -1	and hence $E - U$ has size $< 2^{\aleph_0}$. By Exercise 44.2 there are intervals I_{2n+1} of length a_{2n+1} such that $E - U \subseteq \bigcup_{n=1}^\infty I_{2n+1}$.]
569, line -14	$[A] \Vdash$
576, line 3	$(u, E) \in P_X : u \subseteq a$ and, for every
577, line -13	if $p' \leq p$, $E \subseteq E'$, $u \subseteq u'$, and, for every
582, line -11	The independence of the prime ideal theorem from ZF
592, line 3	in iterated forcing
594, line 12	Mazur formulated the game
594, line 13	Banach then proved
603, line -7	(Y. Bar-Hillel, ed.), pp. 129–145.
606, line -16	Changing measurable into accessible cardinals,
608, line -22	**92** (1970), 1–56.
608, line -6	Sur une définition des ensembles mesurables B sans nombres transfinis,